YEAR 12 *Trial Exam Paper*

2012

MATHEMATICAL METHODS (CAS)

Written examination 2

Worked solutions

This book presents:

- correct solutions with full working
- explanatory notes
- mark allocations
- tips

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Section 1

Question 1

The linear function $f : D \to R, \ f(x) = 7 - 3x$ has the range $[-4, 12]$.
Hence, the domain D is

A. R

B. $[-4, 12]$

C. R^+

D. $\left[-\frac{5}{3}, \frac{11}{3}\right]$

E. $\left[-\frac{11}{3}, \frac{5}{3}\right]$

Answer is D

Worked solution

The graph of the function is a decreasing function over R, so the range of $[-4, 12]$ means that for $7 - 3x = -4, \ x = \frac{11}{3}$; and for $7 - 3x = 12, \ x = -\frac{5}{3}$.

$\therefore$ The domain is $\left[-\frac{5}{3}, \frac{11}{3}\right]$.

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Question 2

Let $g(x) = x^2 + 2x$ and $f(x) = e^{3x-5}$.
Then $f(g(x))$ is given by

A. e^{x^2+2x}

B. e^{3x^2+6x}

C. $\boldsymbol{e^{3x^2+6x-5}}$

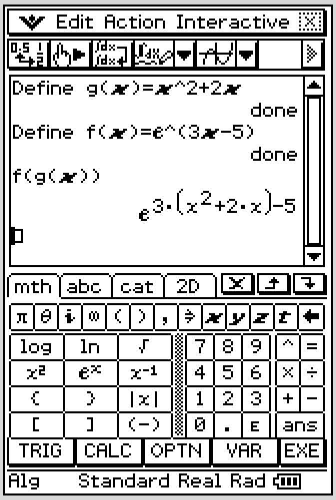
D. e^{x^2+2x-5}

E. $e^{3x^2} + 2e^x - 5$

Answer is C

Worked solution

Using CAS gives:



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Question 3

When $y = \log_a(6x - 2b) + 3$, x is equal to

A. $\frac{1}{6}(a^{-3} + 2b)$

B. $\frac{1}{6}(1 + 2b)$

C. $\frac{y - 3 + \log_a(2b)}{\log_a(6)}$

D. $\frac{y - 3}{6\log_a(2b)}$

E. $\mathbf{\frac{1}{6}(a^{y-3} + 2b)}$

Answer is E

Worked solution

$$y - 3 = \log_a(6x - 2b)$$
$$a^{y-3} = 6x - 2b$$
$$a^{y-3} + 2b = 6x$$
$$x = \frac{1}{6}(a^{y-3} + 2b)$$

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Question 4

The average rate of change of the function with rule $f(x) = x^2 - \sqrt{2x+1}$ between $x = 0$ and $x = 4$ is

A. $\mathbf{3\frac{1}{2}}$

B. $3\frac{1}{3}$

C. 14

D. $1\frac{3}{4}$

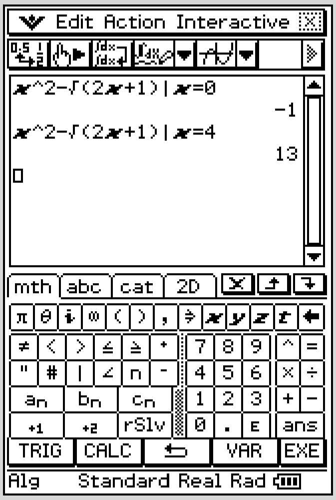
E. $3\frac{1}{6}$

Answer is A.

Worked solution

Average rate of change from $x = 0$ to $x = 4$ is $\dfrac{f(4) - f(0)}{4-0} = \dfrac{13-(-1)}{4} = \dfrac{14}{4} = 3.5.$

Using CAS gives:



Tip

*Be careful not to confuse average rate of change with average value of a function. Alternative **E** is the answer if the average value of the function is calculated from* x = 0 *to* x = 4*.*

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Question 5

The simultaneous linear equations

$$mx + 8y = 16$$
$$8x + my = m$$

has no solution when

A. $\boldsymbol{m = \pm 8}$

B. $m = 8$

C. $m = -8$

D. $m \in R \setminus \{8\}$

E. $m \in R \setminus \{\pm 8\}$

Answer is A

Worked solution

There will be no solution if the lines are parallel lines. This will occur when the gradients of the lines are the same. Rearranging the equations gives:

$$mx + 8y = 16 \;\Rightarrow\; y = \frac{-m}{8}x + 2$$

$$8x + my = m \;\Rightarrow\; y = \frac{-8}{m}x + 1$$

So $\frac{-m}{8} = \frac{-8}{m} \Rightarrow m^2 = 64$

$$m = \pm 8$$

An alternative solution that uses matrices gives:

$\begin{bmatrix} m & 8 \\ 8 & m \end{bmatrix}$ has a determinant of det $= ad - bc = m^2 - 64$. For no solution det = 0.

So $m^2 - 64 = 0$, $m = \pm 8$.

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Question 6

The range of the function $f:[\frac{\pi}{6},\frac{7\pi}{6})\to R,\ f(x)=2\left|\cos(2x)-1\right|+3$ is

A. $\mathbf{[3,7]}$

B. $[1,5]$

C. R

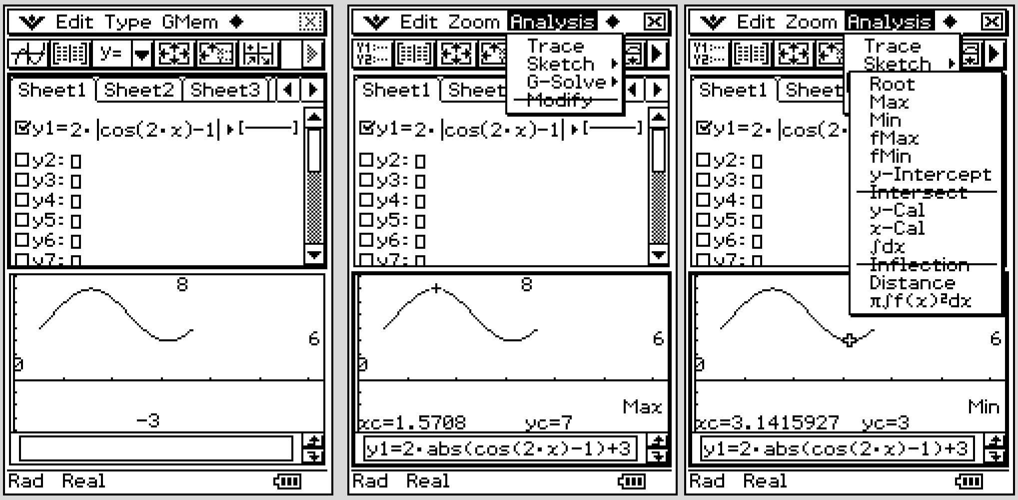
D. R^+

E. $[3,4]$

Answer is A

Worked solution

Using CAS to sketch the graph over the restricted domain, it can be seen that the maximum value is 7 and the minimum value is 3.



So the range is $[3,7]$.

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Question 7

Which one of the following is **not** true about the function $f : R \to R,\ f(x) = |x^2 - 4|$?

A. The graph of f is continuous everywhere.

B. **The graph of f' is continuous everywhere.**

C. $f(x) \geq 0$ for all values of x.

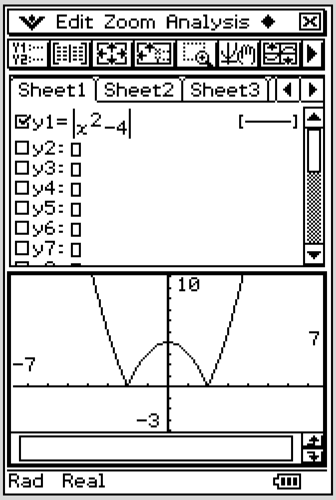
D. $f'(x) = 0$ for $x = 0$.

E. $f(x) = 0$ for $x = 2$ and $x = -2$.

Answer is B

Worked solution

The graph of the function is



This shows that there are cusps at x = 2 and x = –2 and therefore is not differentiable at these points. As the function $f'(x)$ does not exist at these points, it is not continuous everywhere.

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Question 8

If $k = \int_2^5 \frac{2}{x}\,dx$, then $e^{\frac{k}{2}}$ is equal to

A. $\frac{25}{4}$

B. $\mathbf{\frac{5}{2}}$

C. 5

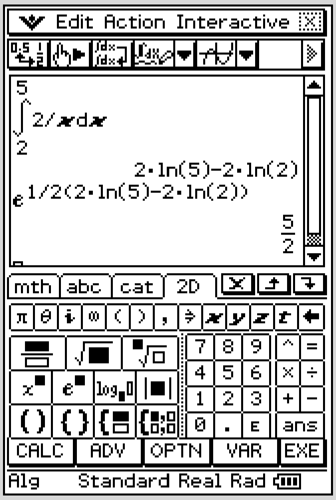
D. $e^5 - e^2$

E. $e^{\frac{5}{2}} - e^1$

Answer is B.

Worked solution

Using CAS gives:



Or alternatively:

$k = [2\log_e x]_2^5$

$= 2\log_e \frac{5}{2}$

So $\frac{k}{2} = \log_e \frac{5}{2}$

$\therefore \quad e^{\frac{k}{2}} = \frac{5}{2}$

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Question 9

The graph of $y = kx - 2$ intersects the graph of $y = x^2 + 6x$ at two distinct points for

A. $k > 0$

B. $k < 6 \pm 2\sqrt{2}$

C. $k < \pm 6 - 2\sqrt{2}$

D. $\boldsymbol{k < 6 - 2\sqrt{2}}$ **or** $\boldsymbol{k > 6 + 2\sqrt{2}}$

E. $6 - 2\sqrt{2} < k < 6 + 2\sqrt{2}$

Answer is D

Worked solution

Let the graphs intersect to get:

$kx - 2 = x^2 + 6x$

$0 = x^2 + 6x - kx + 2$

$0 = x^2 + x(6 - k) + 2$

To have two distinct points of intersection then $\Delta > 0$.

Δ for this quadratic is:

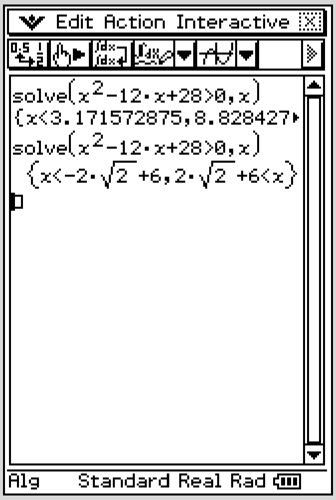
$\Delta = (6 - k)^2 - 4.1.2$

$= 36 - 12k + k^2 - 8$

$= k^2 - 12k + 28$

We want $\Delta > 0$, therefore $k^2 - 12k + 28 > 0$.

So using CAS to solve gives:



$\therefore\ k < 6 - 2\sqrt{2}$ or $k > 6 + 2\sqrt{2}$

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Question 10

The solution set of the equation $e^{6x} - 9e^{3x} + 8 = 0$ over R is

A. R

B. R^+

C. $\{0\}$

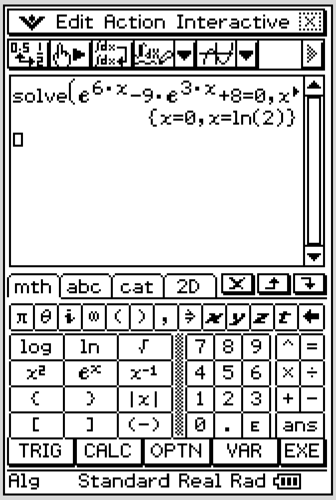
D. $\{\log_e 2\}$

E. $\{\mathbf{0}, \log_e 2\}$

Answer is E

Worked solution

Using CAS gives:



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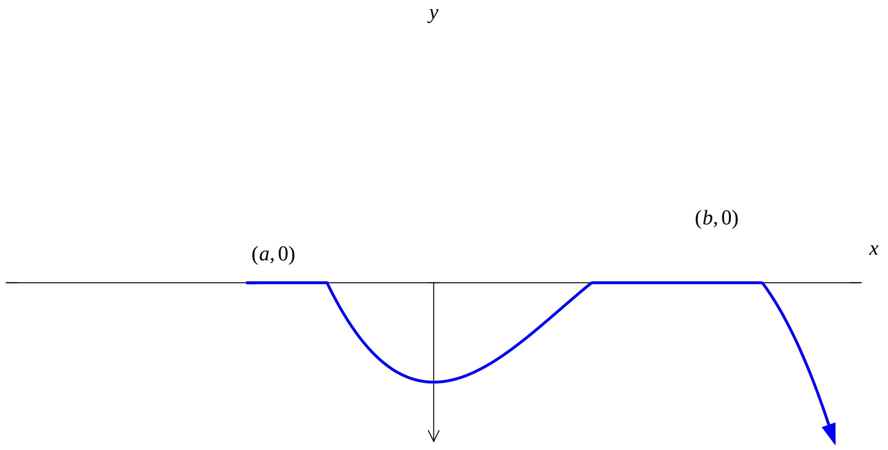
Question 11



For the graph of $y = f(x)$ shown above with $f'(0) = 0$, an interval over which $f(x)$ and $f'(x)$ are simultaneously negative is

A. (a, b)

B. (a, ∞)

C. $(\boldsymbol{a}, \mathbf{0})$

D. $(-\infty, 0)$

E. $(-\infty, a)$

Answer is C

Worked solution

We want the graph to be below the *x*-axis and the gradient to be negative. This occurs for the interval $(a, 0)$, so the answer is option C.

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Question 12

For $y = \log_e(\sqrt{f(x)})$, $\frac{dy}{dx}$ is equal to

A. $\frac{1}{\sqrt{f(x)}}$

B. $\boldsymbol{\frac{f'(x)}{2f(x)}}$

C. $\frac{f'(x)}{f(x)}$

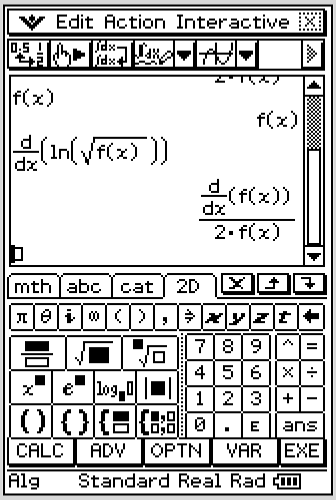
D. $\frac{f'(x)}{2\sqrt{f(x)}}$

E. $\frac{\log_e f(x)}{2}$

Answer is B.

Worked solution

Using CAS gives:



So the derivative is $\frac{f'(x)}{2f(x)}$.

Alternatively, using the chain rule gives:

$$\frac{dy}{dx} = \frac{1}{\sqrt{f(x)}} \times \frac{1}{2}(f(x))^{-\frac{1}{2}} \times f'(x)$$

$$= \frac{f'(x)}{2f(x)}$$

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Question 13

For $f(x)=|\cos(x)|$ over the interval $[-2\pi, 2\pi]$, the derivative $f'(x)$ is defined as

A. $f'(x)=\begin{cases} -\sin(x), & x\in\left[-2\pi,-\frac{3\pi}{2}\right]\cup\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\cup\left[\frac{3\pi}{2},2\pi\right] \\ \sin(x), & x\in\left(-\frac{3\pi}{2},-\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\frac{3\pi}{2}\right) \end{cases}$

B. $f'(x)=\begin{cases} \sin(x), & x\in\left(-2\pi,-\frac{3\pi}{2}\right)\cup\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\cup\left(\frac{3\pi}{2},2\pi\right) \\ -\sin(x), & x\in\left(-\frac{3\pi}{2},-\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\frac{3\pi}{2}\right) \end{cases}$

C. $\boldsymbol{f'(x)=\begin{cases} -\sin(x), & x\in\left(-2\pi,-\frac{3\pi}{2}\right)\cup\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\cup\left(\frac{3\pi}{2},2\pi\right) \\ \sin(x), & x\in\left(-\frac{3\pi}{2},-\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\frac{3\pi}{2}\right) \end{cases}}$

D. $f'(x)=\{-\sin(x), \quad x\in(-2\pi, 2\pi)$

E. $f'(x)=\{\sin(x), \quad x\in(-2\pi, 2\pi)$

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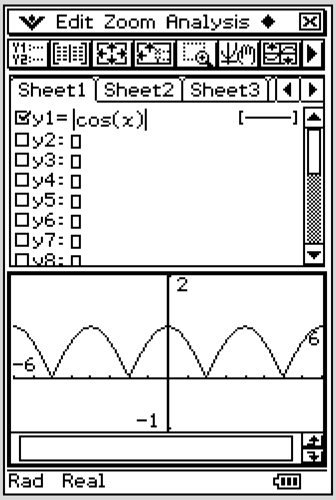
Answer is C

Worked solution

The graph of the function is:



It shows the function behaves as:

$$f(x) = |\cos(x)| = \begin{cases} \cos(x), & x \in \left[-2\pi, -\frac{3\pi}{2}\right] \cup \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \cup \left[\frac{3\pi}{2}, 2\pi\right] \\ -\cos(x), & x \in \left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \end{cases}$$

So the derivative function is defined as:

$$f'(x) = \begin{cases} -\sin(x), & x \in \left(-2\pi, -\frac{3\pi}{2}\right) \cup \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \cup \left(\frac{3\pi}{2}, 2\pi\right) \\ \sin(x), & x \in \left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \end{cases}$$

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Question 14

The maximal domain D of the function $f : D \to R$ with rule $f(x) = \log_e(9 - x^2)$ is

A. [0, 3]

B. (0, 3)

C. [–3, 3]

D. **(–3, 3)**

E. R

Answer is D

Worked solution

For the function to be defined $9 - x^2 > 0$, so for $-3 < x < 3$; i.e. D = (–3, 3).

Question 15

Let $f'(x) = g'(x) - 5$, where $f(0) = 3$ and $g(0) = 1$.
Hence, $f(x)$ is given by

A. $f(x) = g(x) - 5x + 7$

B. $\boldsymbol{f(x) = g(x) - 5x + 2}$

C. $f(x) = g(x) + 2$

D. $f(x) = g(x) - 5x - 2$

E. $f'(x) = g'(x) + 2$

Answer is B

Worked solution

Antidifferentiate both sides to get:

$f'(x) = g'(x) - 5$

$f(x) = g(x) - 5x + c$

$3 = 1 - 0 + c$, since $f(0) = 3$ and $g(0) = 1$.

$\therefore\ c = 2$

So $f(x) = g(x) - 5x + 2$.

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Question 16

The function f satisfies the functional equation $f\left(\frac{x-y}{2}\right)=\frac{1}{2}(f(x)-f(y))$, where x and y are non-zero real numbers. A possible rule for the function is

A. $f(x)=\sin(x)$

B. $f(x)=x^2-4x$

C. $f(x)=e^x$

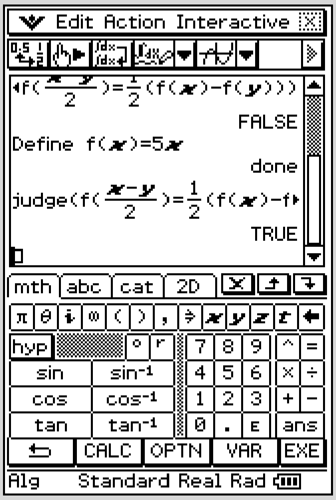
D. $f(x)=\log_e(x)$

E. $\boldsymbol{f(x)=5x}$

Answer is E

Worked solution

Using CAS, define the function and test using the 'judge' or 'test' facility.



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Question 17

The discrete random variable X has a probability distribution as given in the table below.
The mean of X is 4.

x	0	2	4	6	8
$\Pr(X = x)$	0.2	a	0.1	0.4	b

The values of a and b are

A. $a = 0.25,\ b = 0.25$

B. $a = 0.15,\ b = 0.15$

C. $a = 0.1,\ b = 0.2$

D. $\boldsymbol{a = 0.2,\ b = 0.1}$

E. $a = 0.2,\ b = 0.3$

Answer is D

Worked solution

Since the distribution is a pdf:
$\sum p(x) = 1,\ \Rightarrow 0.2 + a + 0.1 + 0.4 + b = 1$
i.e. $a + b = 0.3$

The mean is 4, therefore $0 \times 0.2 + 2 \times a + 4 \times 0.1 + 6 \times 0.4 + 8 \times b = 4$,
So $2a + 8b + 2.8 = 4$

$$2a + 8b = 1.2$$

$$a + 4b = 0.6$$

Solving simultaneously gives:
$0.3 - b + 4b = 0.6$

$$0.3 + 3b = 0.6$$

$$3b = 0.3$$

$$b = 0.1,\ a = 0.2$$

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Question 18

The heights of the teenage girls in a queue for 'Australia's Next Top Model' are normally distributed with mean 180 cm and standard deviation 9.2 cm.
A total of 35% of the girls are not allowed to audition because they are considered too short. Therefore, the minimum acceptable height, correct to the nearest centimetre, is

A. 176

B. **177**

C. 175

D. 183

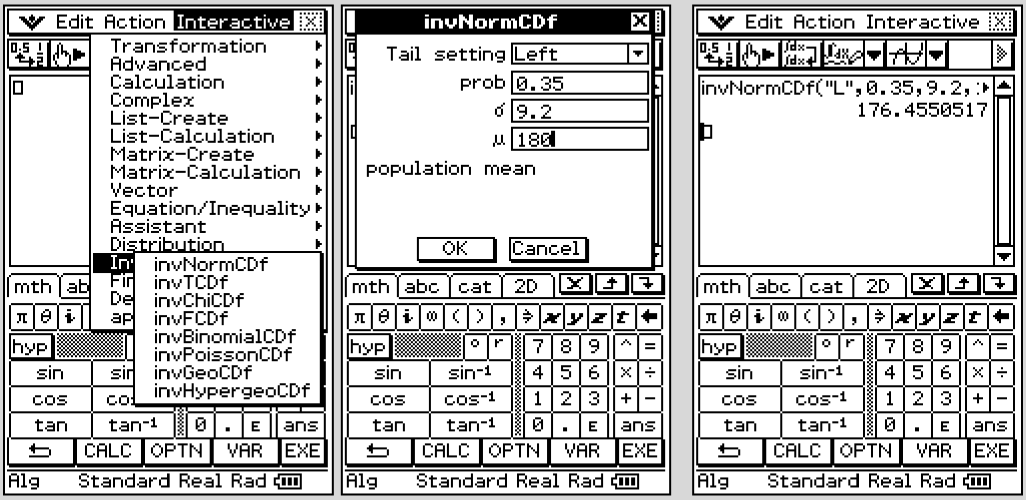
E. 184

Answer is B

Worked solution

$X \sim N(\mu = 180, \sigma = 9.2)$ and $\Pr(X < a) = 0.35$

Using CAS gives:



So any applicant shorter than 176.46 cm is not accepted, meaning a height of 177 cm is the minimum height, correct to the nearest centimetre, that is acceptable.

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Question 19

There are 2000 apples in storage at a fruit shop. Of these, it is found that 250 of them have a weight greater than 125 grams. The weights are normally distributed with a mean of μ grams and a standard deviation of 3.8 grams.
The value of μ is closest to

A. **120.6**

B. 120.7

C. 128.8

D. 122.4

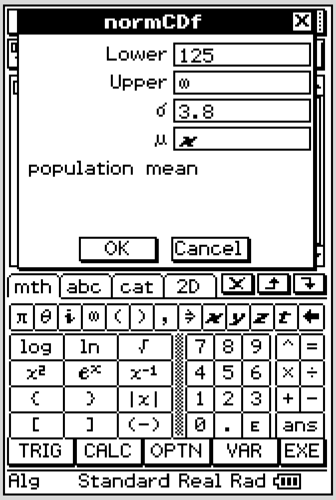
E. 122.5

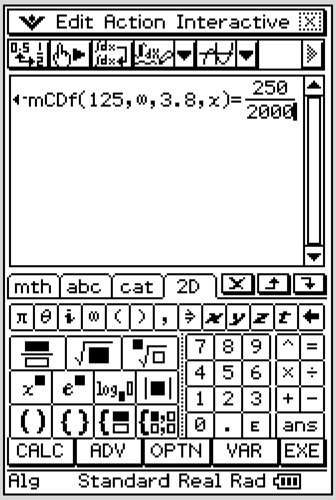
__Answer is A__

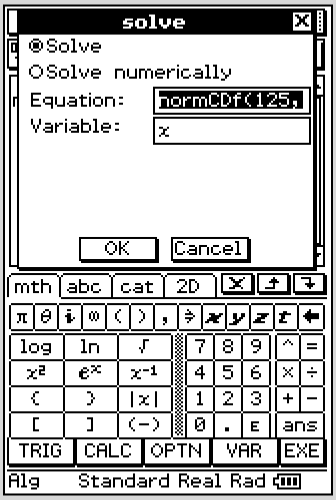
Worked solution

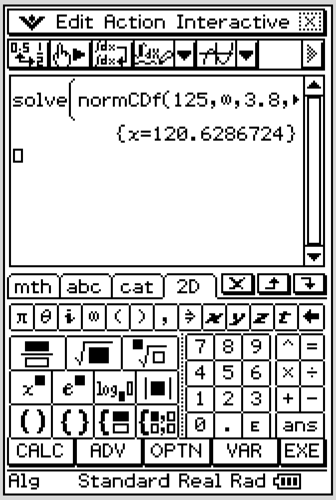
$$X \sim N(\mu = \text{unknown},\ \sigma = 3.8),\ \Pr(X > 125) = \frac{250}{2000}$$

Using CAS gives:









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Question 20

The equation $(x+3)^2(3-x)^3 - w = 0$ has only **one** solution for x when

A. $w > 0$

B. $w < 0$

C. $\boldsymbol{w \geq 269}$ **or** $\boldsymbol{w < 0}$

D. $0 < w \leq 268$

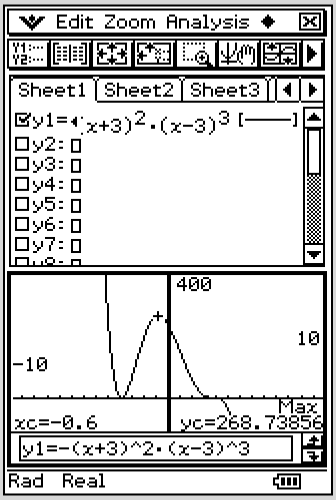
E. $w \geq -269$

Answer is C

Worked solution

Best to look at this problem graphically rather than trying to solve it algebraically.

The equation $(x+3)^2(3-x)^3 - w = 0$ is equivalent to looking at the intersection of the graphs $y = (x+3)^2(3-x)^3$ and the family of horizontal lines given by *y* = *w*.



The graph of $y = (x+3)^2(3-x)^3$ has a local maximum at (–0.6, 268.7), a local minimum at (–3, 0) and a point of inflexion at (3, 0). So at any point at or above *y* = 269 and below *y* = 0, there will be only one point of intersection with the line *y* = *w*.
Therefore, $w \geq 269$ or $w < 0$.

SECTION 1 – continued
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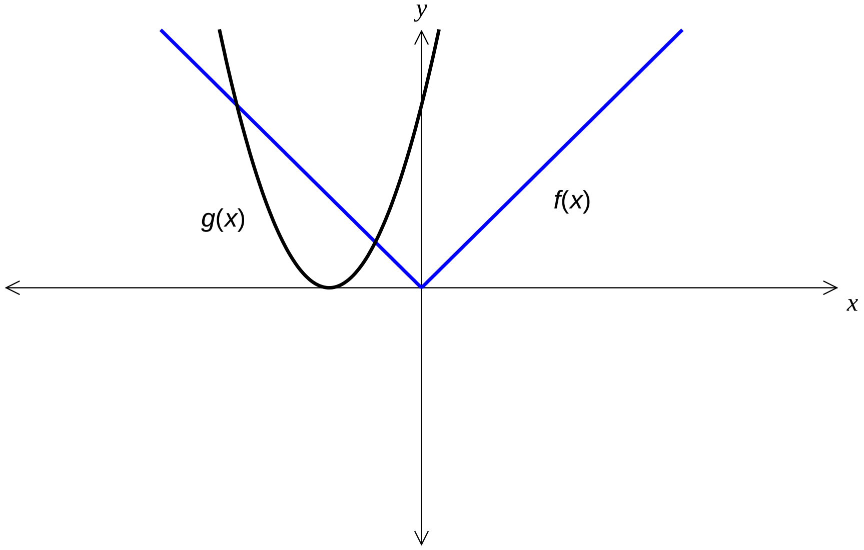
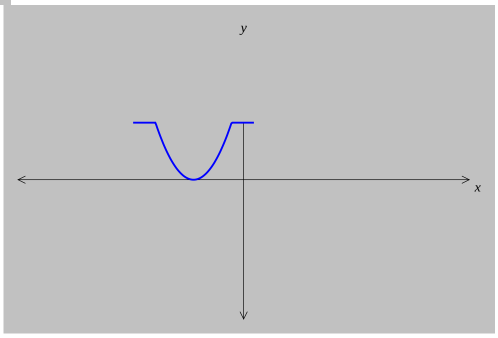
Question 21

The graphs of $y = f(x)$ and $y = g(x)$ are shown below.

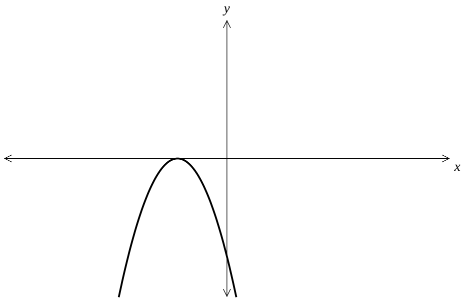


The graph of $y = f(g(x))$ is best represented by

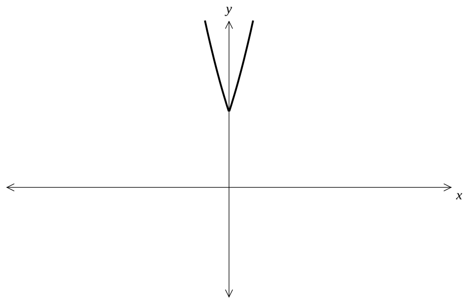
A.



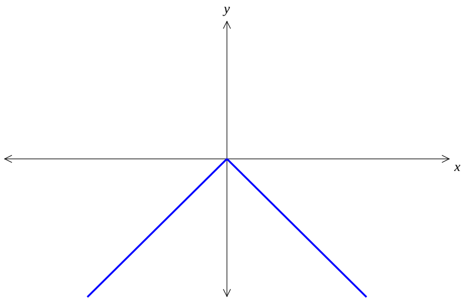
B.



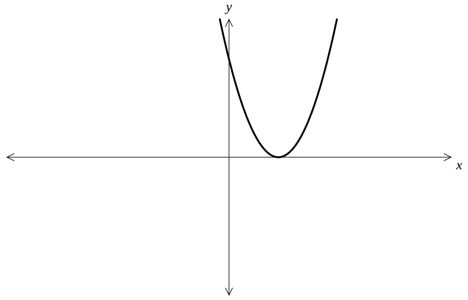
C.



D.



E.



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Answer is A

Worked Solution

The graph of $y = f(x)$ behaves like the absolute value function, so the composite function $y = f(g(x))$ will be the graph of $y = g(x)$ with any part of the graph below the x-axis being reflected across the x-axis to become positive; i.e. $y = \begin{cases} g(x), \text{ for } g(x) \geq 0 \\ -g(x), \text{ for } g(x) < 0 \end{cases}$.

In this case, no part of the graph of $y = g(x)$ falls below the x-axis, so the graph of $y = f(g(x))$ is just the graph of $y = g(x)$.

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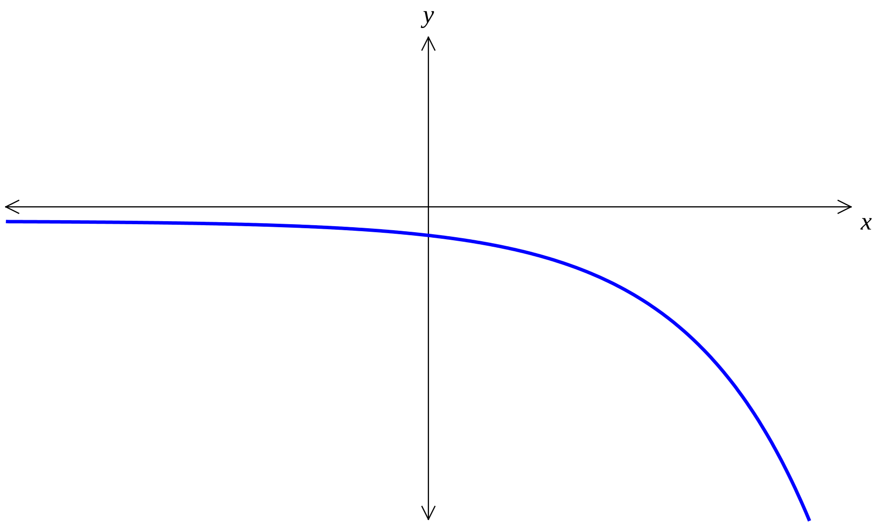
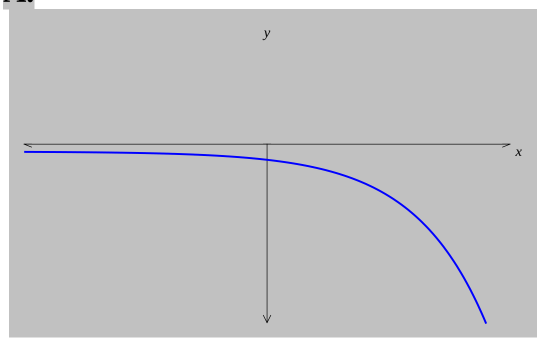
Question 22

The graph of the function f is shown below.

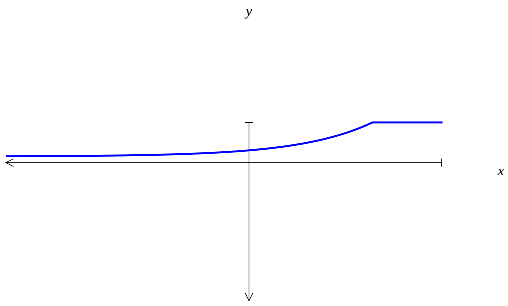


The graph of an antiderivative of f could be

A.



B.



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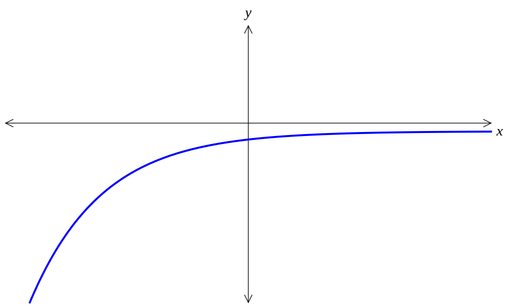

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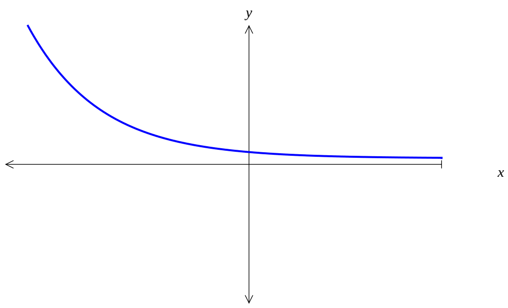
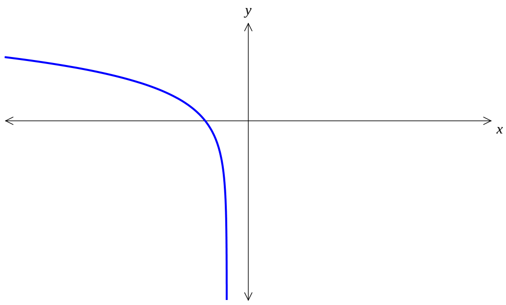


E.



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Answer is A.

Worked solution

The graph of f has the shape of a negative exponential function; e.g. $f'(x) = -e^x$. The antiderivative is $f(x) = -e^x + c,$ and for $c = 0$ the graph of an antiderivative also has the shape of a negative exponential function.

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END OF SECTION 1
TURN OVER

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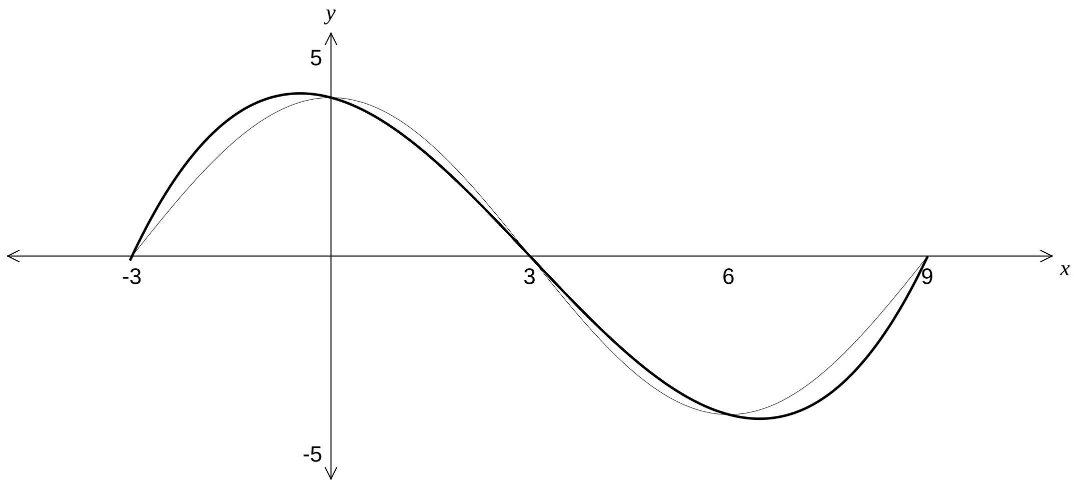
Section 2

Question 1

Shown below are the graphs of the two functions

$f:[-3, 9] \to R,\ f(x) = 4\cos\left(\frac{\pi x}{6}\right)$ and

$g:[-3, 9] \to R,\ g(x) = a(x+3)(x-3)(x-9)$



The point (0, 4) lies on both curves.

a. Show that $a = \frac{4}{81}$.

1 mark

Worked solution

It is given that the point (0, 4) lies on the curve.
Therefore, when $x = 0$, $y = 4$.
Substituting these values gives:

$4 = a(0+3)(0-3)(x-9)$

$4 = a \times 3 \times -3 \times -9$

$4 = 81a$

$a = \frac{4}{81}$

Note: This question is of the type 'show that' and therefore all working steps must be shown and follow logically.

Mark allocation

- 1 mark for setting $x = 0$, $y = 4$, leading to answer.

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b. State the period of the graph of $f(x)$.

1 mark

Worked solution

Period is given by $\dfrac{2\pi}{n} = \dfrac{2\pi}{\frac{\pi}{6}} = 12.$

Mark allocation

- 1 mark for the correct answer.

c. Find the exact value of x such that $g(x)$ is a maximum.

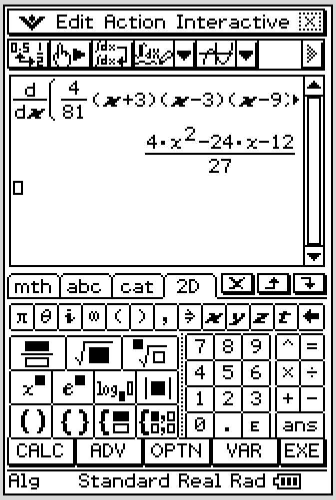
2 marks

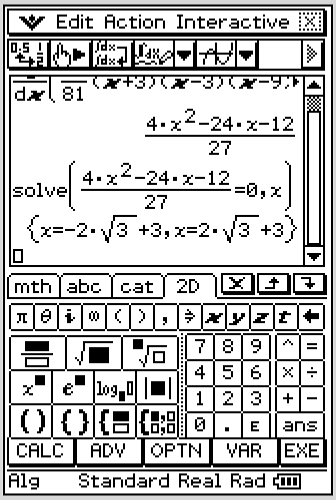
Worked solution

Maximum value of $g(x)$ occurs when $g'(x) = 0$.

So set $\dfrac{d}{dx}\left(\dfrac{4}{81}(x+3)(x-3)(x-9)\right) = 0$ and solve for *x*.

Using CAS, this gives:





$\therefore$ Maximum occurs at $x = -2\sqrt{3} + 3$.

Mark allocation

- 1 mark for setting $\dfrac{d}{dx}\left(\dfrac{4}{81}(x+3)(x-3)(x-9)\right) = 0.$
- 1 mark for the correct answer.

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d. **i.** Write down an integral expression that when evaluated gives the area enclosed between the curves $f(x)$ and $g(x)$ for $-3 \le x \le 9$.

2 marks

Worked solution

The graphs intersect at $x=-3$, $x=0$, $x=3$, $x=6$ and $x=9$.

The area enclosed between the curves can be evaluated with the expression:

$$\int_{-3}^{0} g(x)-f(x)dx+\int_{0}^{3} f(x)-g(x)\,dx+\int_{3}^{6} g(x)-f(x)\,dx+\int_{6}^{9} f(x)-g(x)\,dx$$

Mark allocation

- 1 mark for stating points of intersection.
- 1 mark for expression.

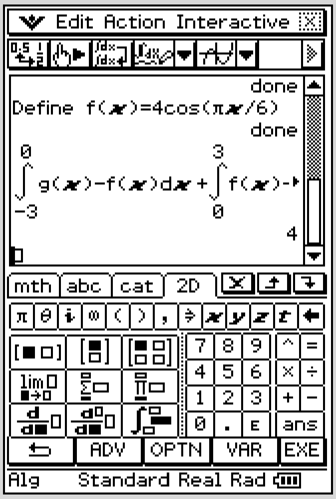
ii. Find the area between the curves $f(x)$ and $g(x)$ for $-3 \le x \le 9$.

1 mark

2 + 1 = 3 marks

Worked solution

Using CAS gives:



∴ Area is 4 sq. units.

Mark allocation

- 1 mark for the correct answer.

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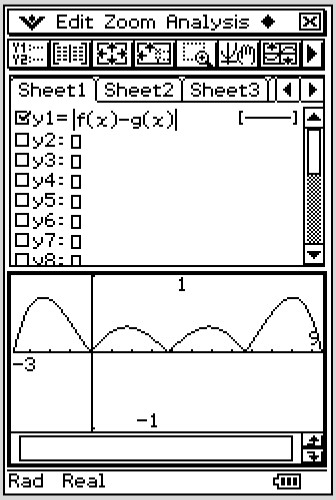
e. **i.** Find the values of x, correct to 2 decimal places, that give the maximum value of $|f(x)-g(x)|$ for $-3 \le x \le 9$.

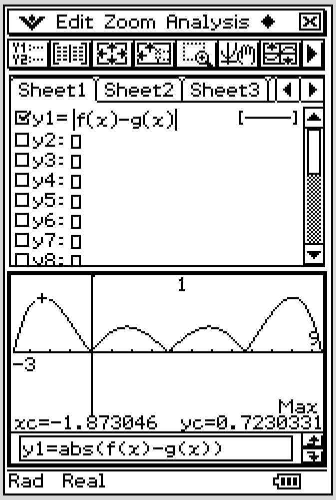
2 marks

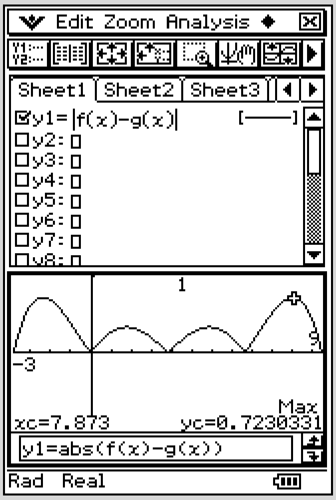
Worked solution

Using CAS, the graph of $|f(x)-g(x)|$ looks like:



And over the domain $-3 \le x \le 9$ there are *two* maximums.





These occur at $x = -1.87$ and $x = 7.87$.

SECTION 2 – continued

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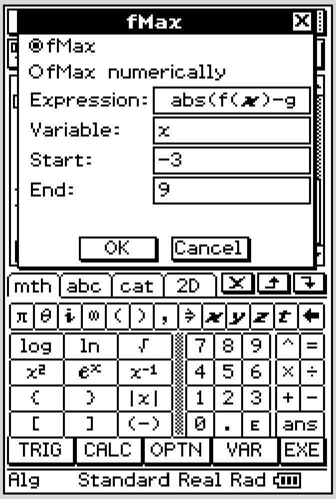
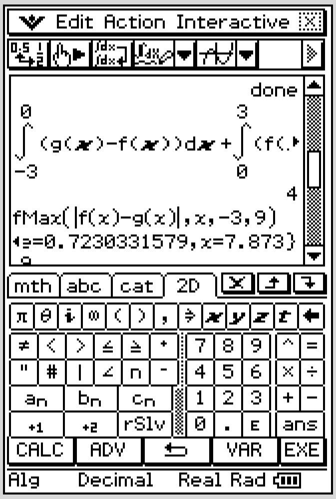
Alternatively, these could be obtained using the main screen, as shown below.





Mark allocation

- 1 mark for $x = -1.87$.
- 1 mark for $x = 7.87$.

ii. State, correct to 2 decimal places, the maximum value of $|f(x) - g(x)|$ for $-3 \le x \le 9$.

1 mark

2 + 1 = 3 marks

Worked solution

From previous part question (i.e. **Q1e i**), the maximum value is 0.72.

Mark allocation

- 1 mark for the correct answer.

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For a different value of a the point (0, 4) no longer lies on the graph of $g(x)$.

f. Find the exact value of a, such that the maximum value of $|f(x)-g(x)|$ occurs when $x = 1$.

3 marks

Worked solution

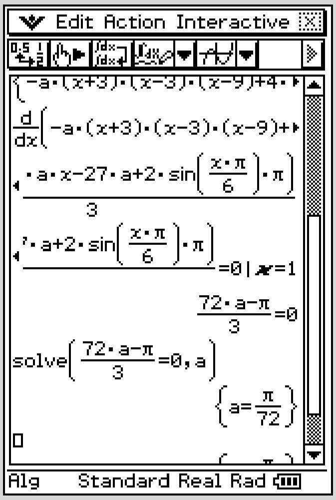
We want the maximum of $|f(x)-g(x)|$ to occur at $x = 1$.

Assume the equation of the graph of $|f(x)-g(x)|$ in this region is $(f(x)-g(x))$.

i.e. $(f(x)-g(x)) = -a(x+3)(x-3)(x-9)+4\cos\left(\frac{\pi x}{6}\right)$

So we want $\frac{d}{dx}\left(-a(x+3)(x-3)(x-9)+4\cos\left(\frac{\pi x}{6}\right)\right)=0$, when $x=1$.

Using CAS, this gives:



So we get $\frac{72a-\pi}{3}=0$.

Using CAS to solve this, gives $a=\frac{\pi}{72}$.

This can be verified by graphing the function $\left|4\cos\left(\frac{\pi x}{6}\right)-\frac{\pi}{72}(x+3)(x-3)(x-9)\right|$

and finding the location of the maximums.

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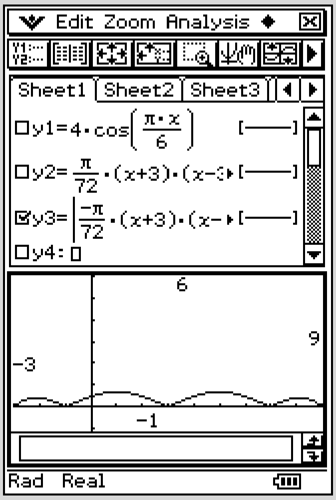

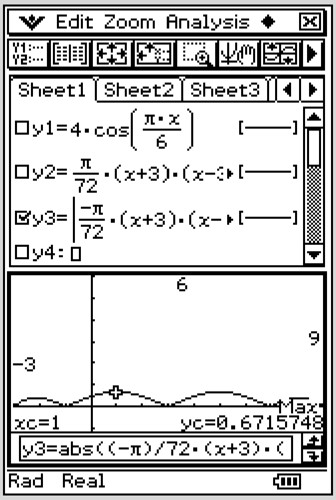

Mark allocation

- 1 mark for getting $\dfrac{d}{dx}\left(-a(x+3)(x-3)(x-9)+4\cos\left(\dfrac{\pi x}{6}\right)\right)=0,$ when $x=1$.
- 1 mark for getting $\dfrac{72a-\pi}{3}=0$.
- 1 mark for the correct answer.

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g. Find the smallest value of *a*, such that there are no local maximum values of $|f(x)-g(x)|$ in the interval $0 \le x \le 6$.

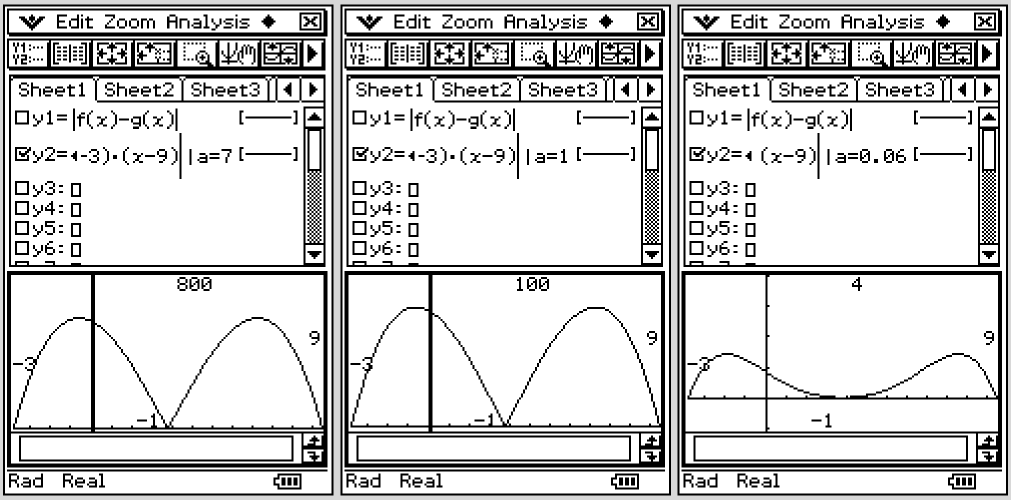
3 marks

Total 16 marks

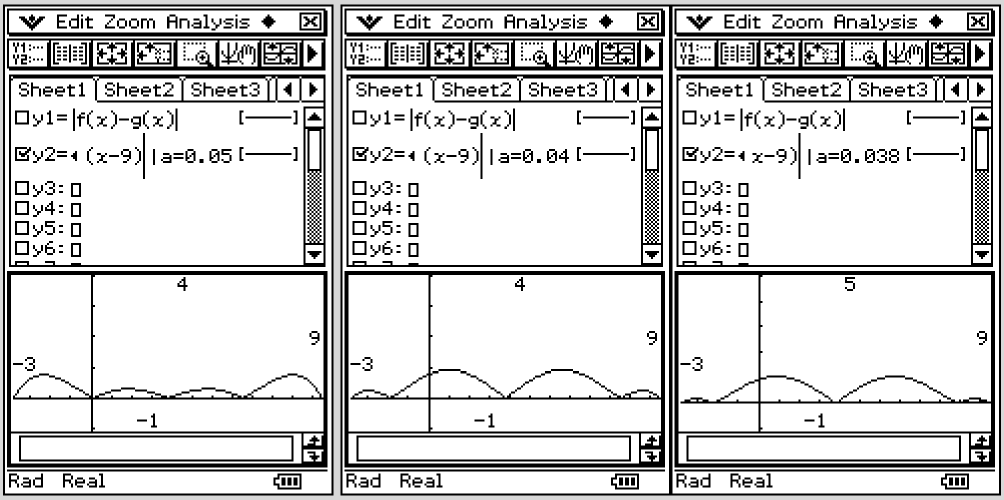
Worked solution

It can be shown that as the value of *a* changes, the shape of the graph of $|f(x)-g(x)|$ also changes.

In particular, as *a* decreases from 7 to 0.06, the shape of the graph has two local maximums outside the interval $0 \le x \le 6$, as shown in the images below.



Beyond a value somewhere between *a* = 0.06 and *a* = 0.05 to *a* = 0.038, the shape changes to have *four* local maximum values with two of them in the interval $0 \le x \le 6$.



For $a \in [0, 0.038)$ there are only *two* local maximums and both are in the interval $0 \le x \le 6$.

SECTION 2 – continued

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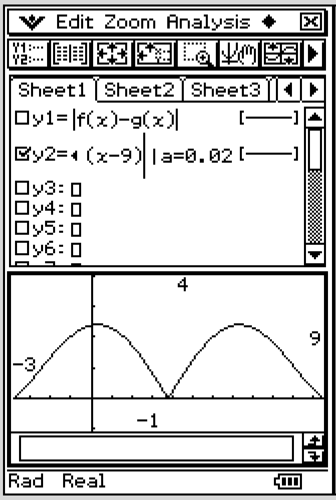
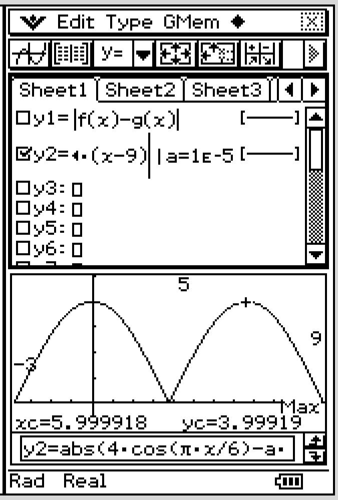
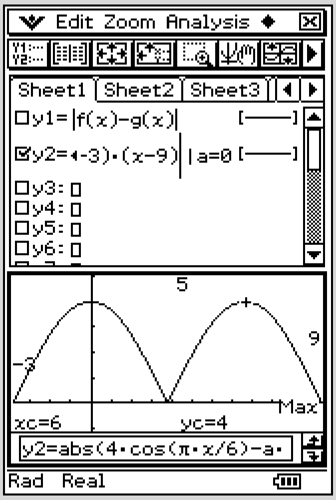
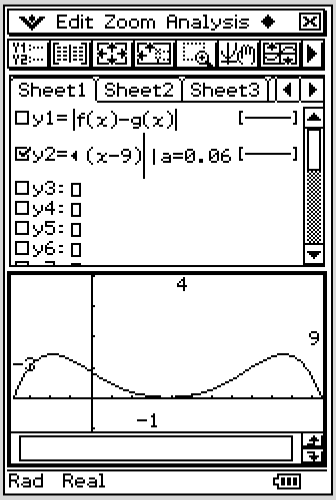


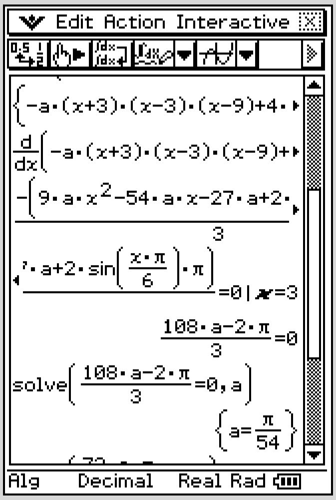




So we are looking for a value of *a* in the region between *a* = 0.06 and *a* = 0.05.
This type of graph has the shape of



with a local minimum turning point at *x* = 3.
Using CAS, set $\dfrac{d}{dx}\left(-a(x+3)(x-3)(x-9)+4\cos\left(\dfrac{\pi x}{6}\right)\right)=0$ when $x=3$.



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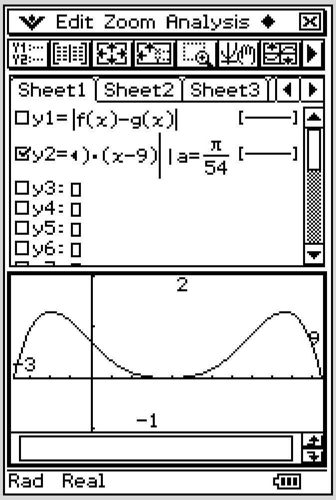
This gives $\frac{108a - 2\pi}{3} = 0$, which when solved gives $a = \frac{\pi}{54}$.

Check this by using CAS to graph $|f(x) - g(x)|$ with $a = \frac{\pi}{54}$.



Mark allocation

- 1 mark for setting $\frac{d}{dx}\left(-a(x+3)(x-3)(x-9) + 4\cos\left(\frac{\pi x}{6}\right)\right) = 0$ when $x = 3$.
- 1 mark for giving an answer in the interval $[0.05, 0.06]$.
- 1 mark for correct exact value answer.

SECTION 2 – continued

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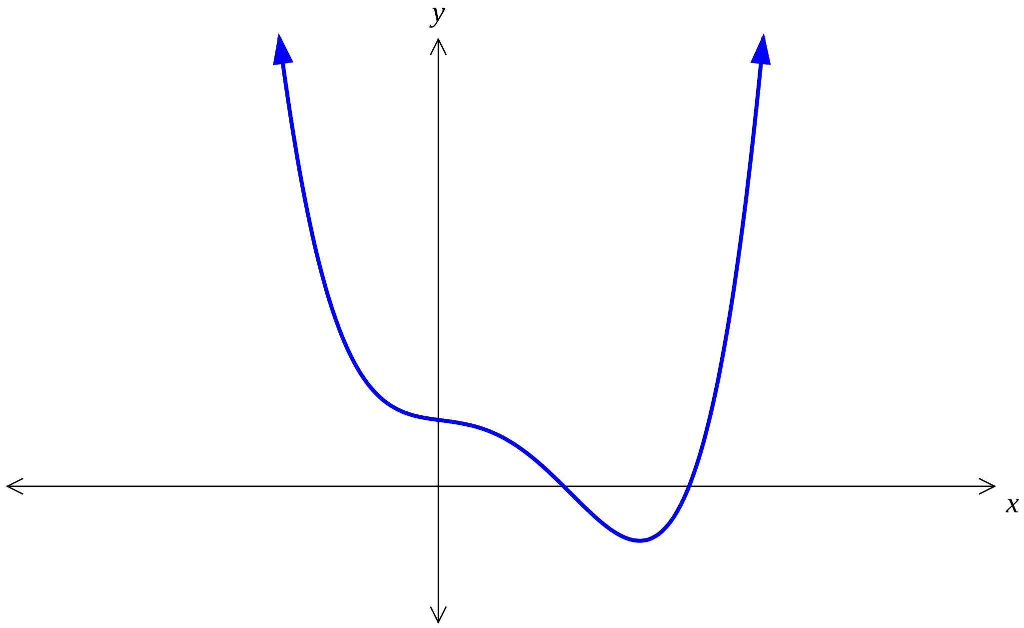
Question 2

The graph of $f : R \to R,\ f(x) = (x-2)(x-4)(3x^2+4x+8)$ is shown below.



a. State the co-ordinates of the x-intercepts.

1 mark

Worked solution

From the equation, the factors $(x - 2)$ and $(x - 4)$, leading to x-intercepts at (2, 0) and (4, 0).

Mark allocation

- 1 mark for giving two intercepts as co-ordinates.

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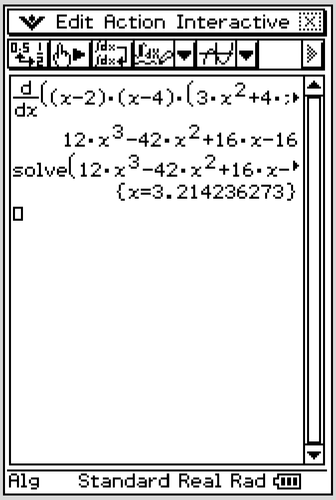
b. State the number of stationary points.

1 mark

Worked solution

Stationary points occur when $f'(x) = 0$.
Solving $f'(x) = 0$ for x shows there is only one stationary point.



Mark allocation

- 1 mark for the correct answer.

The quartic function g is defined by $g: R \to R,\ g(x) = (x-2)(x-4)(3x^2 + ax + 8)$, where a is a real number.

c. If g has m x-intercepts, what possible values can m take?

1 mark

Worked solution

The factors $(x – 2)$ and $(x – 4)$ give two initial x-intercepts.
The quadratic portion of the equation could lead to no, one or two additional x-intercepts. Hence, there are possibly two, three or four x-intercepts, so $m = 2, 3, 4$.

Mark allocation

- 1 mark for the correct answer.

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d. If g has p stationary points, what possible values can p take?

1 mark

Worked solution

There can be one, two or three stationary points.
So p = 1, 2 or 3.

Mark allocation

- 1 mark for the correct answer.

e. Find the values of a such that the graph of $g(x)$ has exactly **three** x-intercepts.

3 marks

Worked solution

The factors $(x-2)$ and $(x-4)$ give two initial x-intercepts.
In certain cases, the quadratic $3x^2+ax+8$ will be able to be factorised to give two distinct linear factors or one linear factor (repeated). This will lead to there being a total of three x-intercepts or four x-intercepts.

To obtain three x-intercepts, either the quadratic $3x^2+ax+8$ factorises to give one linear (repeated) factor or two factors with one being either $(x-2)$ or $(x-4)$; i.e. a repeat of the factors above.

Evaluating Δ for the quadratic $3x^2+ax+8$ gives a^2-96.
When $\Delta = 0$, $a=\pm\sqrt{96}$, then there is one additional x-intercept, so there are three distinct factors in total.

If $3x^2+ax+8$ factorises to give $(x-2)(3x-4)$, then there will be three distinct linear factors in total and so three x-intercepts, in which case $a=-10$. [Note: $(x-2)$ is a repeat of a factor given already.]

If $3x^2+ax+8$ factorises to give $(x-4)(3x-2)$, then there will be three distinct linear factors in total and so three x-intercepts, in which case $a=-14$. [Note: $(x-4)$ is a repeat of a factor given already.]

So $a=-10$, $a=-14$ and $a=\pm\sqrt{96}$.

Mark allocation

- 1 mark for evaluating Δ.
- 1 mark for finding $a=\pm\sqrt{96}$.
- 1 mark for the correct answer.

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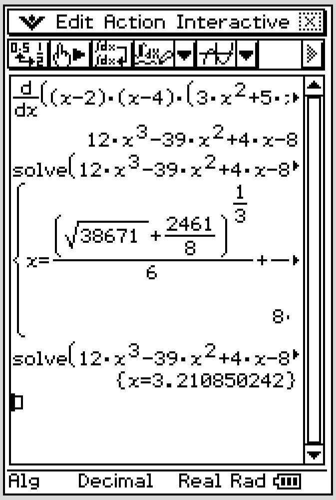
f. For $a = 5$, there is only one stationary point. Write down an equation, the solution of which gives the x value of the stationary point. State the nature and co-ordinates of the stationary point, correct to 3 decimal places.~~For $a = 5$, show that there is only one stationary point. State the nature and co-ordinates of the stationary point, correct to 3 decimal places.~~

3 marks

Worked solution

For $a = 5$, $g(x) = (x-2)(x-4)(3x^2 + ax + 8) = (x-2)(x-4)(3x^2 + 5x + 8)$.
Stationary points occur when $g'(x) = 0$. So by setting $g'(x) = 0$ and using CAS we get $x = 3.211$ and $y = -52.539$; i.e. a coordinate at (3.211, –52.539).

As the curve is a quartic with one stationary point, the nature of this point is a minimum turning point.



Mark allocation

- 1 mark for setting $g'(x) = 0$.
- 1 mark for (3.211, –52.539).
- 1 mark for describing nature of point correctly.

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The graph of $g(x)$ with $a = 5$ undergoes the following transformations.

- A dilation of factor 2 parallel to the x-axis
- A reflection in the x-axis
- A shift of 1 unit in the negative direction of the x-axis.

g. **i.** State the co-ordinates of the x-intercepts and the stationary point.

3 marks

Worked solution

Before the transformation there are x-intercepts at (2, 0) and (4, 0), and a stationary point at (3.211, –52.539).

A dilation of factor 2 parallel to the x-axis means that the x values are doubled, so these points become (4, 0) and (8, 0), and (6.422, –52.539).

A reflection in the x-axis means that the coordinates become (4, 0) and (8, 0), and (6.422, –52.539).

A shift of 1 unit to the left gives (3, 0) and (7, 0), and (5.422, 52.539).

Mark allocation

- 1 mark for each coordinate = 3 marks.

ii. The new equation can be written in the form

$$y = -\frac{1}{16}(x-3)(x-7)(ax^2+bx+c).$$

Find the values of a, b and c.

3 marks

3 + 3 = 6 marks

Total 16 marks

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Worked solution

A dilation of factor 2 parallel to the *x*-axis gives:

$$y=(x-2)(x-4)(3x^2+5x+8)\rightarrow\ y=\left(\frac{x}{2}-2\right)\left(\frac{x}{2}-4\right)\left(3\left(\frac{x}{2}\right)^2+5\left(\frac{x}{2}\right)+8\right)$$

A reflection in the *x*-axis gives:

$$y=\left(\frac{x}{2}-2\right)\left(\frac{x}{2}-4\right)\left(3\left(\frac{x}{2}\right)^2+5\left(\frac{x}{2}\right)+8\right)\rightarrow y=-\left(\frac{x}{2}-2\right)\left(\frac{x}{2}-4\right)\left(3\left(\frac{x}{2}\right)^2+5\left(\frac{x}{2}\right)+8\right)$$

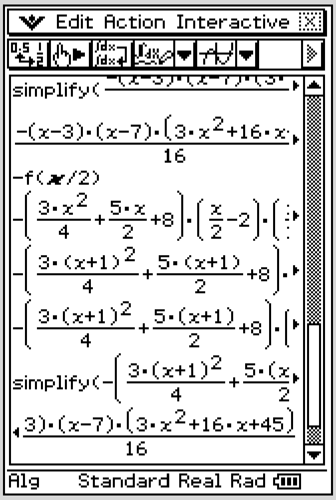
A shift of +1 unit to the left gives:

$$y=-\left(\frac{x}{2}-2\right)\left(\frac{x}{2}-4\right)\left(3\left(\frac{x}{2}\right)^2+5\left(\frac{x}{2}\right)+8\right)\rightarrow\ y=-\left(\frac{x+1}{2}-2\right)\left(\frac{x+1}{2}-4\right)\left(3\left(\frac{x+1}{2}\right)^2+5\left(\frac{x+1}{2}\right)+8\right)$$

simplifying this becomes $y=-\dfrac{(x-3)(x-7)(3x^2+16x+45)}{16}$.

So a = 3, b = 16, c = 45.

Alternatively, using CAS gives:



Mark allocation

- 1 mark for evaluating *a* correctly.
- 1 mark for evaluating *b* correctly.
- 1 mark for evaluating *c* correctly.

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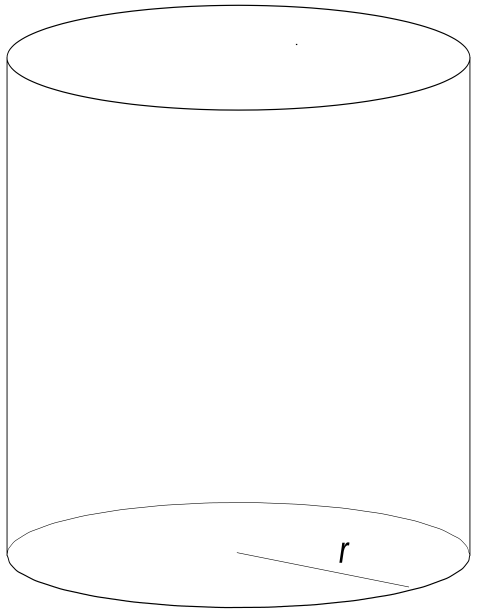
Question 3

Canino, a dog food company, makes dog food in cylindrical cans.
The cans have a radius of r cm and a height of h cm, as shown below.



a. If the volume of the can must be 600 cm^3, show that the surface area of the can is given by $S = 2\pi r^2 + \frac{1200}{r}$.

2 marks

Worked solution

$V = 600 = \pi r^2 h$, so $h = \frac{600}{\pi r^2}$.

$S = 2\pi r^2 + 2\pi rh$

$S = 2\pi r^2 + 2\pi r \frac{600}{\pi r^2}$

$S = 2\pi r^2 + \frac{1200}{r}$

Mark allocation

- 1 mark for $h = \frac{600}{\pi r^2}$.
- 1 mark for substituting into $S = 2\pi r^2 + 2\pi rh$, leading to answer.

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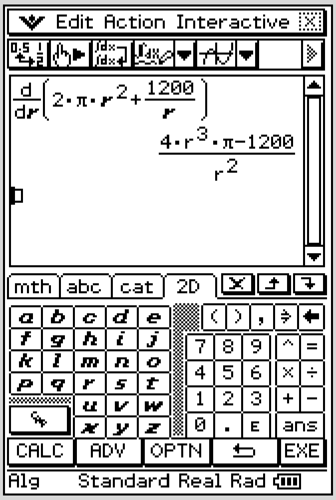
b. **i.** Find $\frac{dS}{dr}$ and, **hence**, state the dimensions of the can that give the minimum surface area.

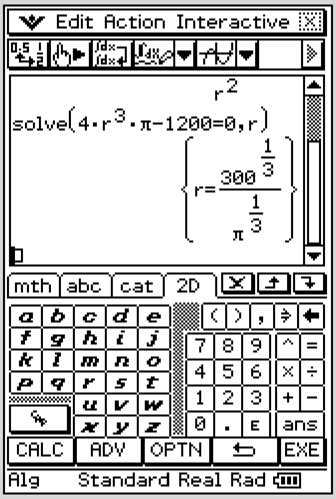
2 marks

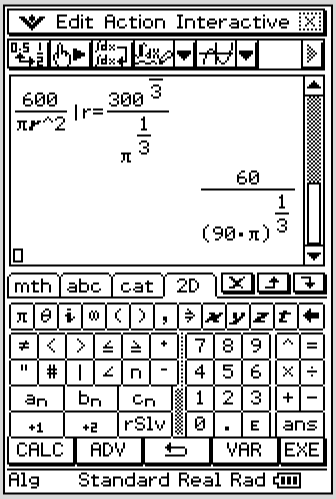
Worked solution

Using CAS, $\frac{dS}{dr} = \frac{4\pi r^3 - 1200}{r^2} = 4\pi r - \frac{1200}{r^2}$.



Solving $\frac{dS}{dr} = 0$ using CAS gives $r = \sqrt[3]{\frac{300}{\pi}}$, and substituting $h = \frac{600}{\pi r^2}$ we get:





Hence, the dimensions of the can are $r = \sqrt[3]{\frac{300}{\pi}}$ and $h = \frac{60}{\sqrt[3]{90\pi}}$.

Mark allocation

- 1 mark for $\frac{dS}{dr}$.
- 1 mark for finding values of *r* and *h*.

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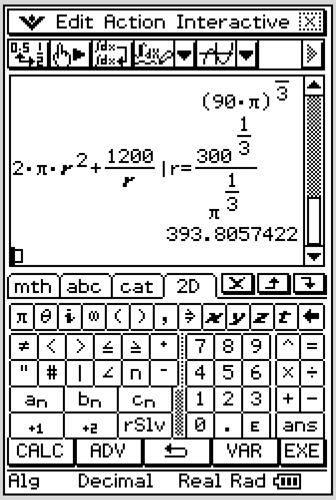
ii. Find the minimum surface area, correct to 2 decimal places.

1 mark
2 + 1 = 3 marks

Worked solution

Using CAS gives:



So the minimum surface area is 393.81 cm^2.

Mark allocation

- 1 mark for the correct answer.

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The lids of the cans are specially designed so that they can be opened without the need for a can opener. This adds to the cost of the cans and means that the cans are made from two different metal materials. One type of metal is used for the curved surface and the base, also called the body of the can, and the other type of metal is used for the lid.

The cost of the metal material for the body of the can is 1 cent per m^2. There is a range of metal materials available for the lid. The cost of the metal material for the lid varies depending on the quality of the metal used and is proportional to the cost of the metal used for the body of the can.

c. Show that the cost of the can is given by $C = \pi r^2 + \frac{1200}{r} + k\pi r^2$, where k is a positive constant of proportionality.

2 marks

Worked solution

$S = \text{surface}_{\text{body}} + \text{surface}_{\text{lid}}$

$S = \pi r^2 + 2\pi rh + \pi r^2$

$S = \pi r^2 + \frac{1200}{r} + \pi r^2$

$C = 1 \times (\pi r^2 + \frac{1200}{r}) + k \times (\pi r^2)$

$C = \pi r^2 + \frac{1200}{r} + k\pi r^2$

Mark allocation

- 1 mark for writing $S = \pi r^2 + 2\pi rh + \pi r^2$.
- 1 mark for writing expression for C.

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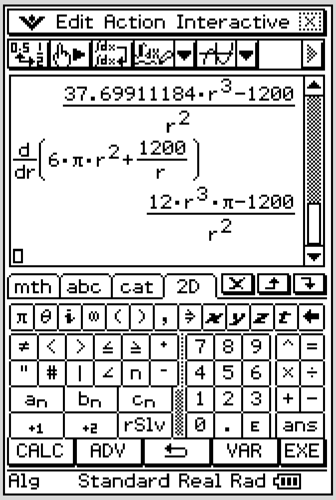
d. **i.** When $k = 5$, find $\frac{dC}{dr}$.

1 mark

Worked solution

$C = \pi r^2 + \frac{1200}{r} + 5\pi r^2 = 6\pi r^2 + \frac{1200}{r}$

Using CAS gives:



$\therefore \frac{dC}{dr} = \frac{12\pi r^3 - 1200}{r^2}$

Mark allocation

- 1 mark for finding $\frac{dC}{dr}$.

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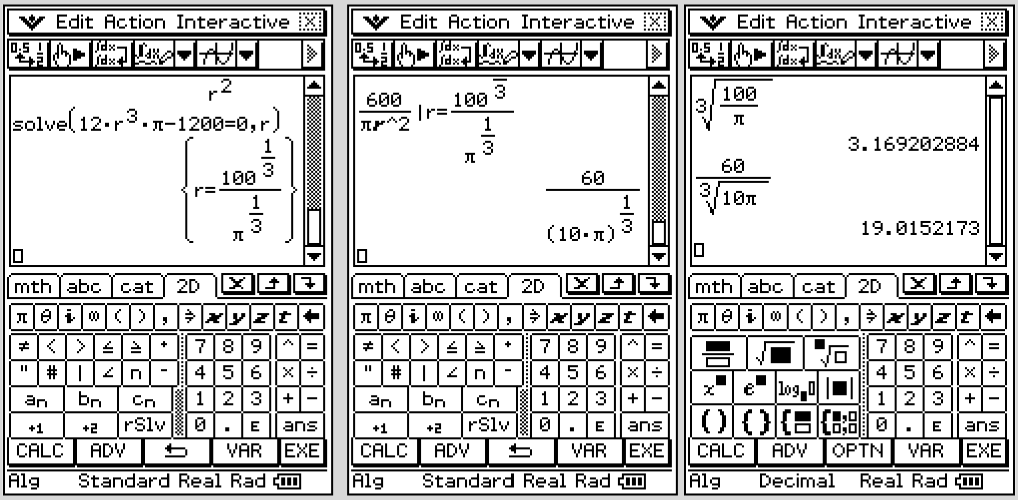
ii. Hence, find the dimensions of the can, correct to 2 decimal places, that will give the minimum cost.

2 marks

1 + 2 = 3 marks

Worked solution

Setting $\dfrac{dC}{dr} = 0$ gives $r = \sqrt[3]{\dfrac{100}{\pi}} = 3.17$ and $h = \dfrac{60}{\sqrt[3]{10\pi}} = 19.02$.



Mark allocation

- 1 mark for evaluating *h*.
- 1 mark for evaluating *r*.

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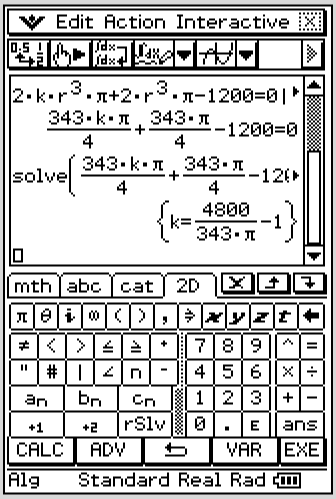
e. For a particular type of metal, the value of k is such that a radius of 3.5 cm will give the minimum cost. Find this value of k.

2 marks

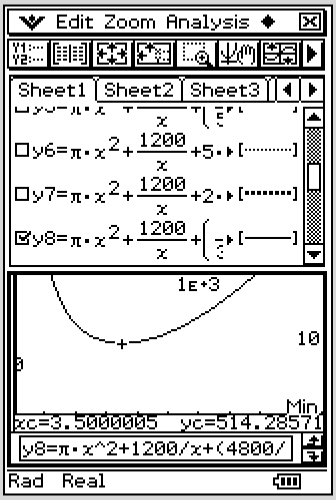
Worked solution

We wish to find $\frac{dC}{dr} = 0$ when r = 3.5. Using CAS gives:



So $k = \frac{4800}{343\pi} - 1.$

Checking using CAS gives:



Showing that the minimum occurs when r = 3.5.

Mark allocation

- 1 mark for setting $\frac{dC}{dr} = 0$ when r = 3.5.
- 1 mark for the correct answer.

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f. It is decided that the radius of the can must not exceed 3.5 cm. Find the values of k that give the minimum cost at $r = 3.5$.

2 marks

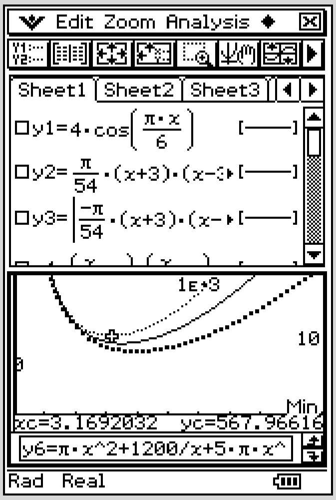
Total 14 marks

Worked solution

From part **e**, it is shown that when $k = \frac{4800}{343\pi} - 1 \approx 3.4544$, the minimum cost occurs at $r = 3.5$.

Graphically, it can be seen that for $k > \frac{4800}{343\pi} - 1$, i.e. $k > 3.4544$, the minimum cost occurs for $r < 3.5$, therefore within the restricted domain.

For example, when $k = 5$, the minimum cost occurs when $r = 3.169$.



Graphically, it can be seen that for $k > \frac{4800}{343\pi} - 1$, i.e. $k > 3.4544$, the minimum cost occurs for $r < 3.5$, therefore outside the domain.

For example, when $k = 2$, the minimum cost occurs at $r = 3.99$. As r must be less than or equal to 3.5, the minimum cost for this situation would occur at $r = 3.5$.

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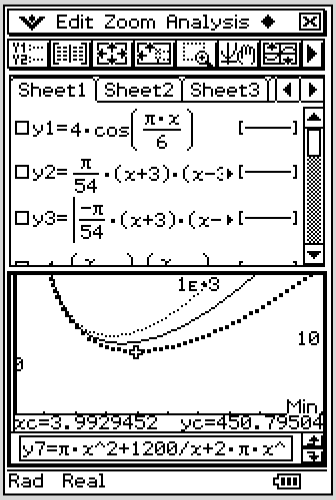

So, for any value of $0 < k < \dfrac{4800}{343\pi} - 1,$ the minimum cost will occur when r = 3.5.

Mark allocation

- 1 mark for $k < \dfrac{4800}{343\pi} - 1.$
- 1 mark for $0 < k < \dfrac{4800}{343\pi} - 1.$

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Question 4

Bikes are manufactured in a bicycle factory and the time, X hours, to produce a bike has the following probability density function.

$$f(x)=\begin{cases}0 & \text{for } x<0\\ \dfrac{1}{8} & \text{for } 0\le x\le 2\\ 0.6e^{-0.8(x-2)} & \text{for } x>2\end{cases}$$

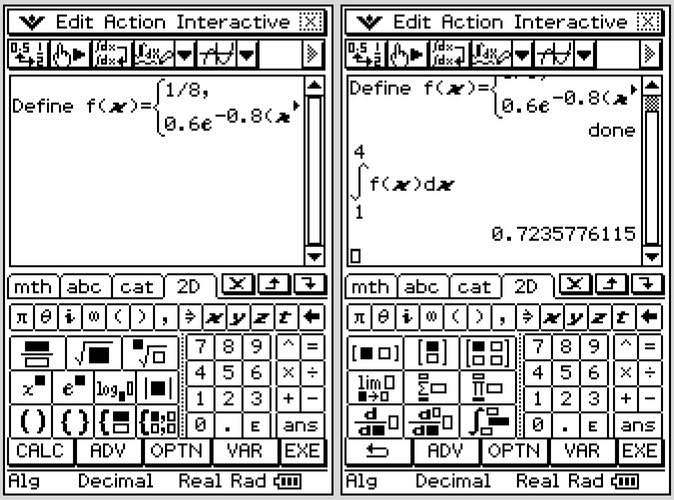
a. Find, correct to 3 decimal places, $\Pr(1 \le X \le 4)$.

2 marks

Worked solution

$$\Pr(1\le X\le 4)=\int_1^4 f(x)\,dx$$

Then using CAS gives:



So $\Pr(1\le X\le 4)$ = 0.724.

Mark allocation

- 1 mark for writing as an integral.
- 1 mark for the correct answer.

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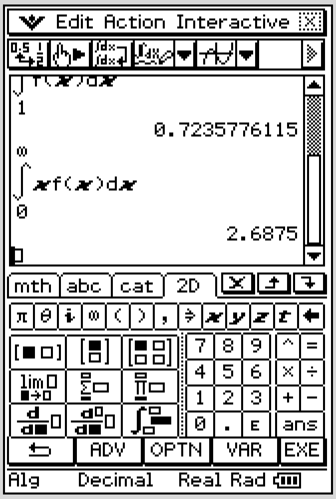
b. Find the mean.

2 marks

Worked solution

The mean = $\int_0^\infty x\, f(x)\, dx$

Using CAS gives:



So, the mean is 2.6875 or $\frac{43}{16}$ or $2\frac{11}{16}$.

Note: *The answer must not be rounded off—the exact value, in either decimal or fraction form, is required.*

Mark allocation

- 1 mark for writing $\int_0^\infty x\, f(x)\, dx$.
- 1 mark for the correct answer.

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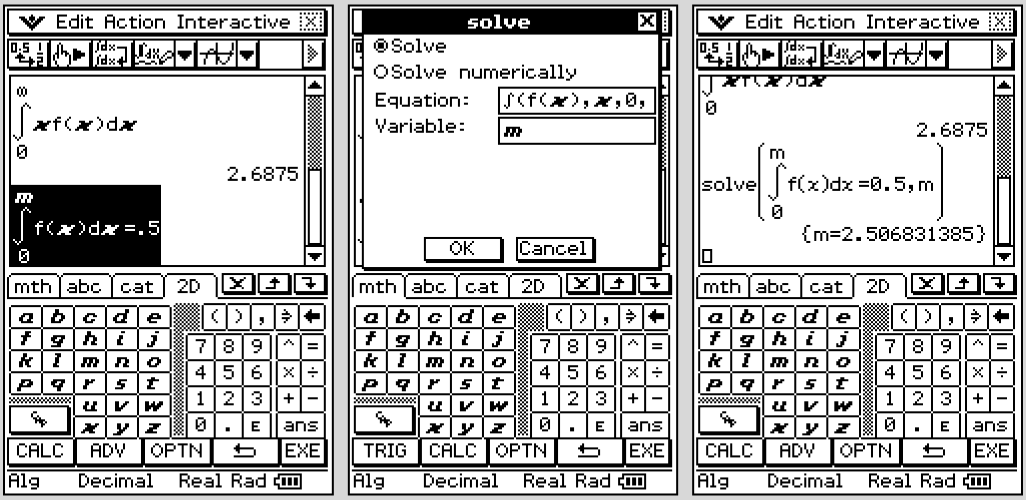
c. Find the median, correct to 3 decimal places.

3 marks

Worked solution

The median is defined as the value of m such that $\int_0^m f(x)\,dx = 0.5$.

Using CAS gives:



So, the median is 2.507.

Mark allocation

- 1 mark for writing m such that $\int_0^m f(x)\,dx = 0.5$.
- 2 marks for the correct answer.

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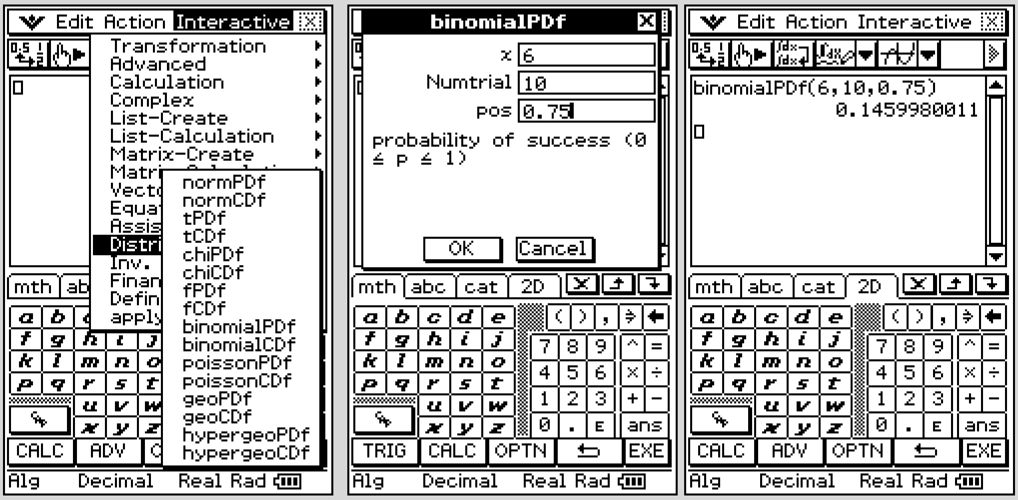
d. It can be shown that $\Pr(X > 2) = 0.75$. A random sample of 10 bikes is chosen. Find the probability, correct to 3 decimal places, that exactly six of these 10 bikes took more than 2 hours to produce.

2 marks

Worked solution

Here, a binomial distribution is introduced. Let the binomial distribution be *Y*, such that $Y \sim Bi(n = 10,\ p = 0.75)$. Find $\Pr(Y = 6)$.

Using CAS gives:



So the $\Pr(Y = 6)$ = 0.146.

Mark allocation

- 1 mark for setting up the binomial and writing $Y \sim Bi(n = 10,\ p = 0.75)$. $\Pr(Y = 6)$.
- 1 mark for the correct answer.

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e. Again, a random sample of 10 bikes is chosen. If it is known that $\Pr(X > b) = a$, where $a, b \in R^+$, and that the probability that no more than one of these 10 bikes took more than b hours to produce was 0.9, find the values of a and b. Give your answer correct to 4 decimal places.

3 marks

Total 12 marks

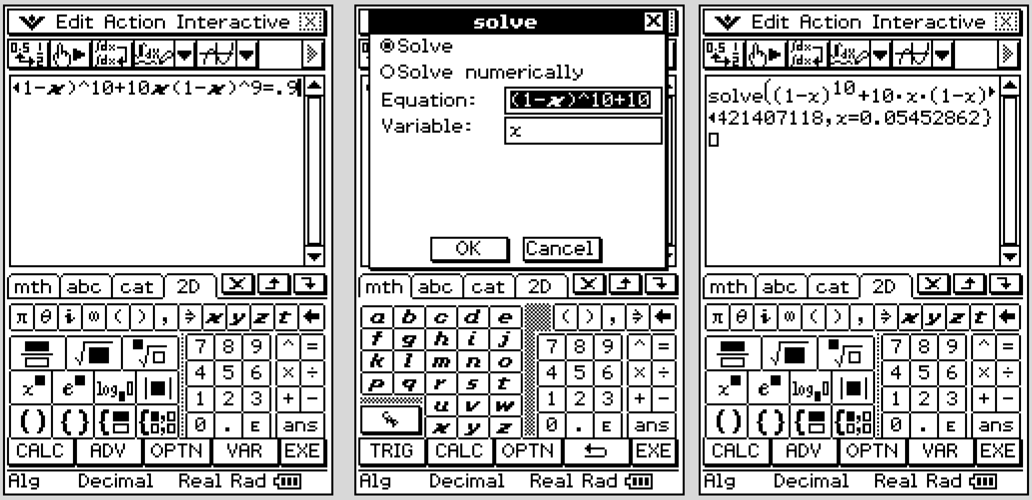
Worked solution

Again, a binomial distribution is involved: $Y \sim Bi(n = 10, p = a)$ and $\Pr(Y \le 1) = 0.9$.

$\Pr(Y \le 1) = \Pr(Y = 0) + \Pr(Y = 1)$

$= (1-a)^{10} + 10a(1-a)^9 = 0.9$

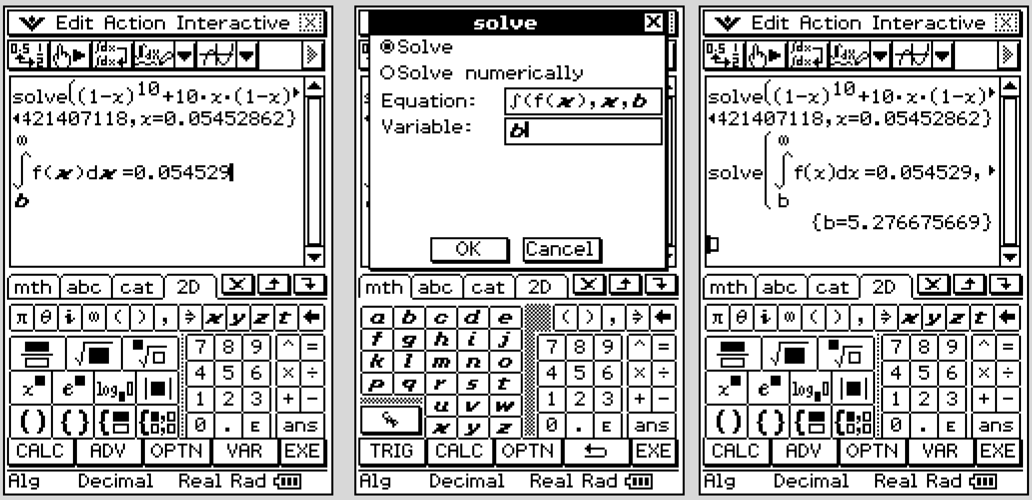
Using CAS gives:



$\therefore\ a = 0.054529$

Then $\Pr(X > b) = a$ gives $\int_b^{\infty} f(x)\,dx = 0.054529.$

Using CAS gives:



$\therefore\ b = 5.2767$

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Mark allocation

- 1 mark for writing $\Pr(Y \le 1) = \Pr(Y = 0) + \Pr(Y = 1)$

$$= (1-a)^{10} + 10a(1-a)^9 = 0.9$$

- 1 mark for evaluating *a*.
- 1 mark for evaluating *b*.

END OF SECTION 2
END OF SOLUTIONS BOOK

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