

Q1a $x^2 + x + 1 = x^2$, $x + 1 = 0$, $x = -1$

Q1b $y = \frac{x^2 + x + 1}{x^2} = \frac{x^2}{x^2} + \frac{x+1}{x^2} = 1 + \frac{x+1}{x^2}$

As $x \rightarrow -\infty$, $y \rightarrow 1 + \frac{x}{x^2} = 1 + \frac{1}{x} \rightarrow 1^-$

As $x \rightarrow \infty$, $y \rightarrow 1 + \frac{x}{x^2} = 1 + \frac{1}{x} \rightarrow 1^+$

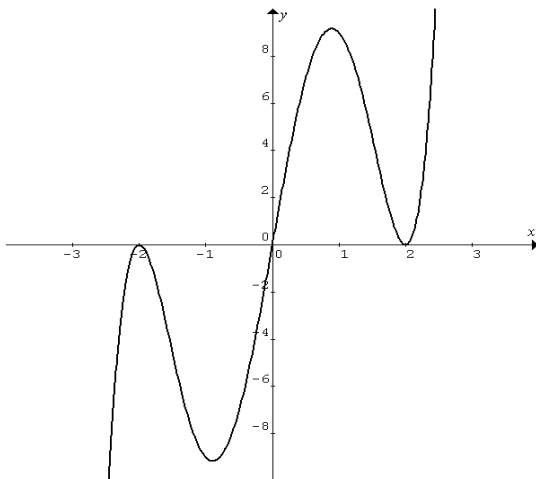
As $x \rightarrow 0^-$, $y \rightarrow \frac{x+1}{x^2} \rightarrow \frac{1}{x^2} \rightarrow \infty$

As $x \rightarrow 0^+$, $y \rightarrow \frac{x+1}{x^2} \rightarrow \frac{1}{x^2} \rightarrow \infty$

Q1c $f(g(x)) = (g(x))^2 + g(x) + 1 = x^4 + x^2 + 1$
 $(f(x))^2 = (x^2 + x + 1)^2 = x^4 + 2x^3 + 3x^2 + 2x + 1$
 $= x^4 + x^2 + 1 + 2x^3 + 2x^2 + 2x = f(g(x)) + 2xf(x)$
 $\therefore f(g(x)) = (f(x))^2 - 2xf(x) = f(x)(f(x) - 2x)$

Q2a All odd polynomials pass through the origin, $\therefore x = 0$ is a root of $P(x) = 0$, $\therefore x$ is a factor of $P(x)$. Since $x = -2$ is a double root, $\therefore (x + 2)$ is a repeated factor of $P(x)$. Since $P(x)$ is an odd polynomial, $\therefore (x - 2)$ is also a repeated factor of $P(x)$.
 $\therefore P(x) = x(x + 2)^2(x - 2)^2$

Q2b



Q3a

$$f(x) + f(-x) = \frac{2}{e^x - e^{-x}} + \frac{2}{e^{-x} - e^x} = \frac{2}{e^x - e^{-x}} - \frac{2}{e^x - e^{-x}} = 0$$

Q3b $y = \frac{2}{e^x - e^{-x}}$, equation of inverse function is $x = \frac{2}{e^y - e^{-y}}$

$$x = \frac{2e^y}{(e^y)^2 - 1}, x(e^y)^2 - 2e^y - x = 0,$$

$$e^y = \frac{2 \pm \sqrt{4 + 4x^2}}{2x} = \frac{1 \pm \sqrt{1 + x^2}}{x}, y = \log_e \frac{1 \pm \sqrt{1 + x^2}}{x}$$

$$\therefore f^{-1}(x) = \begin{cases} \log_e \frac{1 - \sqrt{1 + x^2}}{x} & \text{for } x < 0 \\ \log_e \frac{1 + \sqrt{1 + x^2}}{x} & \text{for } x > 0 \end{cases}$$

Q4 $x^2 + y^2 + z^2 + 10x + 20y - 30z + 350 = 0$

$$x^2 + 10x + 25 + y^2 + 20y + 100 + z^2 - 30z + 225 = 0$$

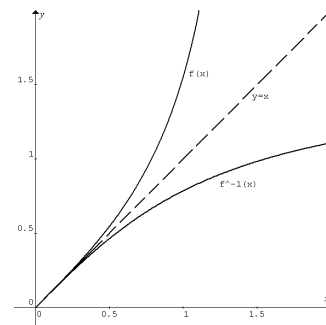
$$(x + 5)^2 + (y + 10)^2 + (z - 15)^2 = 0$$

Since *square* ≥ 0 , $\therefore (x + 5)^2 = 0$, $(y + 10)^2 = 0$ and

$$(z - 15)^2 = 0, \therefore x = -5, y = -10 \text{ and } z = 15$$

Hence $(x - y - z)^2 = (-5 + 10 - 15)^2 = 100$

Q5a $y = f^{-1}(x)$ is the reflection of $y = f(x)$ in the line $y = x$.



Q5b The tangent to the curve $y = f^{-1}(x)$ at $x = 1$ is the reflection (in the line $y = x$) of the tangent to the curve

$$y = f(x) \text{ at } x = \frac{\pi}{4}.$$

Tangent to the curve $y = f(x)$ at $x = \frac{\pi}{4}$, i.e. at $\left(\frac{\pi}{4}, 1\right)$:

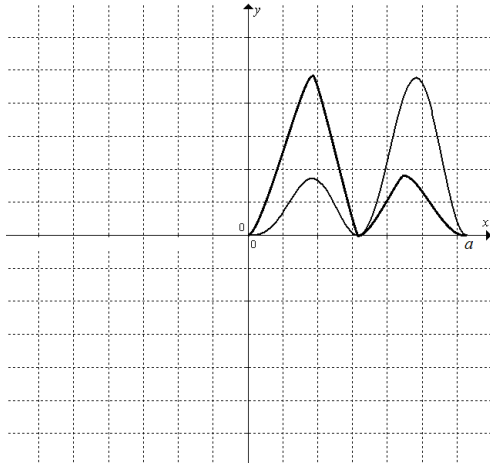
$$f'(x) = \sec^2 x, f'\left(\frac{\pi}{4}\right) = \sec^2 \frac{\pi}{4} = 2$$

$$y - 1 = 2\left(x - \frac{\pi}{4}\right), y = 2x + 1 - \frac{\pi}{2}$$

$\therefore$ the tangent to the curve $y = f^{-1}(x)$ at $x = 1$ is

$$x = 2y + 1 - \frac{\pi}{2}, y = \frac{1}{2}x - \frac{1}{2} + \frac{\pi}{4}$$

Q6a



Q6b $f(x) = \frac{\cos x}{\cos x + \sin x}, x \in \left[0, \frac{\pi}{2}\right]$

$$f\left(\frac{\pi}{2} - x\right) = \frac{\cos\left(\frac{\pi}{2} - x\right)}{\cos\left(\frac{\pi}{2} - x\right) + \sin\left(\frac{\pi}{2} - x\right)} = \frac{\sin x}{\sin x + \cos x} = \frac{\sin x}{\cos x + \sin x}$$

Q6c From part 6a, same area under $y = f(x)$ and $y = f\left(\frac{\pi}{2} - x\right)$

$$\text{for } x \in \left[0, \frac{\pi}{2}\right], \text{ i.e. } \int_0^{\frac{\pi}{2}} \frac{\cos x}{\cos x + \sin x} dx = \int_0^{\frac{\pi}{2}} \frac{\sin x}{\cos x + \sin x} dx$$

$$\int_0^{\frac{\pi}{2}} \frac{\cos x}{\cos x + \sin x} dx + \int_0^{\frac{\pi}{2}} \frac{\sin x}{\cos x + \sin x} dx = \int_0^{\frac{\pi}{2}} \frac{\cos x + \sin x}{\cos x + \sin x} dx = \int_0^{\frac{\pi}{2}} 1 dx$$

$$\therefore 2 \int_0^{\frac{\pi}{2}} \frac{\cos x}{\cos x + \sin x} dx = [x]_0^{\frac{\pi}{2}} = \frac{\pi}{2}$$

$$\therefore \int_0^{\frac{\pi}{2}} \frac{\cos x}{\cos x + \sin x} dx = \frac{\pi}{4}$$

Q7 $k(e^x)^2 - 2e^x + k = 0$

Since e^x is a one to one function, two solutions for $x \rightarrow$ two solutions for e^x in the equation.

$$\therefore \Delta = (-2)^2 - 4(k)(k) > 0 \text{ and } k \neq 0, \therefore 1 - k^2 > 0 \text{ and } k \neq 0$$

$$\therefore \{k : -1 < k < 0\} \cup \{k : 0 < k < 1\}$$

Q8a $f(x) = \frac{\log_e x}{x},$

$$f'(x) = \frac{\left(x\right)\left(\frac{1}{x}\right) - (\log_e x)(1)}{x^2} = \frac{1 - \log_e x}{x^2} = \frac{1}{x^2} - \frac{\log_e x}{x^2}$$

Q8b $\int_1^e f'(x) dx = \int_1^e \frac{1}{x^2} dx - \int_1^e \frac{\log_e x}{x^2} dx$

$$\therefore \int_1^e \frac{\log_e x}{x^2} dx = \int_1^e \frac{1}{x^2} dx - \int_1^e f'(x) dx = \left[-\frac{1}{x}\right]_1^e - \left[\frac{\log_e x}{x}\right]_1^e$$

$$= \left(-\frac{1}{e} + 1\right) - \left(\frac{1}{e}\right) = 1 - \frac{2}{e}$$

Q9 $\Pr(ttc) + \Pr(tct) + \Pr(ctc) + \Pr(cct)$

$$= 0.5 \times 0.6 \times 0.4 + 0.5 \times 0.4 \times 0.7 + 0.5 \times 0.7 \times 0.4 + 0.5 \times 0.3 \times 0.7$$

$$= 0.505$$

Q10a $\sum_i p_i = 1, 0.1a + 0.3(5 - a) = 1, -0.2a + 1.5 = 1, a = 2.5$

Q10b Let m be the median. $(5 - m)0.3 = 0.5, m = \frac{10}{3}$

Q10c $\bar{X} = \int_0^{2.5} 0.1x dx + \int_{2.5}^5 0.3x dx = \left[\frac{0.1x^2}{2}\right]_0^{2.5} + \left[\frac{0.3x^2}{2}\right]_{2.5}^5$

$$= 3.125$$

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