



INSIGHT
YEAR 12 Trial Exam Paper

2012
MATHEMATICAL METHODS
(CAS)

Written examination 1

Worked Solutions

This book presents:

- correct solutions with full working
- explanatory notes
- mark allocations

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Question 1

Let $f(x) = \sqrt{3-2x}$. Write down the rule for $f(\cos x)$.

1 mark

Worked solution

Replace x with $\cos x$ to get $f(x) = \sqrt{3-2\cos(x)}$.

Mark allocation

- 1 mark for correct answer

Question 2

For the function $f : (-1, \infty) \rightarrow \mathbb{R}$, $f(x) = \frac{1}{3} \log_e \left(\frac{x+1}{2} \right)$

- a. Find the rule for the inverse function, f^{-1} .

2 marks

Worked solution

Swap x and y to get $x = \frac{1}{3} \log_e \left(\frac{y+1}{2} \right)$.

Now rearrange to make y the subject:

$$3x = \log_e \left(\frac{y+1}{2} \right)$$

$$e^{3x} = \frac{y+1}{2}$$

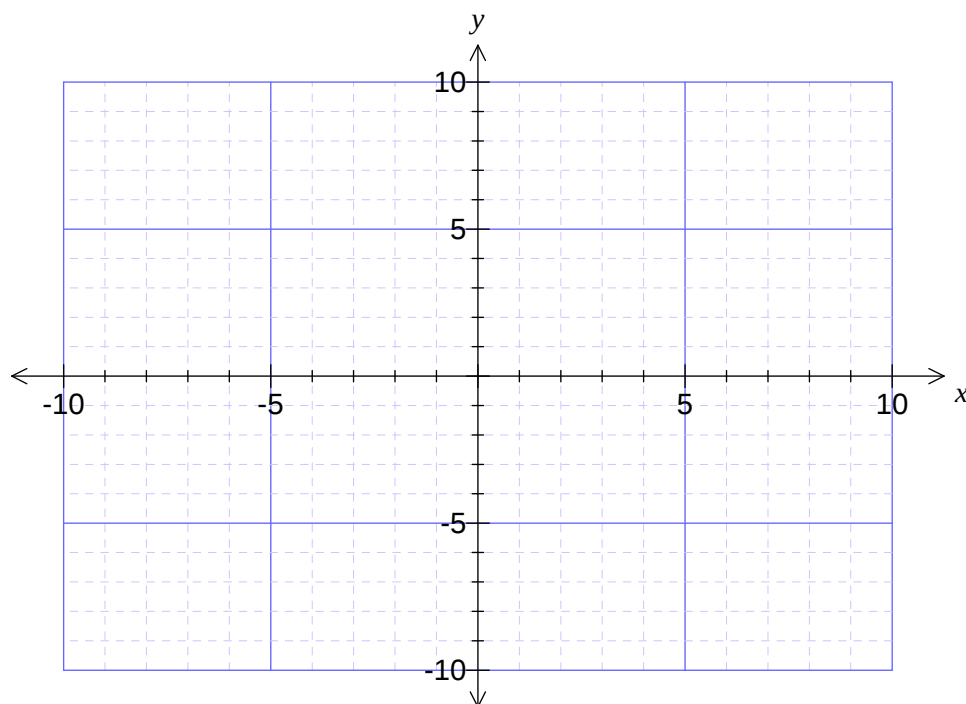
$$2e^{3x} = y+1$$

$$y = 2e^{3x} - 1$$

Mark allocation

- 1 mark for swapping x and y .
- 1 mark for the correct answer.

- b. Sketch the graph of $y = f^{-1}(f(x))$ on the axes below.

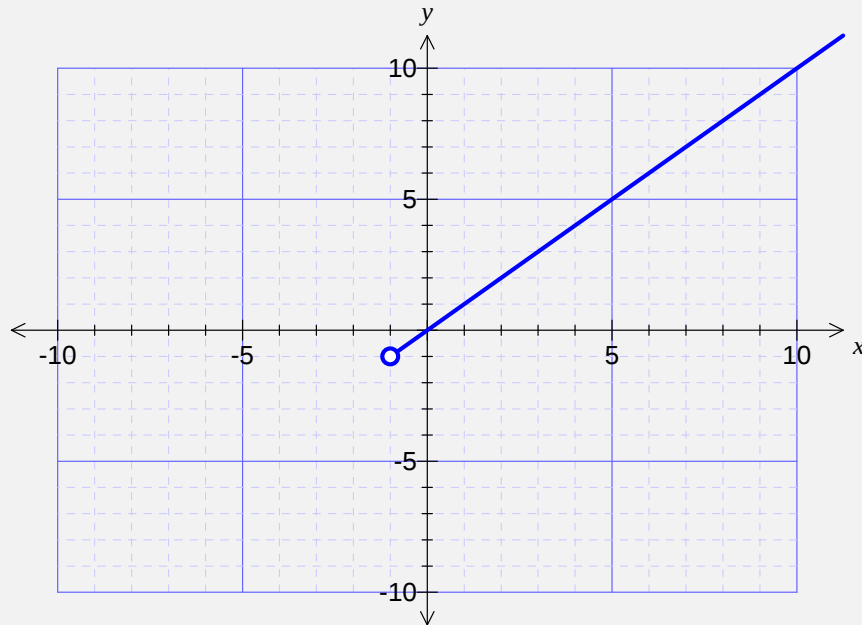


1 mark

Worked solution

The graph of $y = f^{-1}(f(x))$ will be the graph of $y = x$ for the domain of $f(x)$; i.e., $x \in (-1, \infty)$.

So the graph will be

**Mark allocation**

- 1 mark for correctly drawn graph. (It must have an open circle at $x = -1$).

- c. The function $f(x)$ undergoes a transformation as defined by the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$.

State the new equation.

1 mark

Worked solution

The matrix $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ describes a transformation that produces a dilation of factor 2 in the y direction (from the x -axis).

So the new equation is $y = \frac{2}{3} \log_e \left(\frac{x+1}{2} \right)$.

Mark allocation

- 1 mark for the correct answer.

Question 3

a. Let $y = \frac{e^{2x}}{x}$. Find $\frac{dy}{dx}$.

2 marks

Worked solution

Use the quotient rule to differentiate.

$$y = \frac{e^{2x}}{x} = \frac{u(x)}{v(x)}$$

$$\frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$$

$$= \frac{2xe^{2x} - e^{2x}}{x^2}$$

Mark allocation

- 1 mark for using quotient rule.
- 1 mark for correct answer.

b. Let $f(x) = \sqrt{\sin(2x)}$. Find $f'\left(\frac{\pi}{4}\right)$.

3 marks

Worked solution

First, we must use the chain rule.

$$f'(x) = \frac{1}{2}(\sin(2x))^{-\frac{1}{2}} 2\cos(2x)$$

$$= \frac{\cos(2x)}{\sqrt{\sin(2x)}}$$

$$f'\left(\frac{\pi}{4}\right) = \frac{\cos\left(\frac{\pi}{2}\right)}{\sqrt{\sin\left(\frac{\pi}{2}\right)}}$$

$$= \frac{0}{1} = 0$$

Mark allocation

- 1 mark for using chain rule.
- 1 mark for obtaining $f'(x)$.
- 1 mark for correct answer.

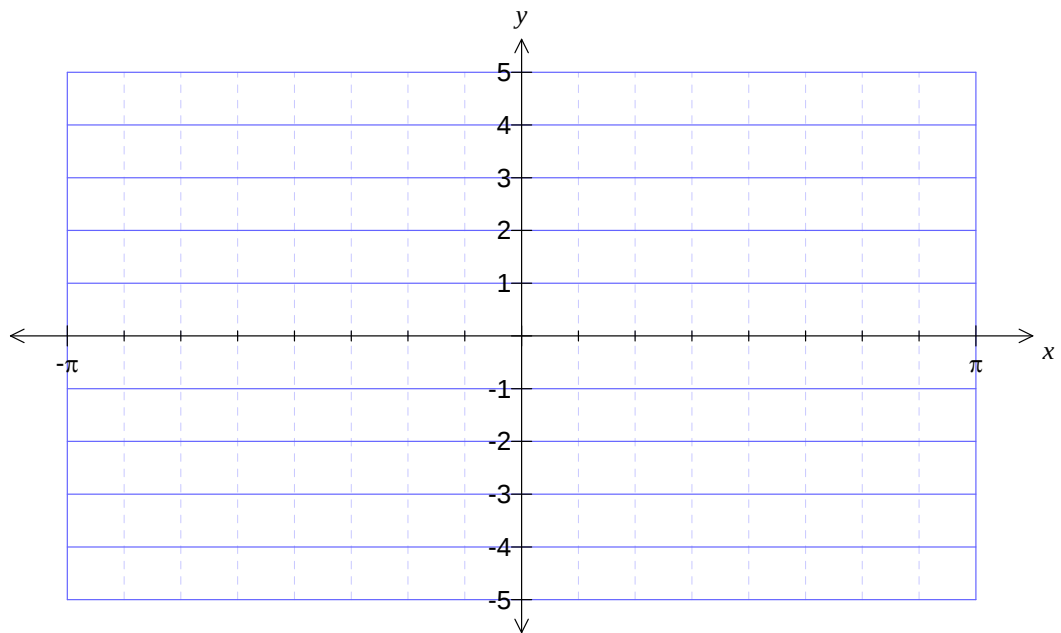
End of Question 3
TURN OVER

Question 4

The graph of $y = \cos(x)$ undergoes the following transformations:

- a dilation of factor $\frac{1}{2}$ from the y-axis
- a translation of +3 units up.

- a. Sketch the transformed graph over the domain $[-\pi, \pi]$ on the axes below.
Label all intercepts and endpoints as co-ordinates.

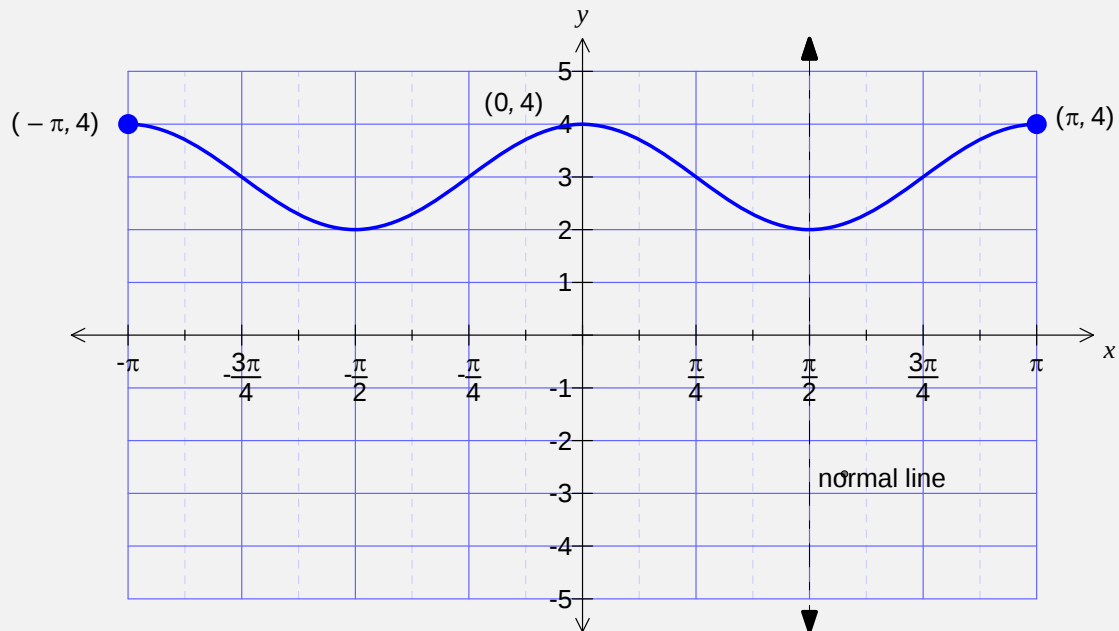


2 marks

Worked solution

The transformations

- a dilation of factor $\frac{1}{2}$ from the y -axis; and
 - a translation of +3 units up
- give a new equation of $y = \cos(2x) + 3$.

**Mark allocation**

- 1 mark for showing two cycles.
- 1 mark for shifting graph up 3 units and labelling points.

- b.** State the equation of the normal to the graph in part **a** at $x = \frac{\pi}{2}$ and sketch the normal on the axes above.

2 marks

Worked solution

The equation of the normal to the curve is $x = \frac{\pi}{2}$ and is shown on the graph.

Mark allocation

- 1 mark for equation.
- 1 mark for line.

End of Question 4
TURN OVER

Question 5

Voicefone, a telephone company, currently has 30% of the market for a new type of home phone system. There are no contracts and customers simply take out a plan for a month at a time. Of the current customers, 80% will still be customers in the next month.

Assume that, for this type of system, $a\%$ of the rest of the market switches to this telephone company from one month to the next.

- a. i. Write a transition matrix for this situation.
- ii. Find the value of a needed in order for Voicefone to maintain its market share.

3 marks

Worked solution

For the transition matrix, treat the value of a as a proportion rather than a percentage.

The transition matrix is $\begin{bmatrix} 0.8 & a \\ 0.2 & 1-a \end{bmatrix}$

and in order to maintain the market share of 30%, then $\frac{a}{a+0.2} = 0.3$, giving:

$$a = 0.3(a + 0.2)$$

$$a = 0.06 + 0.3a$$

$$0.7a = 0.06$$

$$a = \frac{0.06}{0.7} = \frac{6}{70}$$

$$\text{So } a \text{ is } \frac{6}{70} \times 100\% = \frac{60}{7}\%$$

Mark allocation

- 1 mark for matrix.
- 1 mark for relevant method.
- 1 mark for the correct answer.

Voicefone runs an advertising campaign and hopes to eventually hold 60% of the total market, for this type of system.

b. Find the value of a for this situation.

1 mark

Worked solution

$$\begin{bmatrix} 0.8 & a \\ 0.2 & 1-a \end{bmatrix} \text{ and } \frac{a}{a+0.2} = 0.6; \text{ hence:}$$

$$a = 0.6(a + 0.2)$$

$$a = 0.6a + 0.12$$

$$0.4a = 0.12$$

$$a = \frac{12}{40} = 0.3$$

So a is 30%.

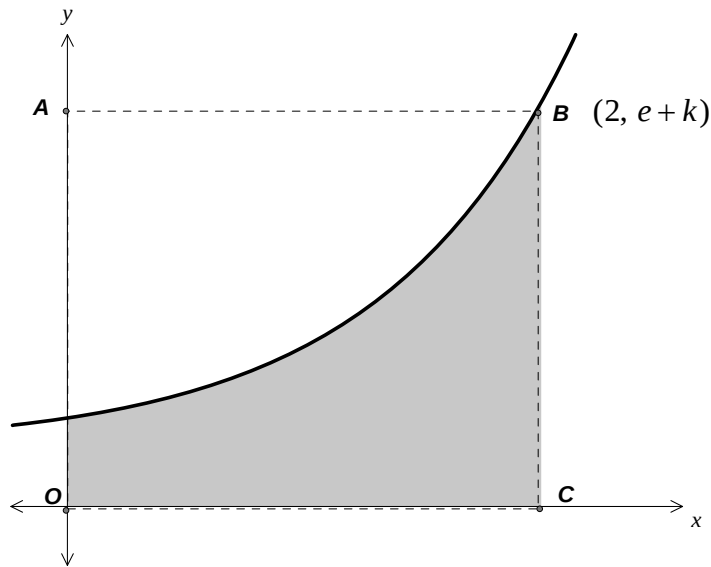
Mark allocation

- 1 mark for the correct answer.

**End of Question 5
TURN OVER**

Question 6

Consider the graph of $y = e^{\frac{x}{2}} + k$. $OABC$ is a rectangle, as shown in the diagram below. If the shaded and unshaded regions are equal in area, find k .



3 marks

Worked solution

Area of the rectangle is $2(e + k)$.

Hence, the area under the curve is equal to $\frac{1}{2}(2(e + k)) = (e + k)$.

$$\text{So, } \int_0^2 e^{\frac{x}{2}} + k \, dx = e + k$$

$$\begin{aligned} \text{LHS} &= [2e^{\frac{x}{2}} + kx]_0^2 \\ &= (2e + 2k) - (2 + 0) \\ &= 2e + 2k - 2 \end{aligned}$$

$$\begin{aligned} \Rightarrow 2e + 2k - 2 &= e + k \\ k &= -e + 2 \end{aligned}$$

Mark allocation

- 1 mark for finding area of rectangle.
- 1 mark for setting up integral.
- 1 mark for correct answer.

End of Question 6

Question 7

a. Find the general solution of $\sqrt{2} \cos(3x) = -1$.

3 marks

Worked solution

$\cos(3x) = \frac{-1}{\sqrt{2}}$, 2nd and 3rd quadrants

$$3x = \frac{3\pi}{4}, \frac{5\pi}{4}$$

$$3x = \pm \frac{3\pi}{4} + 2k\pi$$

$$x = \pm \frac{\pi}{4} + \frac{2k\pi}{3}$$

Mark allocation

- 1 mark for using $\frac{\pi}{4}$ as basic angle.
- 1 mark for finding two quadrants.
- 1 mark for the correct answer.

b. Find the average value of the function $y = 2 \cos(2x)$ over the interval $\left[0, \frac{\pi}{12}\right]$.

3 marks

Worked solution

The average value of a function is defined as $\frac{1}{b-a} \int_a^b f(x) dx$, so in this case

$$\begin{aligned} & \frac{1}{\frac{\pi}{12} - 0} \int_0^{\frac{\pi}{12}} 2 \cos(2x) dx \\ &= \frac{12}{\pi} [\sin(2x)]_0^{\frac{\pi}{12}} \\ &= \frac{12}{\pi} \left[\sin\left(\frac{\pi}{6}\right) - \sin(0) \right] \\ &= \frac{12}{\pi} \left(\frac{1}{2} - 0 \right) \\ &= \frac{6}{\pi} \end{aligned}$$

Mark allocation

- 1 mark for setting up calculation.
- 1 mark for antidifferentiating $2 \cos(2x)$.
- 1 mark for the correct answer.

End of Question 7
TURN OVER

Question 8

A spherical balloon is being inflated. Its volume is increasing at the rate of 4 cm^3 per second. Find the rate, in cm s^{-1} , at which the radius of the balloon is increasing when the radius is 2 cm.

3 marks

Worked solution

The rate equation is $\frac{dV}{dt} = \frac{dr}{dt} \times \frac{dV}{dr}$. We must find $\frac{dV}{dr}$.

As object is a sphere, use

$$V = \frac{4}{3}\pi r^3$$

$$\therefore \frac{dV}{dr} = 4\pi r^2$$

$$\frac{dV}{dt} = \frac{dr}{dt} \times \frac{dV}{dr}$$

$$4 = \frac{dr}{dt} \times 4\pi r^2$$

$$\frac{dr}{dt} = \frac{4}{4\pi r^2}$$

At $r = 2$:

$$\frac{dr}{dt} = \frac{1}{\pi(2)^2} = \frac{1}{4\pi}$$

Mark allocation

- 1 mark for writing rate equation.
- 1 mark for finding $\frac{dV}{dr}$.
- 1 mark for the correct answer.

End of Question 8

Question 9

Let X be a random variable with a normal distribution with mean 6 and variance 4, and let Z be a random variable with the standard normal distribution. If $\Pr(Z > 1) = 0.16$,

a. Find $\Pr(X > 8)$.

1 mark

Worked solution

$$\begin{aligned}\Pr(X > 8) &= \Pr\left(Z > \frac{8-6}{2}\right) \\ &= \Pr\left(Z > \frac{2}{2}\right) \\ &= \Pr(Z > 1) \\ &= 0.16\end{aligned}$$

Mark allocation

- 1 mark for the correct answer.

b. Find $\Pr(X > 8 | X > 6)$.

2 marks

Worked solution

$$\begin{aligned}\Pr(X > 8 | X > 6) &= \frac{\Pr(X > 8 \cap X > 6)}{\Pr(X > 6)} \\ &= \frac{\Pr(X > 8)}{\Pr(X > 6)} = \frac{0.16}{0.5} = 0.32\end{aligned}$$

Mark allocation

- 1 mark for using conditional probability.
- 1 mark for the correct answer.

c. Find a such that $\Pr(Z > a) = \Pr(X < 5)$.

2 marks

Worked solution

$$\Pr(X < 5) = \Pr\left(Z < \frac{5-6}{2}\right) = \Pr\left(Z < -\frac{1}{2}\right)$$

and using symmetry $\Pr\left(Z < -\frac{1}{2}\right) = \Pr\left(Z > \frac{1}{2}\right)$, so $a = \frac{1}{2}$.

Mark allocation

- 1 mark for method.
- 1 mark for the correct answer.

End of Question 9
TURN OVER

Question 10

- a. Using the linear approximation $f(x+h) \approx f(x) + hf'(x)$, where h is 0.03, and $f(x) = \sqrt{x}$, find an approximate value of $\sqrt{16.03}$.

2 marks

Worked solution

$$f(x+h) \approx f(x) + hf'(x) \text{ and } f(x) = \sqrt{x}, f'(x) = \frac{1}{2\sqrt{x}}$$

$$\begin{aligned} \text{So, } \sqrt{16.03} &\approx \sqrt{16} + 0.03 \frac{1}{2\sqrt{16}} = 4 + 0.03 \frac{1}{8} \\ &= 4 + \frac{3}{800} \\ &= 4 \frac{3}{800} \end{aligned}$$

Mark allocation

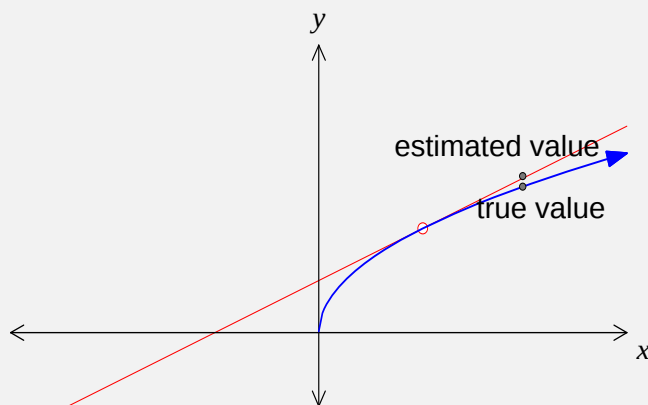
- 1 mark for finding $f'(x) = \frac{1}{2\sqrt{x}}$.
- 1 mark for the correct answer.

- b. Explain why your answer to part a overestimates the value of $\sqrt{16.03}$.

1 mark

Worked solution

The approximate calculation is an overestimation, as it is calculated from a tangent line drawn to the curve $y = \sqrt{x}$ and the tangent line projects above the graph.

**Mark allocation**

- 1 mark for written explanation or using graph to explain.

Question 11

If $f(x) = 3x^2$, show that $f(u+v) + f(u-v) = 2(f(u) + f(v))$.

2 marks

Worked solution

$$f(u+v) + f(u-v) = 2(f(u) + f(v))$$

$$\text{LHS} = 3(u+v)^2 + 3(u-v)^2$$

$$= 3u^2 + 6uv + 3v^2 + 3u^2 - 6uv + 3v^2$$

$$= 6u^2 + 6v^2$$

$$= 2(3u^2 + 3v^2)$$

$$= 2(f(u) + f(v))$$

$$= \text{RHS}$$

Mark allocation

- 1 mark for expanding $3(u+v)^2 + 3(u-v)^2$.
- 1 mark for the correct answer.

END OF SOLUTIONS BOOK