2012 Mathematical Methods (CAS) Trial Exam 2 Solutions

SECTION 1

1	2	3	4	5	6	7	8	9	10	11
B	A	C	C	A	A	E	E	B	C	A

12	13	14	15	16	17	18	19	20	21	22
A	C	C	B	E	D	B	D	B	E	C

Q1 $x+3=0$, $x=-3$ B

Q2 $\log_{\frac{1}{e}}\left(\frac{1}{e^{\sqrt{x}}}\right)=\log_{\frac{1}{e}}\left(\frac{1}{e}\right)^{\sqrt{x}}=\sqrt{x}\log_{\frac{1}{e}}\left(\frac{1}{e}\right)=\sqrt{x}$ A

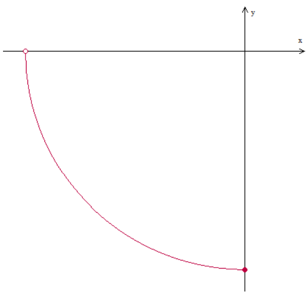
Q3 $\sin(2x)+\cos(2x)=0$, $\frac{\sin(2x)}{\cos(2x)}=-1$ where $\cos(2x)\neq 0$,

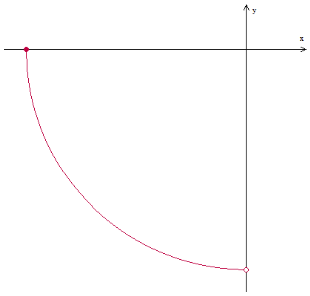
$\tan(2x)=-1$, $2x=n\pi-\frac{\pi}{4}$, $x=\frac{(4n-1)\pi}{8}$ C

Q4 $y=-5$, $x=0$; $y\to 0^-$, $x\to-\infty$; $y\to 2^+$, $x\to 7^-$;

$y\to\infty$, $x\to 2^+$ C

Q5 f f^{-1}





A

Q6 $f(x)=2(x-a)^2(x-b)$,

$g(x)=2e^x(2x-c-a)^2(2x-c-b)=0$

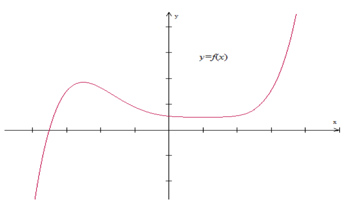
The two unique solutions are $x=\frac{c+a}{2}$ and $x=\frac{c+b}{2}$.

Since $a>b>0$, both solutions are positive if $c>-b$. A

Q7 $g(x)=f\left(\frac{x}{2}+b\right)=2\left(\left(\frac{x}{2}+b\right)+b\right)^3-a=2\left(\frac{x}{2}+2b\right)^3-a$

$=2\left(\frac{1}{2}(x+4b)\right)^3-a=\frac{1}{4}(x+4b)^3-a$ E

Q8



E

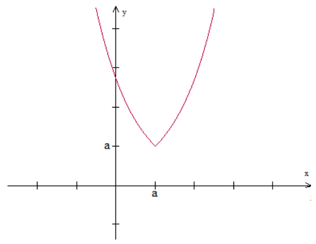
Q9 The intersection of $y=f(x)$ and $y=f^{-1}(x)$ is on the line $y=x$.

Let $x=e-\log_e(\log_e x)$, $\log_e(\log_e x)=e-x$, $x=e$, $\therefore y=e$ B

Q10 $\frac{1}{\sqrt{2}}\leq\sqrt{1-\cos(2x)}\leq 1$, $\frac{1}{2}\leq 1-\cos(2x)\leq 1$,

$0\leq\cos(2x)\leq\frac{1}{2}$, $\frac{\pi}{3}\leq 2x\leq\frac{\pi}{2}$, $\frac{\pi}{6}\leq x\leq\frac{\pi}{4}$ C

Q11



A

Q12 Given $g(x)=f^{-1}(x)$, $f(a)=b$ and $f'(a)=\frac{1}{a}$, then

$g(b)=a$ and $g'(b)=\frac{1}{f'(a)}=a$ A

Q13 Given function $f(x)$, $x\in R$, then $f(|x|)=\begin{cases} f(x), & x\geq 0 \\ f(-x), & x<0 \end{cases}$.

$f(-x)$ for $x<0$ is the reflection (in the y-axis) of $f(x)$ for $x>0$.

$\therefore f'(|x|)=0$ at $(-3,-1)$, $(-1,2)$, $(1,2)$ and $(3,-1)$ C

Q14 $\int_a^b f(x)dx=[F(x)]_a^b=F(b)-F(a)=c$

$\int_{\frac{a-h}{2}}^{\frac{b-h}{2}} 2f(2x+h)dx=\left[\frac{2F(2x+h)}{2}\right]_{\frac{a-h}{2}}^{\frac{b-h}{2}}=F(b)-F(a)=c$ C

Q15 $\int_0^{\frac{\pi}{8}} g(x)dx=\int_0^{\frac{\pi}{8}}\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)dx=0.4824$ by CAS

Average value $=\frac{\int_0^{\frac{\pi}{8}} g(x)dx}{\frac{\pi}{8}-0}\approx\frac{0.4824}{\frac{\pi}{8}}\approx 1.2284$ B

Q16 $f'(x)=\frac{1}{\sqrt{x+10}}$, $f(x)=\int\frac{1}{\sqrt{x+10}}dx=2\sqrt{x+10}$

$bf(a)\approx f(a)+(1.02a-a)f'(a)$

$bf(a)-f(a)\approx 0.02af'(a)$, $(b-1)f(a)\approx 0.02af'(a)$

$\therefore (b-1)2\sqrt{a+10}\approx 0.02a\times\frac{1}{\sqrt{a+10}}$

$b-1\approx\frac{0.01a}{a+10}$, $b\approx\frac{1.01a+10}{a+10}$ E

Q17 $\Pr(JandJtogether)=\frac{2!5!}{6!}=\frac{1}{3}$ D

Q18 $\Pr(1green1blue)=2\times\frac{1}{2}\times\frac{1}{3}=\frac{1}{3}$

Since the two rolls are independent,

$\Pr(green2nd \mid blue1st)=\Pr(green2nd)=\frac{1}{2}$ B

Q19 Binomial: $np=12.3$, $np(1-p)=2.8^2$, .: $n\approx 34$ D

Q20 $B\xrightarrow{\frac{1}{3}}A$, .: $B\xrightarrow{\frac{2}{3}}B$

$A\xrightarrow{\frac{1}{4}}A$, .: $A\xrightarrow{\frac{3}{4}}B$

.: $\Pr(BBAABAB)=1\times\frac{2}{3}\times\frac{1}{3}\times\frac{1}{4}\times\frac{3}{4}\times\frac{1}{3}\times\frac{3}{4}=\frac{1}{96}$ B

Q21 $\Pr(X>12.5)=0.8$, $\Pr(X>18.5 \mid X>12.5)=0.8$

.: $\frac{\Pr(X>18.5\cap X>12.5)}{\Pr(X>12.5)}=0.8$, $\frac{\Pr(X>18.5)}{\Pr(X>12.5)}=0.8$

.: $\Pr(X>18.5)=0.8^2=0.64$, $\Pr\left(Z>\frac{18.5-\mu}{\sigma}\right)=0.64$,

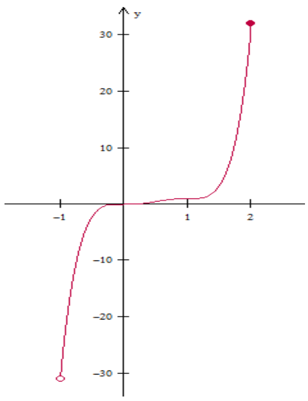
.: $\Pr\left(Z<\frac{18.5-\mu}{\sigma}\right)=0.36$, .: $\frac{18.5-\mu}{\sigma}=-0.3585$

.: $\mu-0.36\sigma\approx 18.5$ E

Q22 The graph is a probability density function, .: $n\neq\Pr(2)$ C

SECTION 2

Q1a



Q1b $f'(x)=ax^2(x-1)^2$, .: $f(x)$ is a degree five polynomial.

x	<0	0	$0<x<1$	1	>0
$f'(x)$	positive	zero	positive	zero	positive

The table shows that there is a stationary inflection point at $x=0$ and another one at $x=1$.

Q1ci $f'(x)=ax^2(x-1)^2=a(x^4-2x^3+x^2)$

.: $f(x)=a\left(\frac{x^5}{5}-\frac{x^4}{2}+\frac{x^3}{3}\right)+c$, $f(0)=0$

.: $f(x)=a\left(\frac{x^5}{5}-\frac{x^4}{2}+\frac{x^3}{3}\right)$

Q1cii $f(1)=1$, .: $a=30$

Q1d $f'(x)=30(x^4-2x^3+x^2)$, $x\in[0,1]$

$f''(x)=30(4x^3-6x^2+2x)=60(2x^3-3x^2+x)$

$=60x(x-1)(2x-1)$

Let $f''(x)=0$ to find the greatest rate of change:

$60x(x-1)(2x-1)=0$, .: $x=\frac{1}{2}$, .: $y=30\left(\frac{x^5}{5}-\frac{x^4}{2}+\frac{x^3}{3}\right)=\frac{1}{2}$

$\left(\frac{1}{2},\frac{1}{2}\right)$

Q1e Since $\left(\frac{1}{2},\frac{1}{2}\right)$ is a point having the greatest rate of change, it is an inflection point. .: total number of inflection points is 3.

Q1f $g(x)=-2f(-x)+2$

$f(x)$ undergoes reflection in both axes, a dilation by a factor of 2 parallel to the *y*-axis and then an upward translation of 2 units.

$(0,0)\rightarrow(0,2)$; $(1,1)\rightarrow(-1,0)$

Q1g $g(x)=-2f(-x)+2$

$=-2\times30\left(\frac{(-x)^5}{5}-\frac{(-x)^4}{2}+\frac{(-x)^3}{3}\right)+2$

$=60\left(\frac{x^5}{5}-\frac{x^4}{2}+\frac{x^3}{3}\right)+2$, a strictly increasing function (by CAS)

Domain of $f(x)$ is $(-1,2]$, .: domain of $g(x)$ is $[-2,1)$.

When $x=-2$, $y=-62$; when $x\rightarrow1$, $y\rightarrow64$.

.: range of $g(x)$ is $[-62,64)$

Q2a $height=2\log_e(a+e)$, $width=2e-a$,

$area$ $A=(2e-a)\times2\log_e(a+e)=2(2e-a)\log_e(a+e)$

Q2bi By the product rule:

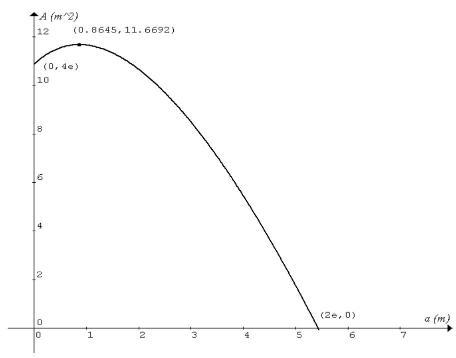
$\frac{dA}{da}=2(2e-a)\times\frac{1}{a+e}+(-2)\log_e(a+e)=\frac{2(2e-a)}{a+e}-2\log_e(a+e)$

Q2bii Maximum cross-sectional area $\Rightarrow$ maximum volume

Let $\frac{dA}{da}=0$, .: $\frac{2(2e-a)}{a+e}-2\log_e(a+e)=0$,

.: $2\log_e(a+e)=\frac{2(2e-a)}{a+e}$, $height=\frac{2(2e-a)}{a+e}$

Q2biii



Q2c The tank has maximum volume when $a = 0.8645$

Volume of water $V = 8(2e - 0.8645)h$,

$\frac{dV}{dh} = 8(2e - 0.8645)$, $\frac{dV}{dt} = 90 - 36h = 54$ when $h = 1$

$\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$, .: $54 = 8(2e - 0.8645)\frac{dh}{dt}$,

$\frac{dh}{dt} = \frac{54}{8(2e - 0.8645)} \approx 1.48$ m/h

Q2d $\frac{dt}{dh} = \frac{1}{2.5 - h}$, $t = \int \frac{1}{2.5 - h} dh = -\log_e |2.5 - h| + c$

Given at time $t = 0$, $h = 0$, .: $c = \log_e 2.5$

.: $t = -\log_e |2.5 - h| + \log_e 2.5$

When $h = 1.25$, $t = -\log_e 1.25 + \log_e 2.5 = \log_e \frac{2.5}{1.25} = \log_e 2$

Q2e Volume of *water* in the tank is maximum when

$\frac{dV}{dt} = 90 - 36h = 0$, $h = 2.5$

$V_{\max} = 8(2e - 0.8645) \times 2.5 \approx 91\,\text{m}^3$

Q2f *Cross-sectional area of the tunnel*

$= \int_0^{2e} 2\log_e(x + e)dx \approx 17.92$

Volume of soil $= (17.92 - 11.67) \times 8 \approx 50$ m^3

Q3a $(0,0)$, $a + c = 0$, .: $c = -a$.

$(10,-5)$, $ae^{10b} + c = -5$, .: $e^{10b} = \frac{-5 - c}{a} = \frac{a - 5}{a}$.

$(20,-8)$, $ae^{20b} + c = -8$, .: $e^{20b} = \frac{-8 - c}{a} = \frac{a - 8}{a}$,

.: $\left(e^{10b}\right)^2 = \frac{a - 8}{a}$, .: $\left(\frac{a - 5}{a}\right)^2 = \frac{a - 8}{a}$.

$a^2 - 10a + 25 = a^2 - 8a$, $2a = 25$, .: $a = \frac{25}{2}$, $c = -\frac{25}{2}$ and

$e^{10b} = \frac{a - 5}{a} = \frac{3}{5}$, $b = \frac{1}{10}\log_e\left(\frac{3}{5}\right)$

Q3aii $y = ae^{bx} + c$, $\frac{dy}{dx} = abe^{bx} = \frac{25}{2} \times \frac{1}{10}\log_e\left(\frac{3}{5}\right)e^{\frac{x}{10}\log_e\left(\frac{3}{5}\right)}$

$= \frac{5}{4}\log_e\left(\frac{3}{5}\right)e^{\log_e\left(\frac{3}{5}\right)^{\frac{x}{10}}} = \left[\frac{5}{4}\log_e\left(\frac{3}{5}\right)\right]\left(\frac{3}{5}\right)^{\frac{x}{10}}$

Q3aiii When $x = 10$,

$y = \left[\frac{5}{4}\log_e\left(\frac{3}{5}\right)\right]\left(\frac{3}{5}\right)^{\frac{x}{10}} = \left[\frac{5}{4}\log_e\left(\frac{3}{5}\right)\right]\left(\frac{3}{5}\right) \approx -0.38$

Q3bi When $t = 6$, $y = 3\sin\left(\frac{\pi t}{6}\right) - 5 = 3\sin\pi - 5 = -5$. The sea level is 5 m below the house.

Q3bii When $t = 6$, $y = -5$ and .: $x = 10$

.: the horizontal distance between the house and the water edge is $2 + 10 = 12$ m

Q3c When $t = 6$, $\frac{dy}{dt} = \frac{\pi}{2}\cos\left(\frac{\pi t}{6}\right) = \frac{\pi}{2}\cos\pi = -\frac{\pi}{2}$

Q3d $\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$, $-\frac{\pi}{2} = \frac{3}{4}\log_e\left(\frac{3}{5}\right) \times \frac{dx}{dt}$

.: $\frac{dx}{dt} = -\frac{2\pi}{3\log_e\left(\frac{3}{5}\right)}$, .: the receding rate is $\frac{2\pi}{3\log_e\left(\frac{3}{5}\right)}$.

Q4a $\Pr(W > 52)$, by CAS *normalcdf* (52, e^99,65,8) ≈ 0.9479

Q4b $\Pr(W > 52 \mid W > 45) = \frac{\Pr(W > 52)}{\Pr(W > 45)} \approx 0.9538$

Q4c *Average price ($) per dozen*

$= \frac{\Pr(52 < W < 60)}{\Pr(W > 52)} \times 1.00 + \frac{\Pr(60 < W < 68)}{\Pr(W > 52)} \times 1.10 + \frac{\Pr(W > 68)}{\Pr(W > 52)} \times 1.20$

≈ 1.11

Q4d Binomial: $n = 12$, success means $65 \le W < 68$,

$p = \frac{\Pr(65 < W < 68)}{\Pr(60 < W < 68)} = \frac{0.14617}{0.380184} \approx 0.384472$

$\Pr(X \ge 6)$ by CAS *binomialcdf* (12,0.384472,6,12) ≈ 0.29

Q4e The average weight of each of the four eggs is less than $\frac{250}{4} = 62.5$.

$\Pr(total < 250) = \frac{\Pr(60 < W < 62.5)}{\Pr(60 < W < 68)} = \frac{0.111345}{0.380184} \approx 0.29$

Q4f Probability density function $f(x) = \frac{1}{8\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-65}{8}\right)^2}$

Mean weight $= 12 \times \frac{\int_{68}^{\infty} xf(x)dx}{\Pr(W > 68)} \approx 12 \times \frac{25.97381}{0.35383} \approx 881$

Q4g $\Pr(W < 45) = 0.05$ and $\Pr(W > 75) = 0.10$

.: $\Pr\left(Z < \frac{45 - \mu}{\sigma}\right) = 0.05$ and $\Pr\left(Z < \frac{75 - \mu}{\sigma}\right) = 0.90$

.: $\frac{45 - \mu}{\sigma} \approx -1.6449$ and $\frac{75 - \mu}{\sigma} \approx 1.2816$

.: $\mu \approx 61.86$ grams, $\sigma = 10.25$ grams

Q4h $\Pr(ABBBBBA) + \Pr(ABBBBAB) + \Pr(ABBBABB)$

$+ \Pr(ABBABBB) + \Pr(ABABBBB) + \Pr(AABBBBB)$

$= (1)(0.64)(0.45)^4(0.55) + (1)(0.64)(0.45)^3(0.55)(0.64)$

$+ (1)(0.64)(0.45)^2(0.55)(0.64)(0.45) + (1)(0.64)(0.45)(0.55)(0.64)(0.45)^2$

$+ (1)(0.64)(0.55)(0.64)(0.45)^3 + (1)(0.36)(0.64)(0.45)^4 \approx 0.11$

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