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Unit 3 and 4 Mathematical Methods (CAS): Exam 1

Practice Exam Solutions

Stop!

Don't look at these solutions until you have attempted the exam.

Found a mistake?

Check the Engage Education website for updated solutions, and then email practiceexams@ee.org.au.

Marks allocated are indicated by a number in square brackets, for example, [1] indicates that the line is worth one mark.

Question 1a

$$f'(x) = 3 \cos(3x) e^{\sin(3x)} \quad [2]$$

Question 1b

$$f'(x) = 2 \cos(2x) e^{3x} + 3 \sin(2x) e^{3x} \quad [1]$$

$$f'\left(\frac{\pi}{2}\right) = -5e^{\frac{3\pi}{2}} \quad [1]$$

Question 2a

$$\int \frac{1}{x} dx = \log_e |x| + c \quad [1]$$

$$\log_e 3 + c = 10$$

$$c = 10 - \log_e 3$$

Therefore the antiderivative is $\log_e \left(\frac{x}{3}\right) + 10 \quad [1]$

Question 2b

$$\int_1^c 3x^2 dx = [x^3]_1^c = c^3 - 1 \quad [1]$$

$$c^3 - 1 > 2$$

$$c^3 > 3$$

$$c > \sqrt[3]{3} \quad [1]$$

Question 3a

$f^{-1}(x)$ is formed by swapping x and y [1 mark for statement or use of this fact]

$$x = \log_e \frac{(y-3)}{2} + 5 \quad [1]$$

$$y = 2e^{(x-5)} + 3 = f^{-1}(x) \quad [1]$$

Question 3b

$$g(f(x)) = 2e^{\left(\log_e \frac{(x-3)}{2} + 5 - 5\right)} + 3 \quad [1]$$

$$= 2e^{\left(\log_e \left(\frac{x-3}{2}\right)\right)} + 3$$

$$= 2 * \frac{x-3}{2} + 3$$

$$= x \quad [1]$$

Question 4

First deduce that the transformations necessary are $(x, y) \rightarrow \left(\frac{x}{2} - \frac{\pi}{4}, \pi y\right) \quad [2]$

$$\text{Now we have } \begin{bmatrix} \frac{x}{2} - \frac{\pi}{4} \\ \pi y \end{bmatrix} = \mathbf{A} \begin{bmatrix} x \\ y \end{bmatrix} + \mathbf{b}$$

Hence $\mathbf{A} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{x}{2} - \frac{\pi}{4} \\ \pi y \end{bmatrix} = \mathbf{b}$ and so $\mathbf{b} = \begin{bmatrix} -\frac{\pi}{4} \\ 0 \end{bmatrix}$ [1]

Then $\mathbf{A} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{x}{2} \\ \pi y \end{bmatrix}$ and so $\mathbf{A} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \pi \end{bmatrix}$ [1]

Question 5

$3x + 3\pi = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) + n\pi$, where $n \in \mathbb{Z}$ [1]

$3x + 3\pi = \frac{\pi}{6} + n\pi$

$x + \pi = \frac{(1 + 6n)\pi}{18}$

$x = \frac{(6n-17)\pi}{18}$ [1]

$x = -\frac{17\pi}{18}, -\frac{11\pi}{18}, -\frac{5\pi}{18}, \frac{\pi}{18}, \frac{7\pi}{18}, \frac{13\pi}{18}$ [1]

Question 6a

Let μ' be the mean of aX , and σ' be the standard deviation of aX

$\mu' = \mu a = 27.2a$ by the properties of the normal distribution.

$\text{Var}(aX) = a^2 \text{Var}(X) = a^2 \times 27.2^2 = (\sigma')^2$

$\therefore \sigma' = 27.2a$ [1 for giving the correct expressions for the mean and standard deviation for aX]

As $\Pr(aX \geq 108.8) = 0.025$, 108.8 must be 2 standard deviations away from the mean. [1]

Therefore, $\frac{108.8 - \mu'}{\sigma'} = 2$

$\frac{108.8 - 27.2a}{27.2a} = 2$

$108 - 27.2a = 54.4a$

$108.8 = 81.6a$

$a = \frac{108.8}{81.6} = \frac{4}{3}$ [1]

Question 6b

$\frac{0-27.2}{27.2} = \frac{b-27.2a}{27.2a}$ [1]

$-27.2a = b - 27.2a$

$b = 0$ [1]

Question 7a

$f(x) \geq 0$ for all x [1]

$\int_0^\infty \frac{1}{\theta} e^{\left(\frac{-x}{\theta}\right)} dx$

$$\begin{aligned}
&= \lim_{t \rightarrow \infty} \left[-e^{\left(\frac{-x}{\theta}\right)} \right]_0^t \\
&= \lim_{t \rightarrow \infty} \left(-e^{\left(\frac{-t}{\theta}\right)} + e^0 \right) \\
&= 1 \text{ [1]}
\end{aligned}$$

Question 7b

$$\Pr(X \leq \theta \mid X \leq 3\theta) = \frac{\int_0^{\theta} \frac{1}{\theta} e^{\left(\frac{-x}{\theta}\right)} dx}{\int_0^{3\theta} \frac{1}{\theta} e^{\left(\frac{-x}{\theta}\right)} dx} \text{ [1]}$$

$$= \frac{\left[-e^{\left(\frac{-x}{\theta}\right)} \right]_0^{\theta}}{\left[-e^{\left(\frac{-x}{\theta}\right)} \right]_0^{3\theta}} = \frac{-e^{-1} + 1}{-e^{-3} + 1} \text{ [1]}$$

Question 8a

$$E(X) = 0 \text{ [1]}$$

Question 8b

$$E(X^2) = 1.2 \text{ [1]}$$

Question 8c

$$\text{Var}(X) = E(X^2) - (E(X))^2 = 1.2 - 0 = 1.2$$

$$\text{Var}(2X) = 4 \text{Var}(X) = 4.8 \text{ [1]}$$

Question 9a

$$f'(x) = 2x$$

$$f'(3) = 6 \text{ [1]}$$

Hence the tangent is given by:

$$y = 6(x - 3) + f(3)$$

$$y = 6x - 18 + 5$$

$$y = 6x - 13 \text{ [1]}$$

Question 9b

$$f(3.02) = f(3 + 0.02) \text{ [1]}$$

$$f(3 + 0.02) \approx f(3) + 0.02f'(3)$$

$$= 5 + 0.02 \times 6 = 5.12 \text{ [1]}$$

Question 10a

$$\frac{dS}{dt} = \frac{dS}{dr} \frac{dr}{dt} \text{ [1]}$$

$$\frac{dS}{dr} = 8\pi r \text{ and } \frac{dr}{dt} = 3 \times 10^8 \text{ [1]}$$

$$\frac{dS}{dt} = 8\pi r \times 3 \times 10^8 = 24\pi r \times 10^8 \text{ [1]}$$

Question 10b

$$\frac{dS}{dt} = \frac{8}{3}\pi r \times 10^8 = 10^{19} \text{ [1]}$$

$$r = \frac{10^{19}}{\frac{8}{3}\pi \times 10^8} = \frac{3}{8\pi} \times 10^{11} \text{ m [2]}$$

Question 10c

The ratio is given by $\frac{s(10^{10})}{s(10^{12})}$ [1]

$$= \frac{4\pi(10^{10})^2}{4\pi(10^{12})^2}$$

$$= \frac{1}{10^4} \text{ [1]}$$