

The Mathematical Association of Victoria
Maths Methods CAS 2011
Trial Written Examination 2 – SOLUTIONS

- | | | | | | | | |
|--------------|--------------|--------------|--------------|--------------|--------------|--------------|--------------|
| 1. C | 2. A | 3. B | 4. C | 5. E | 6. D | 7. A | 8. B |
| 9. A | 10. E | 11. C | 12. B | 13. B | 14. C | 15. D | 16. B |
| 17. A | 18. C | 19. E | 20. D | 21. A | 22. C | | |

Question 1**Answer C**

The domain of fg is the intersection of the domain of f and the domain of g .

The domain of f is $\left(-\infty, \frac{1}{2}\right]$ and the domain of g is $R \setminus \left\{\frac{1}{2}\right\}$.

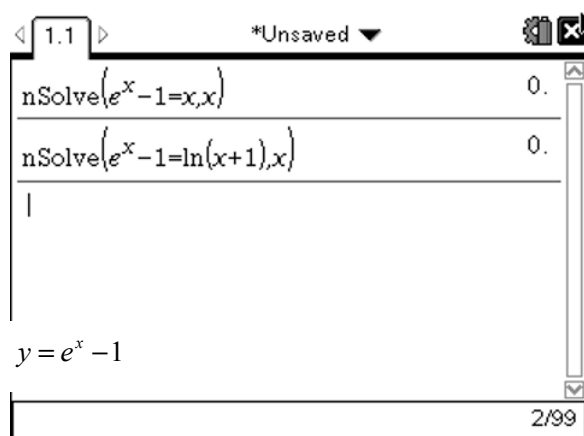
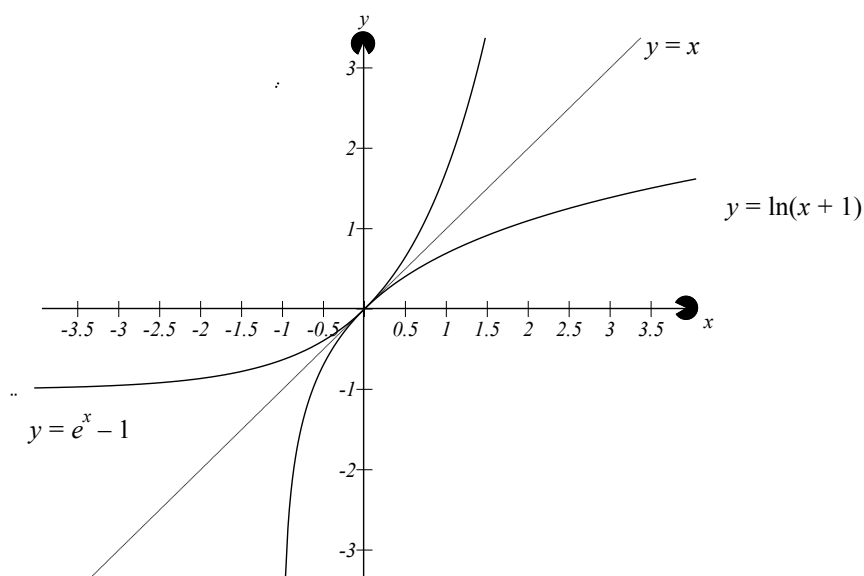
Hence the domain of fg is $\left(-\infty, \frac{1}{2}\right)$.

Question 2**Answer A**

f and its inverse, f^{-1} will have two points of intersection when f and $y = x$ have two points of intersection.

This will occur when $k < -1$.

The graphs with equations $y = e^x - 1$, $y = x$ and $y = \ln_e(x+1)$, where $k = -1$ are shown below. There is only one point of intersection which occurs when $e^x - 1 = x$. Hence $x = 0$ as $e^0 - 1 = 0$.

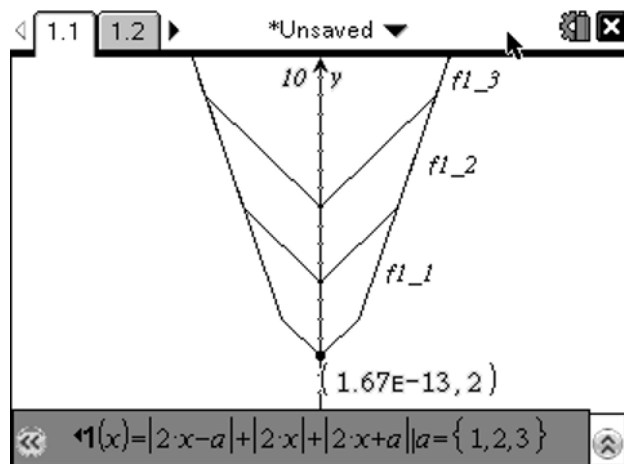


Question 3**Answer B**

$$f(x) = |2x + a| + |2x| + |2x - a|.$$

The minimum value occurs when $x = 0$. $f(x) = |a| + |-a| = 2a$.

The minimum value is $2a$.

**Question 4****Answer C**

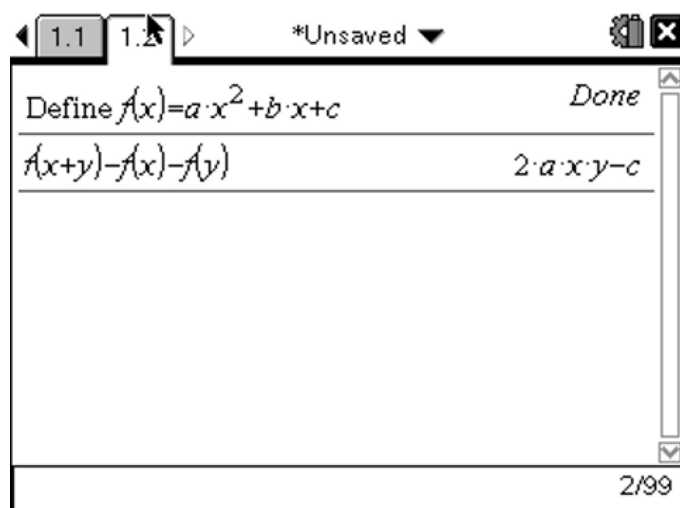
$$f(x) = ax^2 + bx + c, \quad f(y) = ay^2 + by + c$$

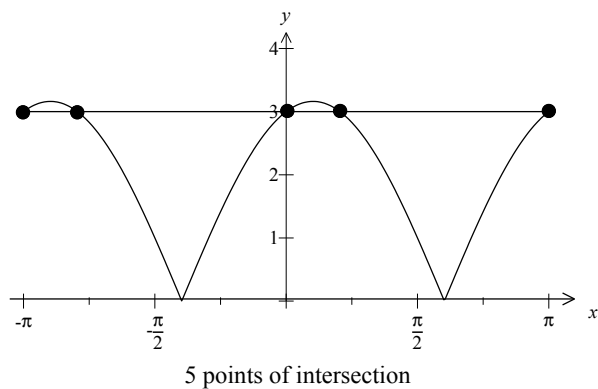
$$f(x+y) = a(x+y)^2 + b(x+y) + c$$

$$= ax^2 + 2axy + ay^2 + bx + by + c$$

$$= ax^2 + bx + c + ay^2 + by + 2axy$$

$$= f(x) + f(y) - c + 2axy$$



Question 5**Answer E****Question 6****Answer D**

$$2\sin(2x) = 1$$

$$\sin(2x) = \frac{1}{2}$$

$$2x = 2n\pi + \frac{\pi}{6}, (2n+1)\pi - \frac{\pi}{6}$$

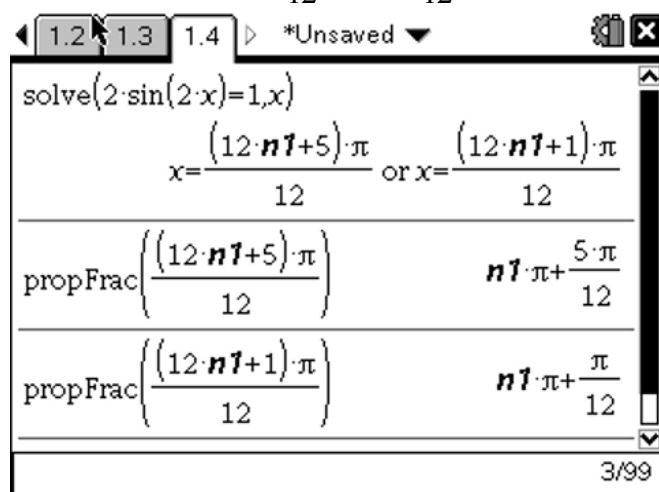
$$x = n\pi + \frac{\pi}{12}, n\pi + \frac{\pi}{2} - \frac{\pi}{12}$$

$$= n\pi + \frac{\pi}{12}, n\pi + \frac{5\pi}{12}, n \in \mathbb{Z}$$

OR

$$x = \frac{(12n+5)\pi}{12}, \frac{(12n+1)\pi}{12}$$

$$= n\pi + \frac{5\pi}{12}, n\pi + \frac{\pi}{12}, n \in \mathbb{Z}$$



Question 7**Answer A**

$$f(x) = 2 \tan\left(3x + \frac{\pi}{2}\right) + 1, -\frac{\pi}{3} \leq x \leq \frac{\pi}{3}$$

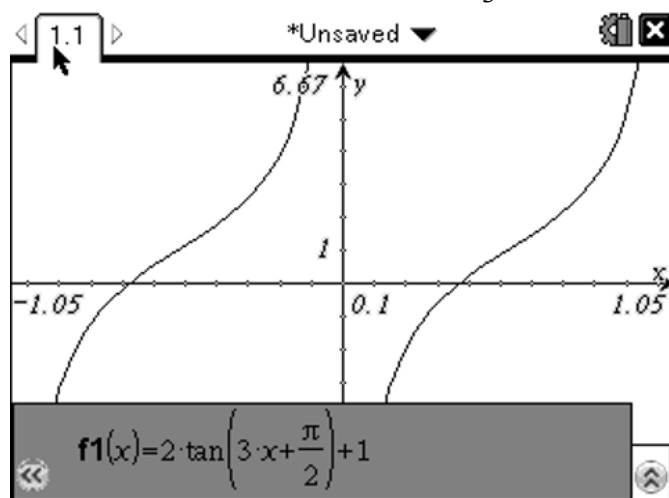
$$= 2 \tan\left(3\left(x + \frac{\pi}{6}\right)\right) + 1, -\frac{\pi}{3} \leq x \leq \frac{\pi}{3}$$

The period is $\frac{\pi}{3}$.

The graph of g with equation $g(x) = 2 \tan(3x) + 1$ has asymptotes at $x = -\frac{\pi}{6}$, $x = \frac{\pi}{6}$ and $x = \frac{\pi}{2}$.

Translate $\frac{\pi}{6}$ units to the left.

Thus asymptotes have equations $x = -\frac{\pi}{3}$, $x = 0$ and $x = \frac{\pi}{3}$.

**Question 8****Answer B**

The equation of f can be written in the form $f(x) = A(x - B)^3 + C$.
There is a stationary point of inflection at $(2, 4)$.

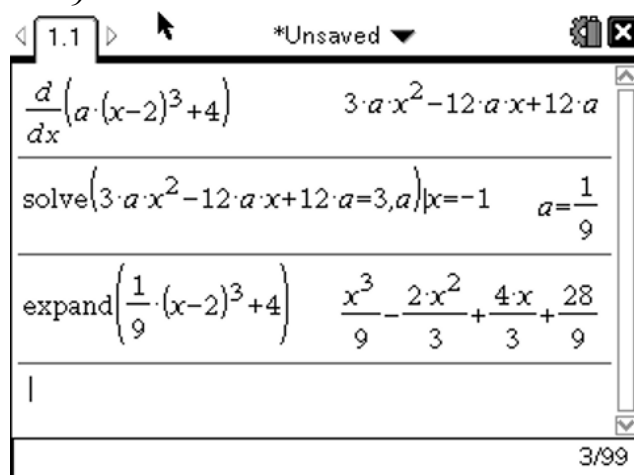
Hence $f(x) = A(x - 2)^3 + 4$.

$$f'(-1) = 3$$

$$f'(x) = 3A(x - 2)^2$$

$$27A = 3$$

$$A = \frac{1}{9}$$

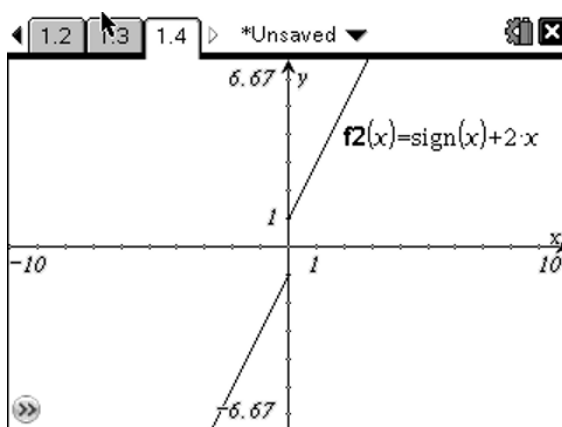
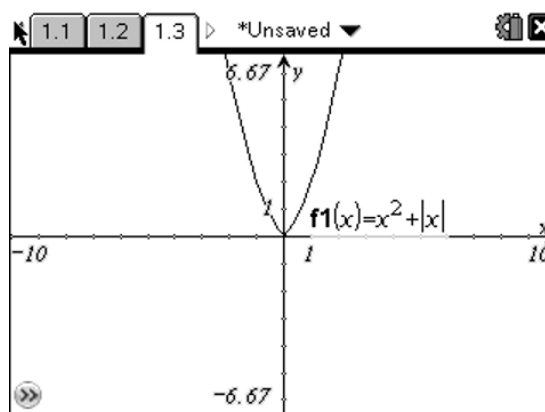
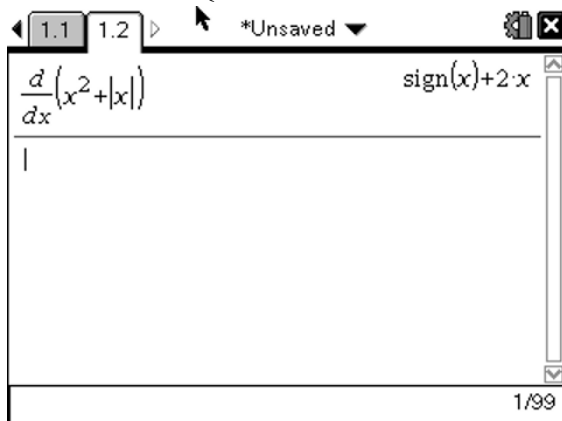


Question 9**Answer A**

$$h(x) = x^2 + x \text{ and } g(x) = |x|$$

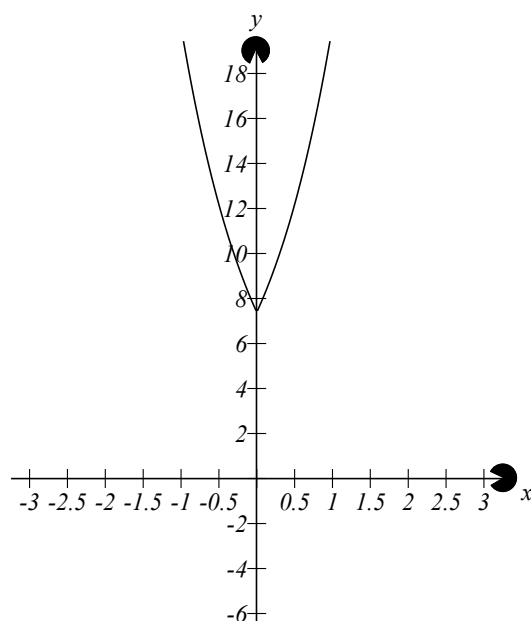
$$h(g(x)) = (|x|)^2 + |x| = x^2 + |x|$$

$$h'(g(x))g'(x) = \begin{cases} 2x+1, & x > 0 \\ 2x-1 & x < 0 \end{cases}$$

**Question 10****Answer E**

$$f(x) = e^{|x|+2}$$

By close inspection of the graph near $x = 0$, f cannot be differentiated at $x = 0$.



Question 11

$$y = f(x) = 2 \log_e(1-x) + 3$$

$$f'(x) = \frac{-2}{1-x}$$

Gradient of the tangent

$$f'(0) = -2$$

Gradient of the normal is $\frac{1}{2}$.**Answer C**

Calculator interface showing the derivative of $2 \cdot \ln(1-x) + 3$ at $x=0$ is -2 . The normal line equation is $\frac{x}{2} + 3$. A warning message at the bottom states: "Domain of the result might be larger than the do..."

Question 12

$$\text{Average rate of change} = \frac{f(\pi) - f(0)}{\pi - 0} = 0$$

Answer B

Calculator interface showing the definition of $f(x) = x^2 \cdot \sin(2 \cdot x)$. The calculation of $\frac{f(\pi) - f(0)}{\pi - 0}$ results in 0 . The status bar at the bottom shows $1/2$.

Question 13**Answer B**

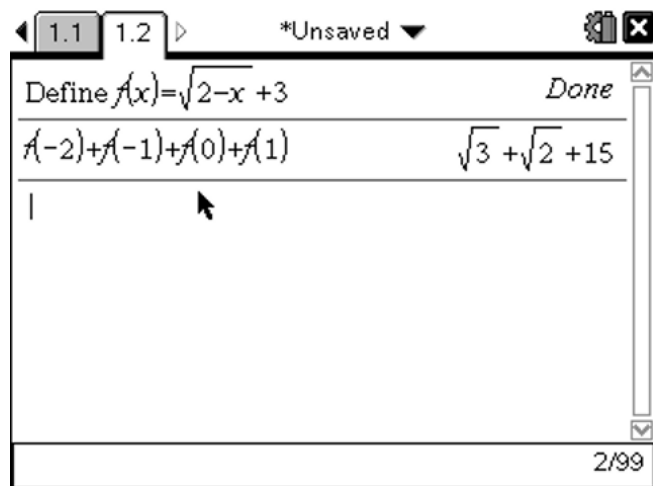
$$\text{Area} \approx f(-2) + f(-1) + f(0) + f(1)$$

$$= 5 + \sqrt{3} + 3 + \sqrt{2} + 3 + 4$$

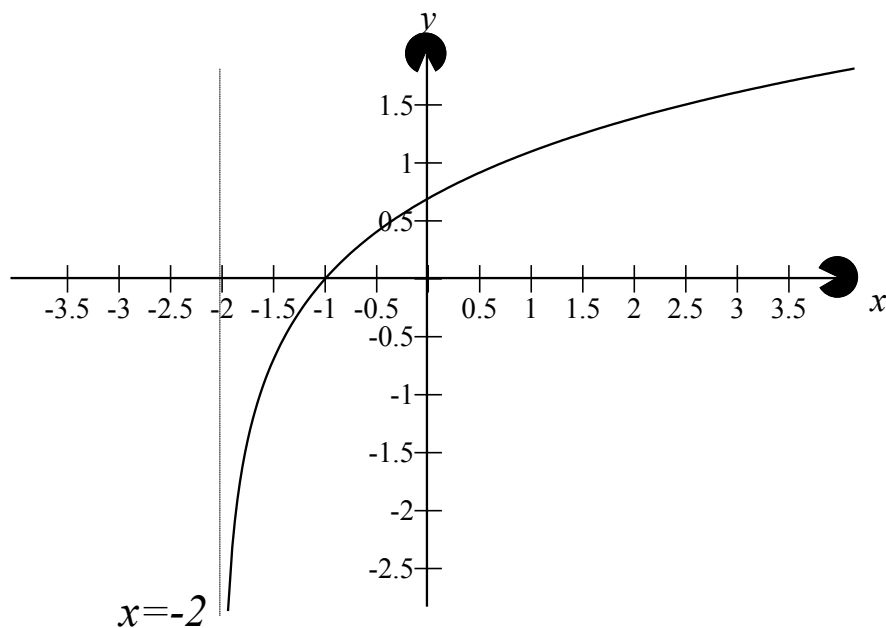
$$= 15 + \sqrt{3} + \sqrt{2}$$

The rectangles are above the curve.

Hence an overestimate of the actual area.

**Question 14****Answer C**

The graph of $y = \log_e(x+2)$ is shown below.



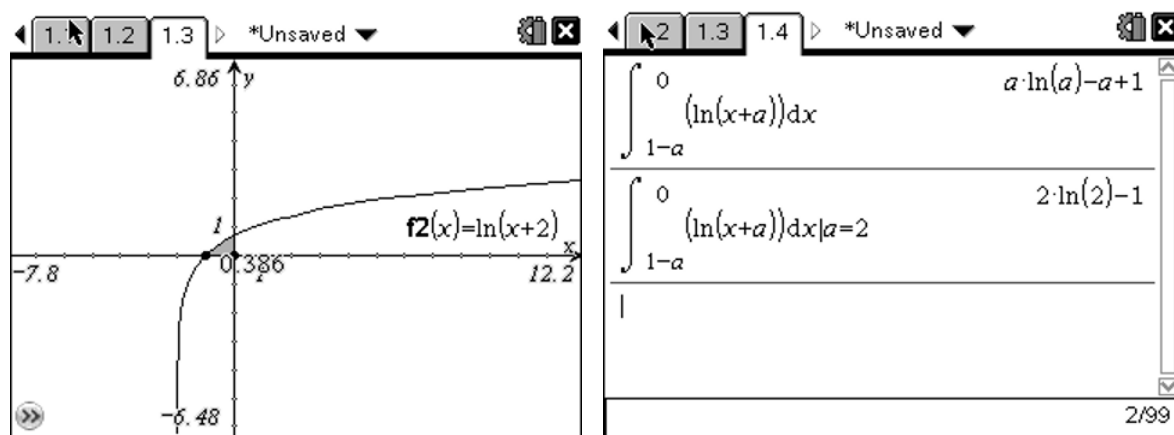
$$x = -2$$

Solve $\log_e(x+a) = 0$ for x .

$$x+a = e^0$$

$$x = 1 - a$$

$$\text{Area} = \int_{1-a}^0 f(x) dx$$

**Question 15****Answer D**

$$\frac{d(x \sin(x))}{dx} = \sin(x) + x \cos(x)$$

$$\Rightarrow x \sin(x) = \int (\sin(x) + x \cos(x)) dx$$

$$x \sin(x) = \int (\sin(x)) dx + \int (x \cos(x)) dx$$

$$\Rightarrow \int (x \cos(x)) dx = x \sin(x) - \int (\sin(x)) dx$$

Question 16**Answer B**

Find coordinates of B.

$$\sin(x) + 2x = 2x$$

$$\sin(x) = 0$$

$$x = 0, \pi, 2\pi$$

B has x value of π

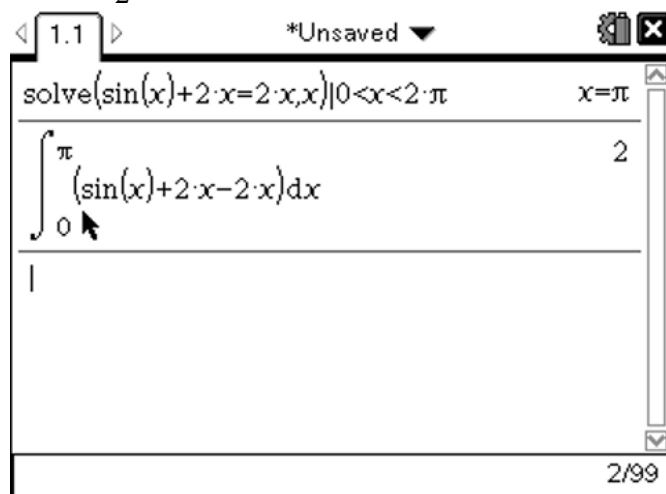
Area between curves:

$$A = \int_0^\pi (\sin(x) + 2x - 2x) dx$$

$$= [-\cos(x)]_0^\pi$$

$$= -\cos(\pi) + \cos(0)$$

$$= 2$$



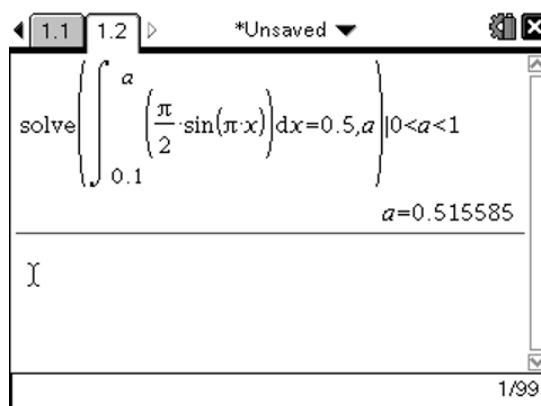
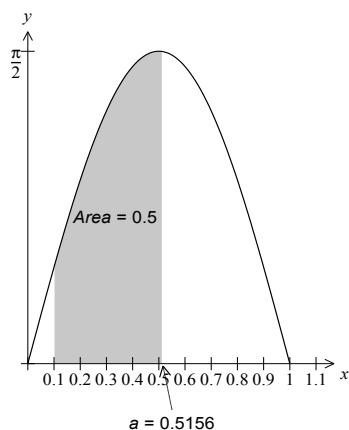
Question 17**Answer A**

$$\int_{0.1}^a \frac{\pi}{2} \sin(\pi x) dx = 0.5$$

$$a = -0.5156; 0.5156$$

By domain restriction,

$$a = 0.5156$$

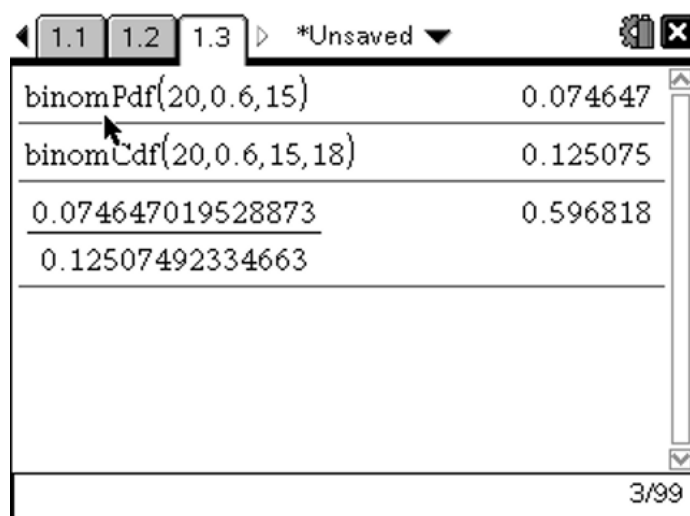
**Question 18****Answer C**Binomial $n = 20$, $p = 0.6$

$$\Pr(X = 15 | 15 \leq X \leq 18) = \frac{\Pr(X = 15)}{\Pr(15 \leq X \leq 18)}$$

$$= \frac{\Pr(X = 15)}{\Pr(X = 15) + \Pr(X = 16) + \Pr(X = 17) + \Pr(X = 18)}$$

$$= \frac{0.074647}{0.074647 + 0.034991 + 0.01235 + 0.003087}$$

$$= 0.5968$$



Question 19

Answer E

$$\Pr(A) \times \Pr(B) = 0.4 \times 0.3 = 0.12$$

$$\Pr(A \cap B) = 0.2$$

A and B are not independent

Question 20

Answer D

The initial state matrix is $S_0 = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$

$$R_0 \quad L_0$$

The transition matrix is $T = \begin{matrix} R_1 \\ L_1 \end{matrix} \begin{bmatrix} 0.8 & 0.6 \\ 0.2 & 0.4 \end{bmatrix}$

The next state matrix is given by the following.

$$R_0 \quad L_0$$

$$S_1 = \begin{matrix} R_1 \\ L_1 \end{matrix} \begin{bmatrix} 0.8 & 0.6 \\ 0.2 & 0.4 \end{bmatrix} \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$$

Question 21

Answer A

$$E(X) = (-1 + 2 + 3 - 4 + 5 - 6) \times \frac{1}{6} = -\frac{1}{6}$$

The player losses $\$ \frac{1}{6}$.

Question 22

Answer C

Score A 92; score a pass 57

$\text{invNorm}(0.85, 76, 15)$	91.5465
$\text{invNorm}(0.1, 76, 15)$	56.7767

Solutions to the Extended Answer Section**Question 1**

$$\text{a. i. } h(x) = \begin{cases} -2x + 2, & 0 \leq x \leq 1 \\ 0, & 1 < x \leq 2 \\ 2x - 4, & 2 < x \leq 3 \end{cases}$$

$$-2x + 2$$

1A

$$0$$

1A

$$2 < x \leq 3$$

1A

$$\text{ii. } \int_0^3 h(x) dx = \text{Area of the two triangles} = 2 \text{ units}^2$$

1A

$$3 \times \text{Average Value} = 2$$

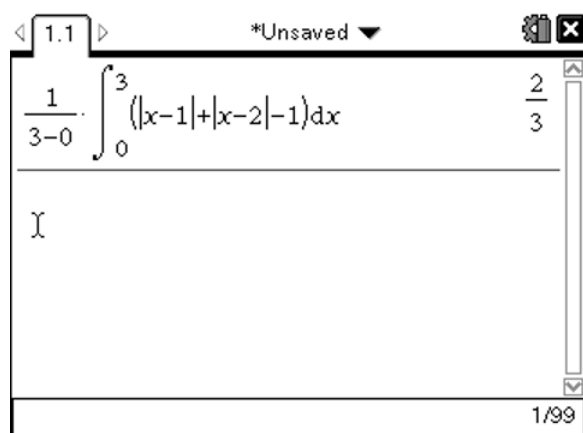
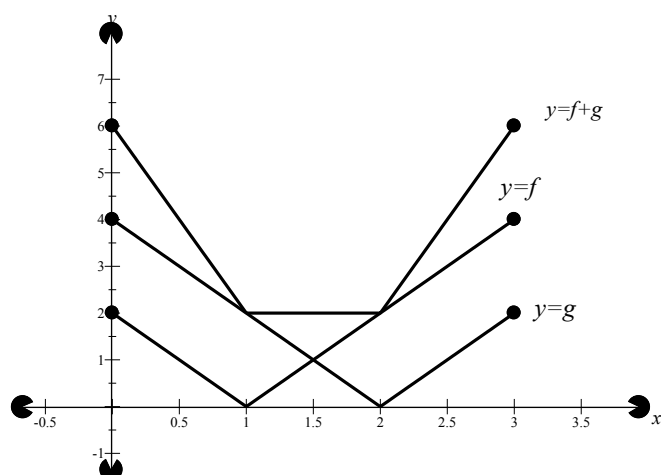
$$\text{Average Value} = \frac{2}{3}$$

1A**OR****OR**

$$\frac{1}{3-0} \int_0^3 h(x) dx$$

1A

$$= \frac{2}{3}$$

1A**b. i.**

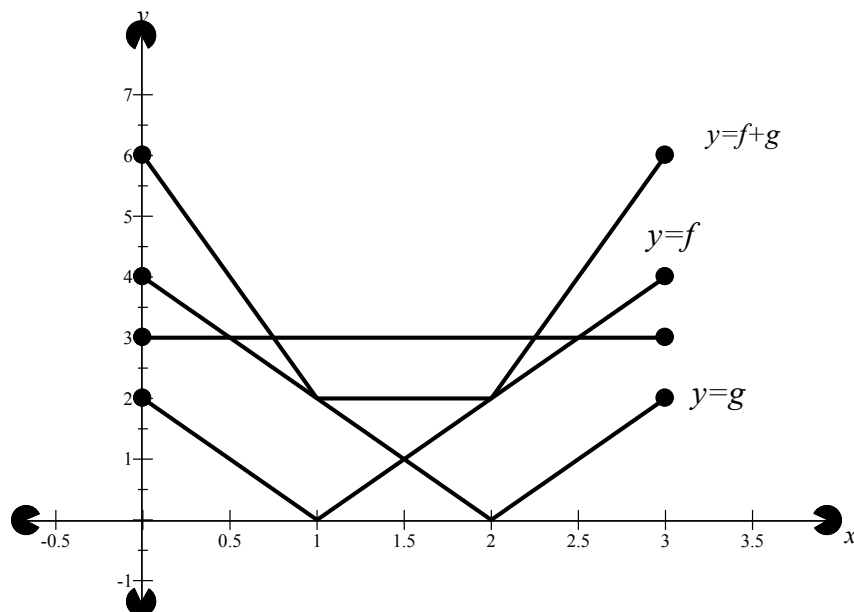
Correct shape for $y = f + g$

1A

Correct endpoints and closed circles

1A

ii.

Correct line $y = \frac{10}{3}$

1A

iii. There are many different approaches to this question. Some are outlined below.

The required area = the area of the triangles above the line $y = \frac{10}{3}$ because $\frac{10}{3}$ is the average value.

The equation of the first line segment is $y = -4x + 6$.

$$\frac{10}{3} = -4x + 6$$

$$x = \frac{2}{3}$$

1A

Area of both triangles

= base $\times$ height

$$= \frac{2}{3} \times \left(6 - \frac{10}{3}\right)$$

1A

$$= \frac{16}{9} \text{ units}^2$$

1A

OR

OR

$$f(x) + g(x) = |2(x-1)| + |2(x-2)|$$

$$\text{Solve } f(x) + g(x) = \frac{10}{3}$$

$$x = \frac{2}{3} \text{ or } x = \frac{7}{3}$$

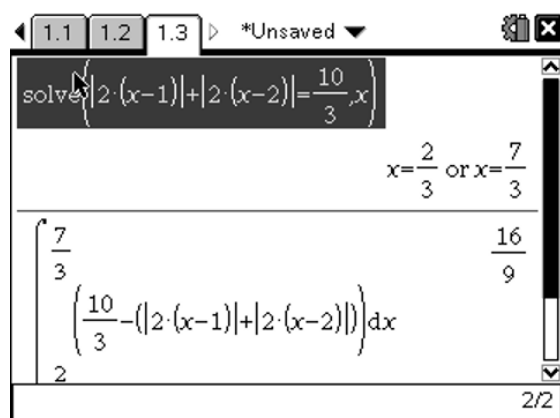
1A

(or could use the first method to get $x = \frac{2}{3}$ and then the other x value is $2 + \frac{1}{3}$)

$$= \int_{\frac{2}{3}}^{\frac{7}{3}} \left(\frac{10}{3} - (f(x) + g(x)) \right) dx$$

1A

$$= \frac{16}{9} \text{ units}^2$$

1A**OR****OR**

$$\text{Area of the trapezium} = \frac{h(a+b)}{2}$$

The equation of the first line segment is $y = -4x + 6$.

$$\frac{10}{3} = -4x + 6$$

$$x = \frac{2}{3} \text{ or } x = 2 + \frac{1}{3} = \frac{7}{3}$$

1A

$$\text{Area} = \frac{\left(\frac{10}{3} - 2\right)\left(2 + \frac{7}{3} - \frac{2}{3}\right)}{2}$$

1A

$$= \frac{16}{9} \text{ units}^2$$

1A

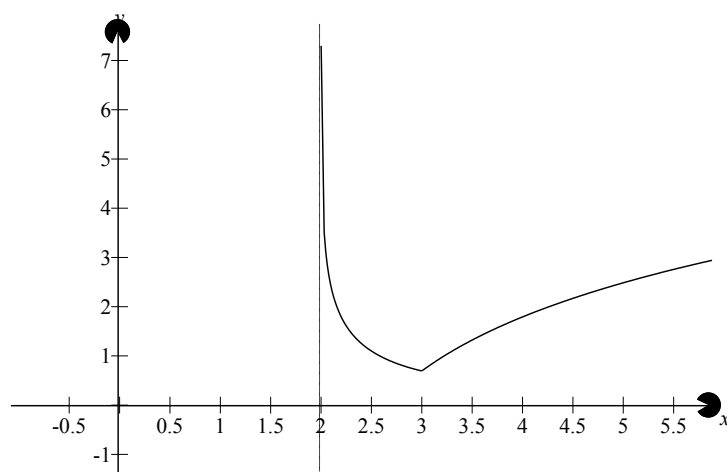
c. i. There is an asymptote at $x = 2$.

$$a = 2$$

1A

$$\text{dom}(p) = \text{dom}(y = |\log_e(x-1)|) \cap \text{dom}(y = |\log_e(x-2)|)$$

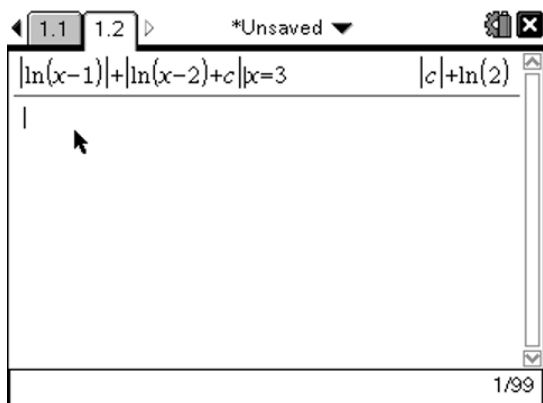
The graph of $p(x) = |\log_e(x-1)| + |\log_e(x-2)|$



ii. $(3, \log_e(2) + c)$

1A

$$p(3) = |\log_e(3-1)| + |\log_e(3-2)| + c = \log_e(2) + c$$



Question 2

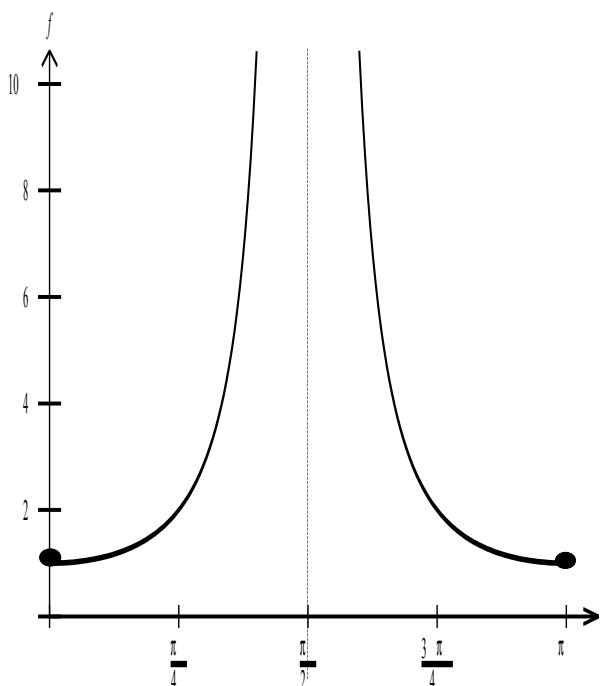
a. $\frac{dy}{dx} = \frac{\cos(x)\cos(x) + \sin(x)\sin(x)}{\cos^2(x)}$ quotient rule

Since $\cos^2(x) + \sin^2(x) = 1$

$$\frac{dy}{dx} = \frac{1}{\cos^2(x)}$$

1M

b.



Correct Shape and endpoints
Asymptote

1A

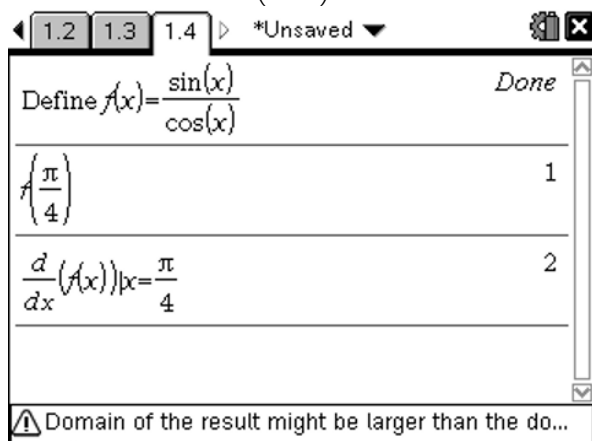
1A

c. i. $y = \tan\left(\frac{\pi}{4}\right) = 1$

1A

ii. $\frac{dy}{dx} = \frac{1}{\cos^2\left(\frac{\pi}{4}\right)} = \frac{1}{\left(\frac{1}{\sqrt{2}}\right)^2} = 2$

1A



d. domain: $\theta \in \left[0, \frac{\pi}{2}\right)$

1A

e. Let the path of the rocket be h km.

$$\tan \theta = \frac{h}{3} \Rightarrow h = 3 \tan(\theta)$$

1M

f. From **part a.**, $\frac{dh}{d\theta} = \frac{3}{\cos^2(\theta)}$

1A

The rate at which the height of the rocket is changing with respect to the angle, θ .

1A

g. $\frac{d\theta}{dt} = 20 \times \frac{\pi}{180}$ radians/second

1M

$$\frac{dh}{dt} = \frac{dh}{d\theta} \times \frac{d\theta}{dt}$$

1M

$$= \frac{3}{\cos^2(\theta)} \times \frac{\pi}{9}$$

$$= \frac{2\pi}{3} \text{ kilometres/second}$$

1A

h. i. $\tan \alpha = \frac{3}{4} \Rightarrow \alpha = 0.6435$ radians

1A

ii. $h = 4 \tan(\alpha) \Rightarrow \frac{dh}{d\alpha} = \frac{4}{\cos^2(\alpha)}$

$$\frac{d\alpha}{dt} = \frac{d\alpha}{dh} \times \frac{dh}{dt}$$

1A

$$= \frac{\cos^2(\alpha)}{4} \times \frac{2\pi}{3}$$

1M

$$= \frac{\pi \times \cos^2(0.6435)}{6}$$

$$= 0.34 \text{ radians/second (2 dp)}$$

1A

Question 3

- a. X is the weight of chocolate statues

$$X \sim N(1000, 16)$$

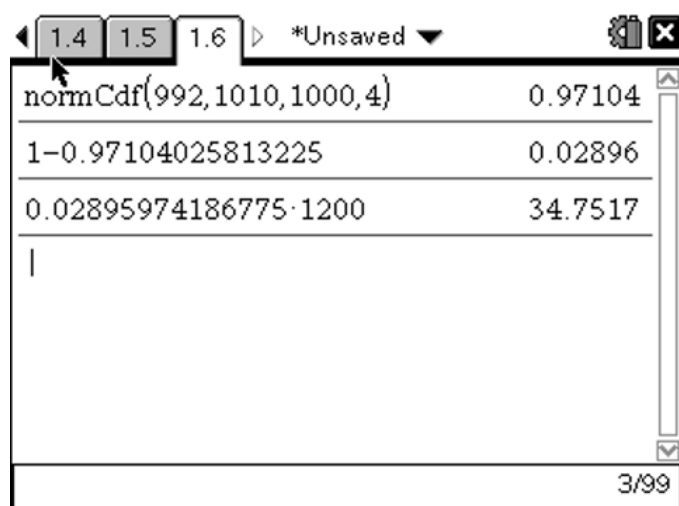
$$\Pr(992 < X < 1010) = 0.97104 = 0.9710 \text{ (4 dp)}$$

1A

- b. $\Pr(\text{rejected}) = 1 - 0.97104 = 0.0290$

$$\text{Number rejected} = 1200 \times 0.0290 = 34.7 = 35 \text{ statues}$$

1A



The calculator screen shows the following calculations:

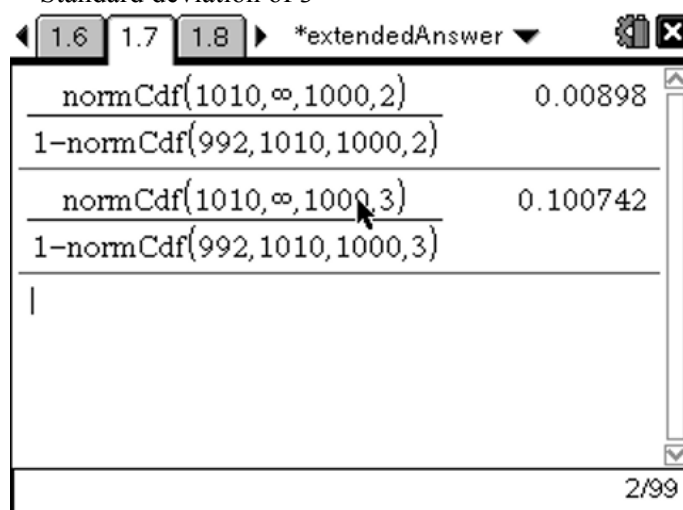
normCdf(992,1010,1000,4)	0.97104
1-0.97104025813225	0.02896
0.02895974186775 · 1200	34.7517

- c. Trial and error (technology) using $N(1000, \{2,3,4\})$ with $\frac{\Pr(X > 1010)}{1 - \Pr(992 < X < 1010)}$.

1M

Standard deviation of 3

1A



The calculator screen shows the following calculations:

normCdf(1010,∞,1000,2)	0.00898
1-normCdf(992,1010,1000,2)	
normCdf(1010,∞,1000,3)	0.100742
1-normCdf(992,1010,1000,3)	

- d. Y is the number rejected out of 5.

$$Y \sim \text{Bi}(5, 0.0290)$$

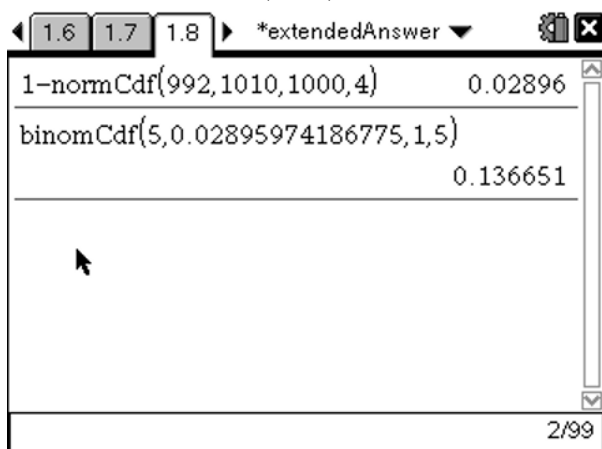
$$Y = \{0, 1, 2, 3, 4, 5\}$$

1M

$$\Pr(Y \geq 1) = 1 - \Pr(Y = 0)$$

$$= 1 - 0.9710^5$$

$$= 0.13665 = 0.137 \text{ (3 dp)}$$

1A

e. i. $\Pr(b|b) = 0.85$; $\Pr(b|b') = 0.3$; $\Pr(b'|b) = 0.15$; $\Pr(b'|b') = 0.7$

transition matrix $T = \begin{bmatrix} 0.85 & 0.3 \\ 0.15 & 0.7 \end{bmatrix}$

$$\Pr(bbb|b) = 0.85^3 = 0.614125 = 0.6141 \text{ (4 dp)}$$

1A

ii. $\Pr(b'bb|b) + \Pr(bb'b|b) + \Pr(bbb'|b)$

1M

$$0.15 \times 0.3 \times 0.85 + 0.85 \times 0.15 \times 0.3 + 0.85^2 \times 0.15 = 0.18485 = 0.1849 \text{ (4 dp)}$$

1A

iii. $\begin{bmatrix} 0.85 & 0.3 \\ 0.15 & 0.7 \end{bmatrix} \begin{bmatrix} x \\ 1-x \end{bmatrix} = \begin{bmatrix} x \\ 1-x \end{bmatrix}$

$$x = \frac{2}{3}$$

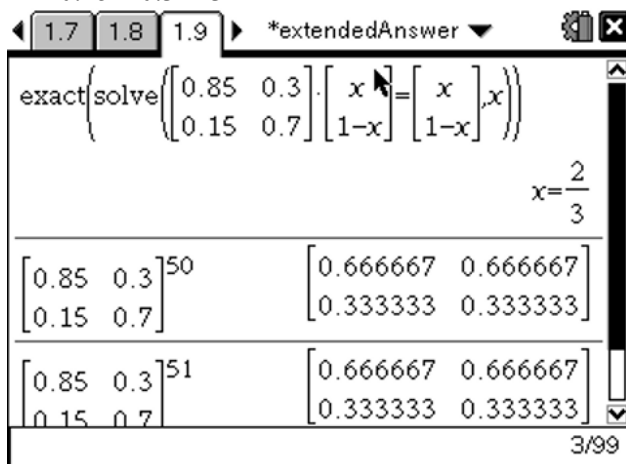
1A**OR****OR**

$$t = \begin{bmatrix} 0.85 & 0.3 \\ 0.15 & 0.7 \end{bmatrix} \Rightarrow t^\infty = \begin{bmatrix} 0.6667 & 0.6667 \\ 0.3333 & 0.3333 \end{bmatrix} \text{ (technology)}$$

$$x = \frac{2}{3}$$

1A**OR****OR**

$$x = \frac{0.3}{0.15 + 0.3} = \frac{2}{3}$$

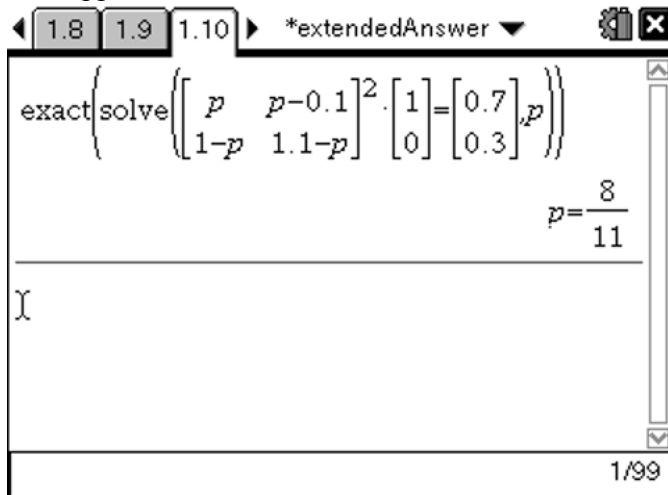
1A

f. Let $t = \begin{bmatrix} p & p-0.1 \\ 1-p & 1-(p-0.1) \end{bmatrix} = \begin{bmatrix} p & p-0.1 \\ 1-p & 1.1-p \end{bmatrix}$ $s = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ **1A**

$$\Pr(\text{buy 3rd month}): t^2 s = \begin{bmatrix} 1.1p-0.1 \\ \dots \end{bmatrix}$$

$$1.1p - 0.1 = 0.7$$

$$p = \frac{8}{11} \quad \text{1M}$$



g. Let $T = \begin{bmatrix} \frac{8}{11} & \frac{8}{11} - 0.1 \\ \frac{3}{11} & 1.1 - \frac{8}{11} \end{bmatrix} = \begin{bmatrix} \frac{8}{11} & \frac{69}{110} \\ \frac{3}{11} & \frac{41}{110} \end{bmatrix}$ **1M**

$$\Pr(\text{no buys}) = \Pr(b'b' | b) = \frac{3}{11} \times \frac{41}{110} = \frac{123}{1210}$$

$$\Pr(1 \text{ buy}) = \Pr(bb' | b) + \Pr(b'b | b) = \frac{8}{11} \times \frac{3}{11} + \frac{3}{11} \times \frac{69}{110} = \frac{447}{1210}$$

$$\Pr(2 \text{ buys}) = \Pr(bb | b) = \frac{8}{11} \times \frac{8}{11} = \frac{64}{121}$$

x	0	1	2
$\Pr(X = x)$	$\frac{123}{1210}$	$\frac{447}{1210}$	$\frac{64}{121}$

1M

$$E(X) = 0 \times \frac{1123}{12100} + 1 \times \frac{447}{1210} + 2 \times \frac{64}{121} = 1.427... = 1.4 \text{ month (1 dp)}$$

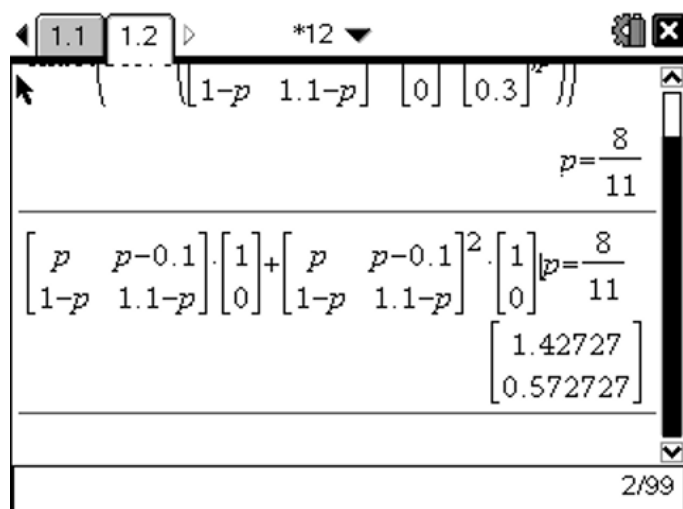
1A**OR****OR**

$$\begin{bmatrix} \frac{8}{11} & \frac{69}{110} \\ \frac{3}{11} & \frac{41}{110} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{8}{11} & \frac{69}{110} \\ \frac{3}{11} & \frac{41}{110} \end{bmatrix}^2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{157}{110} \\ \frac{63}{110} \end{bmatrix} \approx \begin{bmatrix} 1.427 \\ 0.573 \end{bmatrix}$$

2M

1.4 months

1A

**Question 4****a. i.** x -intercept, $f(x) = 0$

$$0 = a(x-b)^5 + c$$

$$(x-b)^5 = -\frac{c}{a}$$

$$x = b + \sqrt[5]{-\frac{c}{a}}$$

$$\left(b + \sqrt[5]{-\frac{c}{a}}, 0 \right)$$

1A

$$\text{or } \left(b - \sqrt[5]{\frac{c}{a}}, 0 \right)$$

Note the TI-nspire CAS did not solve this equation

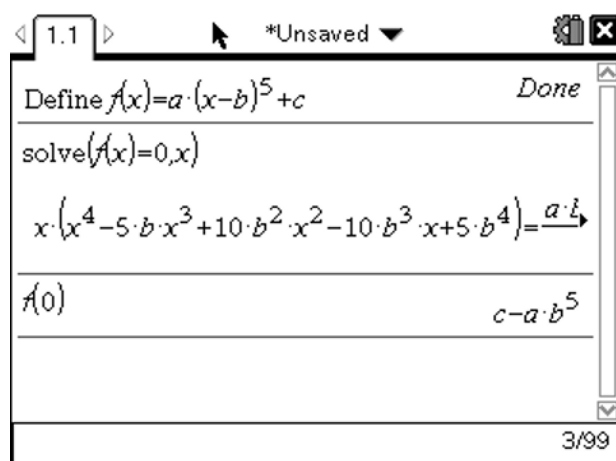
ii. y -intercept, $x = 0$

$$f(x) = a(x-b)^5 + c$$

$$f(0) = a(-b)^5 + c = -ab^5 + c$$

$$(0, -ab^5 + c)$$

1A



iii. The value of the x -intercept is the same as the y -intercept when the graph passes through $(0, 0)$ only.

The graph of f is a quintic polynomial function which has its stationary point of inflection in the first quadrant. Hence when the y -intercept is positive, the x -intercept will be negative and vice-versa.

The stationary point of inflection is at (b, c) .

1A

Solve the y -intercept equal to 0 for c

$$-ab^5 + c = 0$$

$$c = ab^5$$

The stationary point of inflection is at (b, ab^5)

1A

iv. The x -intercept is $b + \sqrt[5]{-\frac{c}{a}}$

$$b + \sqrt[5]{-\frac{c}{a}} < 0$$

Solve $2 + \sqrt[5]{-\frac{3}{a}} < 0$ for a

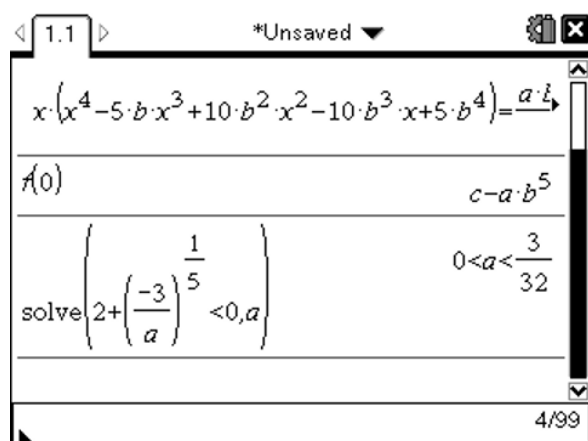
1A

$$\sqrt[5]{-\frac{3}{a}} < -2$$

$$-\frac{3}{a} < -32, a > 0$$

$$0 < a < \frac{3}{32}$$

1A



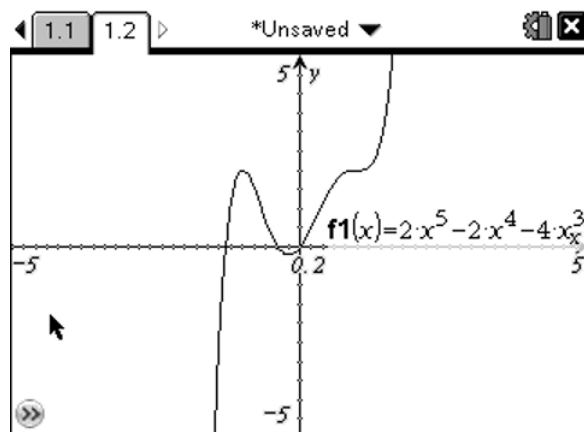
b. i. There are three stationary points.

1A

Since g can be written in the form $g(x) = A(x - B)^3(x - C)^2 + D$ and $A > 0$,

there is a local maximum at (C, D) , then a local minimum and then a stationary point of inflection at (B, D)

This can also be seen by graphing g .



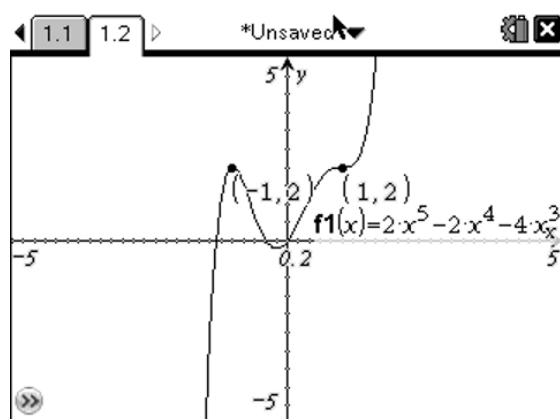
ii. $A = 2$

Method 1

From the graph of g ,

Any 2 correct **1A** All correct **2A**

$B = 1$, $C = -1$ and $D = 2$



Note that this method does not always give exact values.

OR Method 2

$$A = 2$$

Solve the derivative of g equal to zero.

$$B = 1, C = -1$$

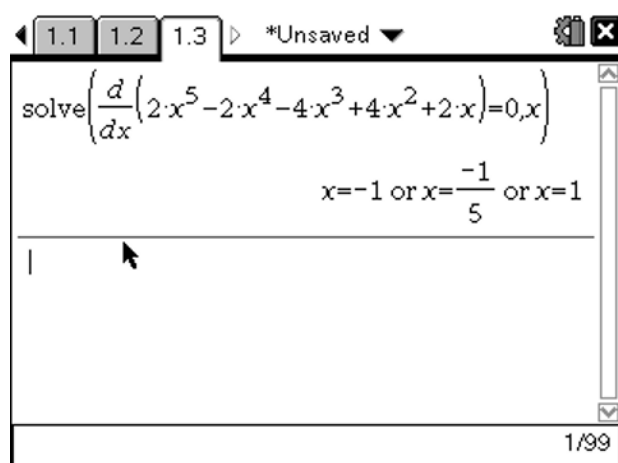
$$g(x) = 2(x-1)^3(x+1)^2 + D$$

The graph of g passes through $(0, 0)$

$$-2 + D = 0$$

$$D = 2$$

Any 2 correct **1A** All correct **2A**

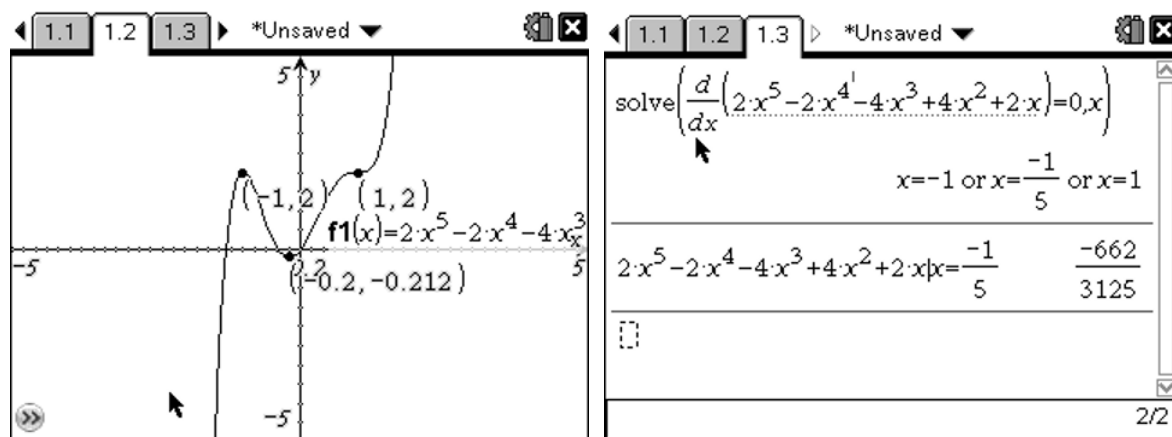


iii. The exact value for the y -coordinate is not given on the graph screen.

$$g(-0.2) = -\frac{662}{3125}$$

$$\left(-\frac{1}{5}, -\frac{662}{3125}\right)$$

1A



iv.

$$(-\infty, -1] \cup [-\frac{1}{5}, \infty)$$

2A

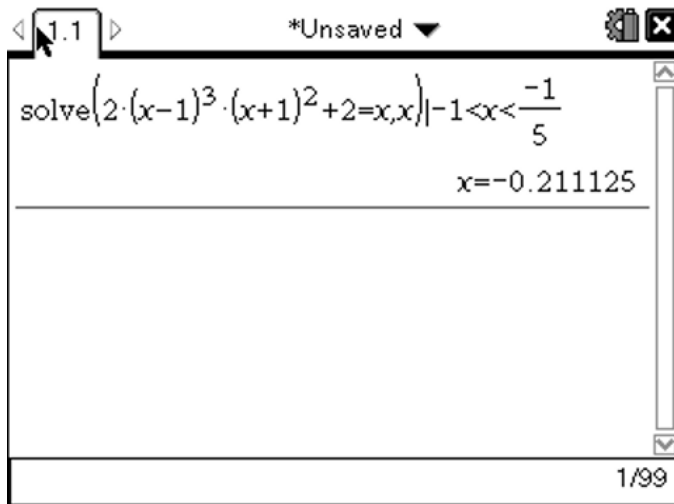
v. There is no need to find the equation of the inverse.

Solve $g_1^{-1} = x$

1M

$$x = -0.211$$

$$(-0.211, -0.211)$$

1A**END OF SOLUTIONS**