

Year 2011
VCE
Mathematical Methods
CAS
Solutions
Trial Examination 2



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SECTION 1

ANSWERS

1	A	B	C	D	E
2	A	B	C	D	E
3	A	B	C	D	E
4	A	B	C	D	E
5	A	B	C	D	E
6	A	B	C	D	E
7	A	B	C	D	E
8	A	B	C	D	E
9	A	B	C	D	E
10	A	B	C	D	E
11	A	B	C	D	E
12	A	B	C	D	E
13	A	B	C	D	E
14	A	B	C	D	E
15	A	B	C	D	E
16	A	B	C	D	E
17	A	B	C	D	E
18	A	B	C	D	E
19	A	B	C	D	E
20	A	B	C	D	E
21	A	B	C	D	E
22	A	B	C	D	E

SECTION 1

Question 1

Answer A

$$|x - 2k| < k$$

$$-k < x - 2k < k$$

$$k < x < 3k \quad \text{or} \quad (k, 3k)$$

Question 2

Answer A

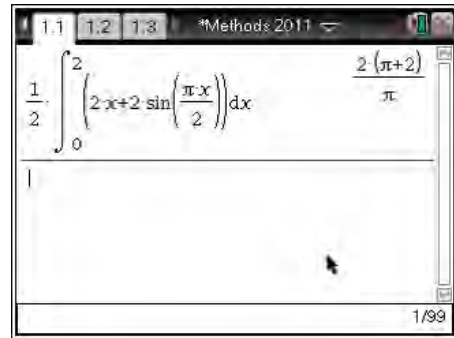
average value $\bar{y} = \frac{1}{b-a} \int_a^b f(x) dx$

$$\bar{y} = \frac{1}{2-0} \int_0^2 \left(2x + 2 \sin\left(\frac{\pi x}{2}\right) \right) dx$$

$$\bar{y} = \frac{1}{2} \left[x^2 - \frac{4}{\pi} \cos\left(\frac{\pi x}{2}\right) \right]_0^2$$

$$\bar{y} = \frac{1}{2} \left[\left(4 - \frac{4}{\pi} \cos(\pi) \right) - \left(0 - \frac{4}{\pi} \cos(0) \right) \right] = \frac{1}{2} \left[4 + \frac{8}{\pi} \right]$$

$$\bar{y} = 2 + \frac{4}{\pi} = \frac{2(\pi + 2)}{\pi}$$



Question 3

Answer B

$$y = \frac{ax+b}{x+c} = a + \frac{b-ac}{x+c}$$

so $y = a$ is a horizontal asymptote and $x = -c$ is a vertical asymptote.

Question 4

Answer D

$$f(x) = x^2, \quad g(x) = \frac{1}{x} \quad \text{and} \quad h(x) = \cos(x)$$

$$\frac{d}{dx}(\tan(x)) = \frac{1}{\cos^2(x)}$$

$$\text{now } g(f(h(x))) = g(f(\cos(x))) = g(\cos^2(x)) = \frac{1}{\cos^2(x)}$$

$$\text{so } \frac{d}{dx}(\tan(x)) = \frac{1}{\cos^2(x)} = g(f(h(x)))$$

Question 5

Answer B

$$f(x) = \sqrt{x-a} + \sqrt{b-x}$$

the domain requires $x-a \geq 0$ and $b-x \geq 0$

that is $x \geq a$ and $x \leq b$ since $b > a > 0$

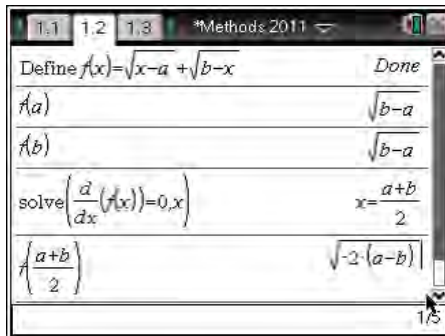
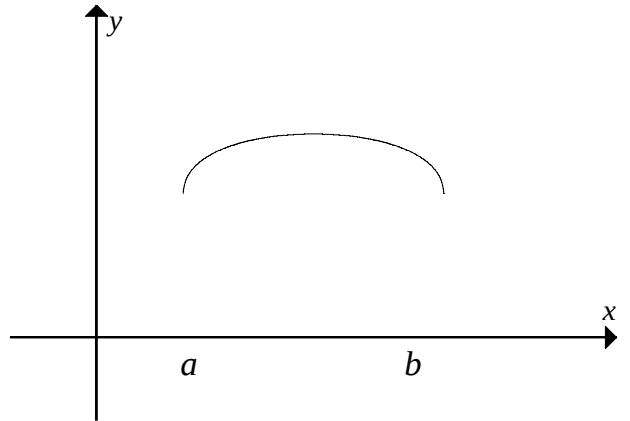
the domain is $[a, b]$

$$\text{now } f(a) = f(b) = \sqrt{b-a}$$

$$f'\left(\frac{a+b}{2}\right) = 0 \Rightarrow \text{turning point at } x = \frac{a+b}{2}$$

$$\text{and } f\left(\frac{a+b}{2}\right) = \sqrt{2(b-a)}$$

the range is $[\sqrt{b-a}, \sqrt{2(b-a)}]$



Question 6

Answer E

$$\Delta = \begin{vmatrix} p & 3 & 1 \\ 2 & -1 & 2p \\ 1 & 4 & p \end{vmatrix} = -9(p^2 - 1)$$

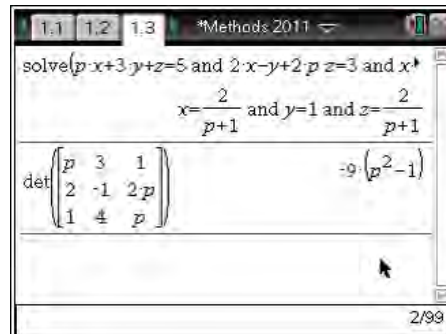
solving using CAS gives

$$x = \frac{2}{p+1} \quad y = 1 \quad \text{and} \quad z = \frac{2}{p+1}$$

Since $\Delta = 0 \Rightarrow p = \pm 1$, there is no unique solution when $p^2 = 1$,

there is a unique solution when $p^2 \neq 1$. When $p = -1$ there is no solution and when $p = 1$ there is an infinite number of solutions.

Only option E. is correct.



Question 7

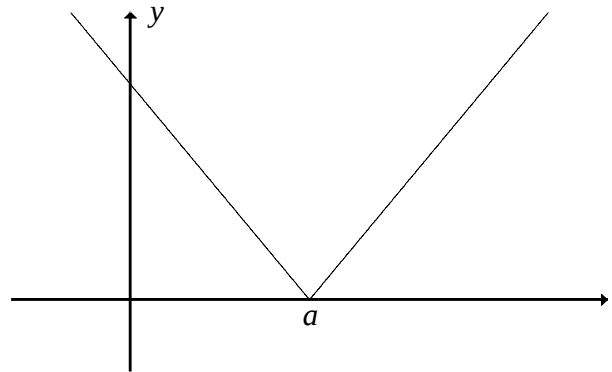
Answer A

When the point $(2, -3)$ is reflected in the x -axis, it becomes, the point $(2, 3)$, when it is translated one unit, to the left parallel to the x -axis, or away from the y -axis, it becomes $(1, 3)$, finally it is translated one unit up parallel to the y -axis or away from the x -axis, it becomes $(1, 4)$ under $y = 1 - f(x + 1)$.

Question 8

Answer C

The function is continuous at $x = a$,
all other options are true.



Question 9

Answer B

$$z = 3$$

$$x + y = 5 \quad \text{rewrite the equations as}$$

$$y - x = -1$$

$$x + y = 5$$

$$x - y = 1 \quad \text{in matrix form these become}$$

$$z = 3$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 3 \end{bmatrix}$$

Question 10

Answer E

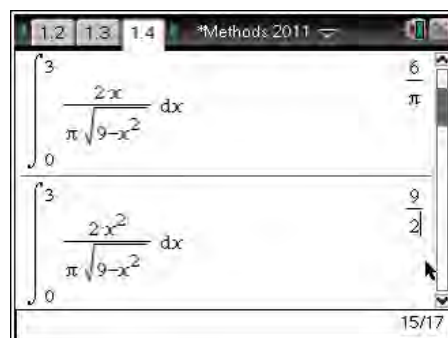
Since it is a probability density function

$$\int_0^3 \frac{2}{\pi\sqrt{9-x^2}} dx = 1$$

$$E(X) = \int_0^3 \frac{2x}{\pi\sqrt{9-x^2}} dx = \frac{6}{\pi}$$

$$E(X^2) = \int_0^3 \frac{2x^2}{\pi\sqrt{9-x^2}} dx = \frac{9}{2} \quad \text{by CAS}$$

$$\text{var}(X) = E(X^2) - (E(X))^2 = \frac{9}{2} - \left(\frac{6}{\pi}\right)^2 \approx 0.85$$



Question 11**Answer D**

$$\frac{dy}{dx} = 4e^{-\frac{x}{2}} \Rightarrow y = \int 4e^{-\frac{x}{2}} dx$$

$$y = -8e^{-\frac{x}{2}} + c \quad \text{now when } x=0 \quad y=0$$

$$0 = -8 + c \Rightarrow c = 8$$

$$y = 8\left(1 - e^{-\frac{x}{2}}\right)$$

Question 12**Answer B**

$$\text{Let } f(x) = e^{-x} \quad \text{Now } \frac{1}{e^{0.99}} = e^{-0.99} = e^{-(1-0.01)}$$

$$\text{with } x=1 \quad \text{and } h=-0.01,$$

$$\text{using } f(x+h) \approx f(x) + hf'(x)$$

$$\frac{1}{e^{0.99}} = f(1) - 0.01f'(1)$$

Question 13**Answer C**

$$y = 8\cos\left(\frac{x}{2}\right) \Rightarrow \frac{dy}{dx} = -4\sin\left(\frac{x}{2}\right)$$

$$\text{gradient of the tangent } m_T = \left. \frac{dy}{dx} \right|_{x=\frac{2\pi}{3}} = -4\sin\left(\frac{\pi}{3}\right) = -2\sqrt{3}$$

$$\text{gradient of the normal } m_N = \frac{1}{2\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{6}$$

The screenshot shows a CAS calculator interface. The top bar indicates '1.3 1.4 1.5' and '*Methods 2011'. The main display shows the derivative of $8 \cdot \cos\left(\frac{x}{2}\right)$ as $-4 \sin\left(\frac{x}{2}\right)$. Below this, it evaluates the derivative at $x = \frac{2\pi}{3}$, resulting in $-2\sqrt{3}$. The next line shows the negative reciprocal of $-2\sqrt{3}$, which is $\frac{\sqrt{3}}{6}$. The bottom right corner shows '3/99'.

Question 14**Answer A**

$$f(x) = \frac{x^2}{g(x)} \quad \text{using the quotient rule}$$

$$f'(x) = \frac{2xg(x) - x^2g'(x)}{[g(x)]^2}$$

$$f'(3) = \frac{6g(3) - 9g'(3)}{[g(3)]^2} \quad \text{now } g(3) = 2 \quad \text{and } g'(3) = 1$$

$$f'(3) = \frac{6 \times 2 - 9 \times 1}{2^2} = \frac{3}{4}$$

Question 15

Answer A

$$\int_2^0 (2x - f(x)) dx$$

$$= \left[x^2 \right]_2^0 - \int_2^0 f(x) dx = (0 - 4) + \int_0^2 f(x) dx = -4 + 2 = -2$$

Question 16

Answer B

$$\Pr(-a < Z < -b)$$

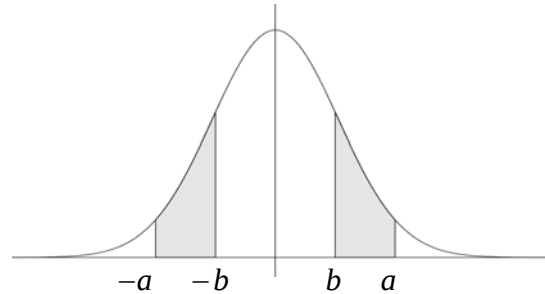
$$= \Pr(b < Z < a)$$

$$= \Pr(Z < a) - \Pr(Z < b)$$

$$= (1 - \Pr(Z > a)) - (1 - \Pr(Z > b))$$

$$= (1 - A) - (1 - B)$$

$$= B - A$$



Question 17

Answer D

The shaded area, with the x-axis is

$$A = \int_a^b (y_2 - y_1) dx \quad \text{with } a=0 \quad b=1 \quad y_2=2 \quad \text{and } y_1=x^2+1$$

$$A = \int_0^1 (1 - x^2) dx$$

however this is none of the alternatives,
the area with the y-axis, is

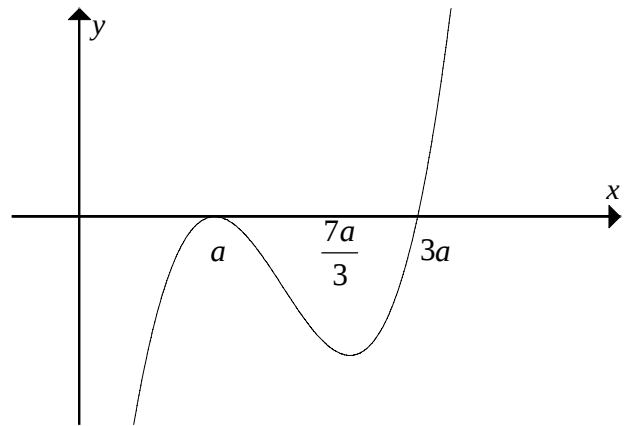
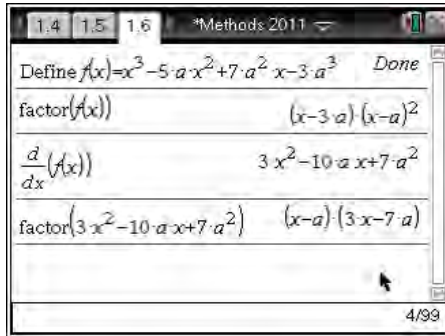
$$A_y = \int_c^d x dy \quad \text{with } c=1 \quad \text{and } d=2 \quad y=x^2+1$$

$$\Rightarrow x^2 = y - 1 \quad \text{and } x = \sqrt{y - 1} \quad \text{since } x > 0$$

$$A = \int_1^2 \sqrt{y - 1} dy = \int_1^2 \sqrt{x - 1} dx \quad \text{using dummy variable property.}$$

Question 18

Answer E



$$f(x) = x^3 - 5ax^2 + 7a^2x - 3a^3 = (x-a)^2(x-3a)$$

$$f'(x) = 3x^2 - 10ax + 7a^2 = (x-a)(3x-7a)$$

there are turning points at $x=a$ and $x=\frac{7a}{3}$,

for the function to be one-one, the only correct option is the restricted interval $\left(\frac{7a}{3}, \infty\right)$

Question 19

Answer D

Since A and B are independent events, $\Pr(A \cap B) = \Pr(A)\Pr(B) = ab$

	A	A'	
B	ab	$b-ab$	b
B'	$a-ab$	$1-a-b+ab$	$1-b$
	a	$1-a$	

$$\begin{aligned} \Pr(A' \cup B') &= \Pr(A') + \Pr(B') - \Pr(A' \cap B') \\ &= (1-a) + (1-b) - (1-a-b+ab) \\ &= 1-ab \end{aligned}$$

Question 20

Answer C

Three right rectangles, each of width $h = \frac{\pi}{6}$

x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$y = \sin(x)$	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1

The shaded area of the three rectangles is $A = \frac{\pi}{6} \left(\frac{1}{2} + \frac{\sqrt{3}}{2} + 1 \right) = \frac{\pi}{6} \left(\frac{3 + \sqrt{3}}{2} \right) = \frac{\pi(3 + \sqrt{3})}{12}$

Question 21

Answer C

$$X \sim \text{Bi}(n, p)$$

$$\Pr(\text{more than two}) = \Pr(X > 2)$$

$$= 1 - [\Pr(X = 0) + \Pr(X = 1) + \Pr(X = 2)]$$

$$= 1 - \left[q^n + npq^{n-1} + \frac{n(n-1)}{2} p^2 q^{n-2} \right]$$

$$= 1 - (0.7^{10} + 10 \times 0.7^9 \times 0.3 + 45 \times 0.7^8 \times 0.3^2)$$

$$\Rightarrow n = 10, \quad q = 0.7 \quad \text{and} \quad p = 0.3$$

Question 22

Answer E

One solution when $n = 0$ is $x = -\frac{\pi}{6}$ so that

$$2x = -\frac{\pi}{3} \quad \text{and} \quad \tan(2x) = -\sqrt{3}$$

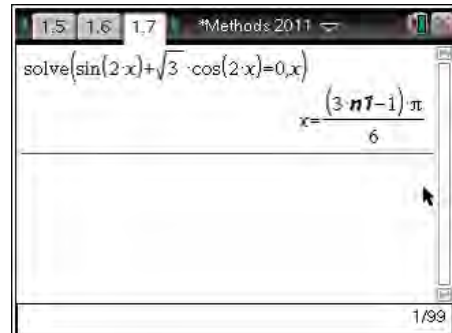
$$\frac{\sin(2x)}{\cos(2x)} = -\sqrt{3} \quad \text{or} \quad \sin(2x) = -\sqrt{3} \cos(2x)$$

$$\text{or} \quad \sin(2x) + \sqrt{3} \cos(2x) = 0$$

so the general solution of

$$\sin(2x) + \sqrt{3} \cos(2x) = 0 \quad \text{is} \quad x = \frac{n\pi}{2} - \frac{\pi}{6} = \frac{(3n-1)\pi}{6} \quad \text{where} \quad n \in \mathbb{Z}$$

$$a = 1 \quad \text{and} \quad b = \sqrt{3}$$



END OF SECTION 1 SUGGESTED ANSWERS

SECTION 2

Question 1

a.i. 2 marks, for correct transformations

- reflection in the x -axis
- dilation by a factor of 4 parallel to the y -axis (or away from the x -axis)
- translation by 3 units to the right parallel to the x -axis, (or away from the y -axis)
- translation by 4 units up and parallel to the y -axis, (or away from the x -axis)

ii. $m: y = 4 - \frac{4}{(x-3)^2}$ interchanging x and y

$$m^{-1}: x = 4 - \frac{4}{(y-3)^2}$$

M1

$$\frac{4}{(y-3)^2} = 4 - x$$

$$(y-3)^2 = \frac{4}{4-x}$$

$$y-3 = \frac{\pm 2}{\sqrt{4-x}}$$

Since the range of m^{-1} is $(-\infty, 3)$, the same as the domain of m , we

must take the negative, so $y = 3 - \frac{2}{\sqrt{4-x}}$ A1

Now the domain of m^{-1} is the same as the range of m , that is $(-\infty, 4)$.

To state the function, we need to state both the domain and the rule.

$$m^{-1}: (-\infty, 4) \rightarrow \mathbb{R}, \quad m^{-1}(x) = 3 - \frac{2}{\sqrt{4-x}} \quad \text{A1}$$

iii. the graph of m^{-1} crosses the x -axis at $\left(\frac{32}{9}, 0\right)$, since $m(0) = 4 - \frac{4}{9} = \frac{32}{9}$

and crosses the y -axis at $(0, 2)$, since $m^{-1}(0) = 3 - \frac{2}{\sqrt{4}} = 2$ A1

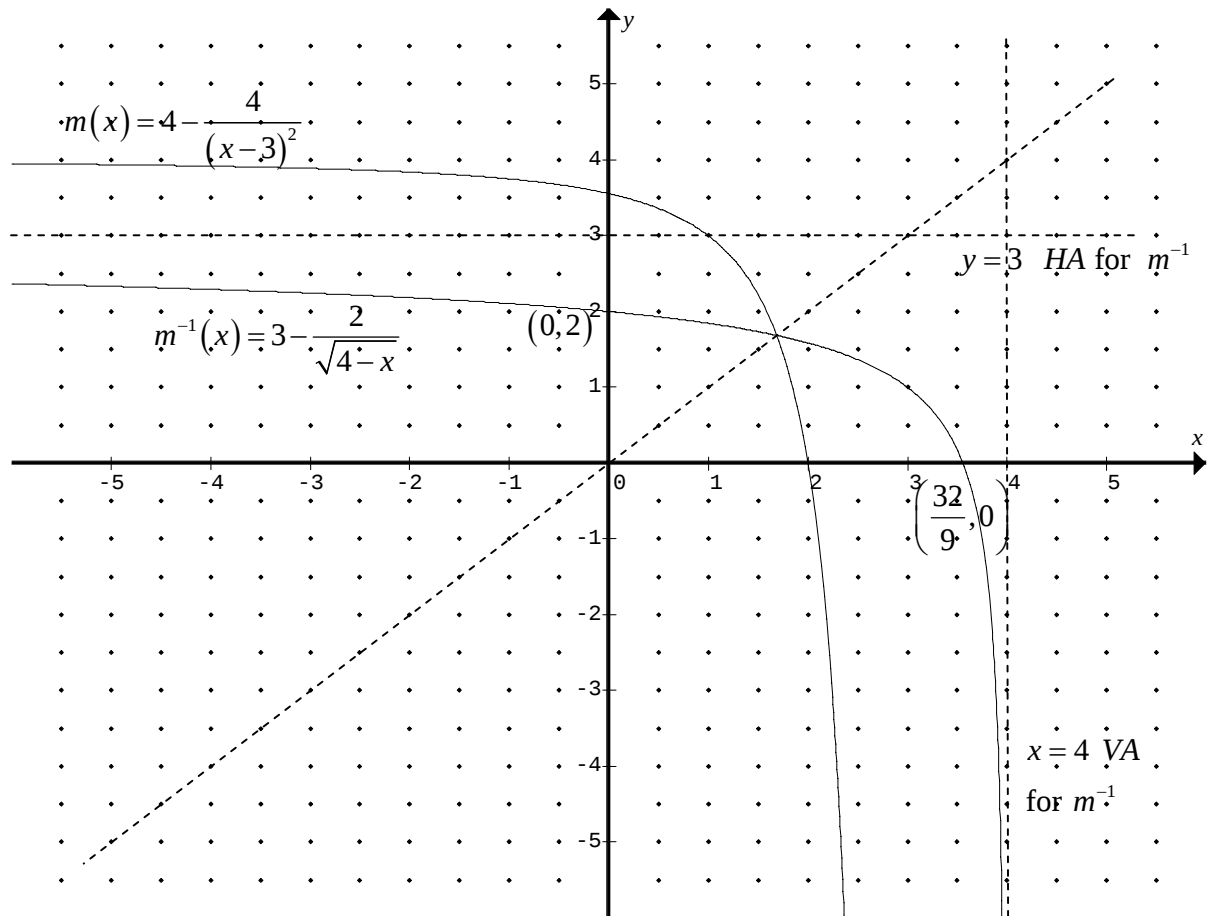
for the graph of $m(x)$ $x = 3$ is a vertical asymptote and $y = 4$ is a horizontal

asymptote, so for the graph of $m^{-1}(x)$ $y = 3$ is a horizontal asymptote and

$x = 4$ is a vertical asymptote. A1

correct graph, shape, reflection in the line $y = x$, and the intersection of m and

m^{-1} must be on the line $y = x$. A1



b.i. since $f(0) = \frac{32}{9} = 3\frac{5}{9}$, $f(2) = 0$ and $g(2) = g(10) = 0$ and $g(6) = \frac{32}{9}$

amplitude $\Rightarrow b = \frac{32}{9}$, the phase shift is 2 units, to the right, so that $c = 2$ A1

the sine squared wave is half a cycle $\Rightarrow T = \frac{\pi}{n} = 8 \Rightarrow n = \frac{\pi}{8}$ A1

the graph of h is the reflection in the line, $x = 6$ $h(10) = 0$ and $h(12) = \frac{32}{9}$

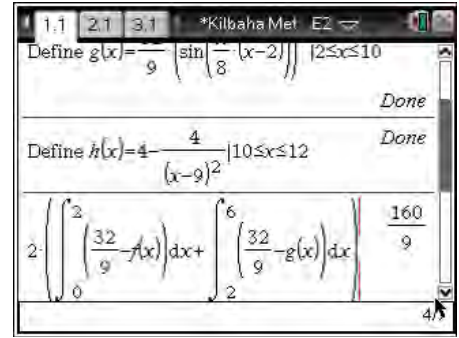
$h(x) = 4 - \frac{4}{(x-9)^2}$ so that $p = 4$, $r = -4$ and $s = 9$ A1

ii. using symmetry, in terms of two definite integrals, the area between the curves

$$A = 2 \left[\int_0^2 \left(\frac{32}{9} - \left(4 - \frac{4}{(x-3)^2} \right) \right) dx + \int_2^6 \left(\frac{32}{9} - b \sin^2(n(x-c)) \right) dx \right] \quad \text{A2}$$

or alternatively, other equivalent answers are possible.

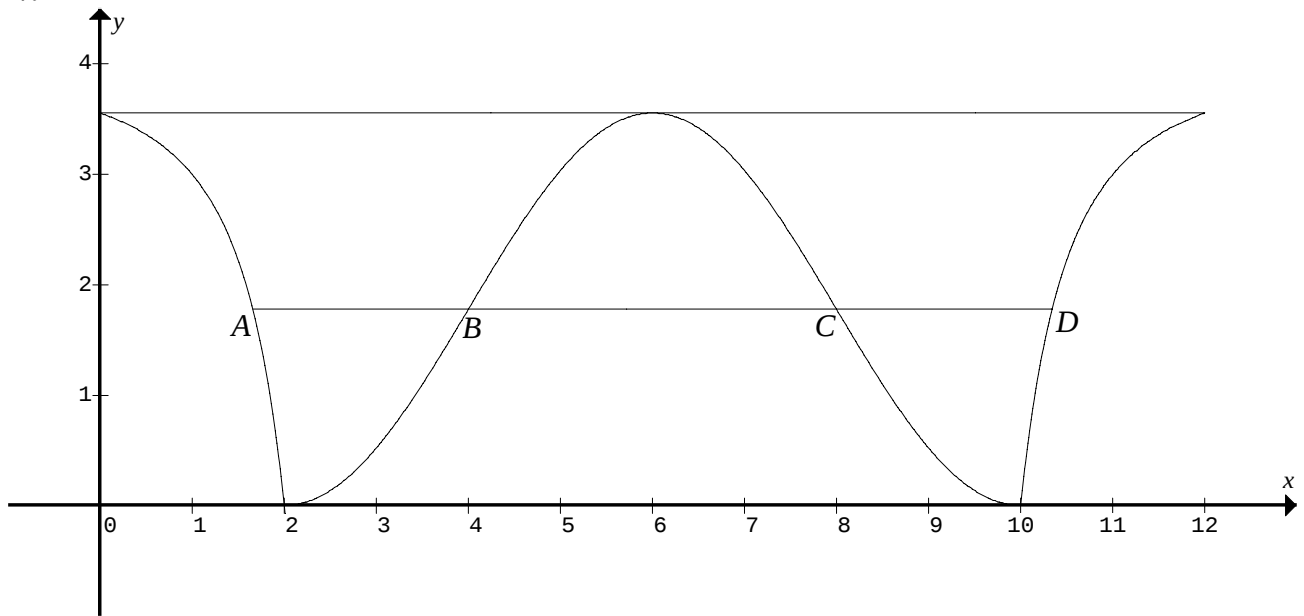
$$A = 2 \left[\int_0^2 \left(\frac{4}{(x-3)^2} - \frac{4}{9} \right) dx + \int_2^6 \left(\frac{32}{9} \cos^2 \left(\frac{\pi}{8}(x-2) \right) \right) dx \right]$$



iii. $A = \frac{160}{9} = 17\frac{7}{9}$ metres² using CAS

A1

iv.



the line through $ABCD$ is half the maximum value, that is

$$y = \frac{16}{9}, \text{ solving } \frac{16}{9} = 4 - \frac{4}{(x-3)^2} \text{ with } 0 < x < 2$$

gives $x = 1.65836$ solving

$$\frac{16}{9} = \frac{32}{9} \sin^2 \left(\frac{\pi}{8}(x-2) \right) \text{ with } 2 < x < 10$$

gives $x = 4$ and $x = 8$

$$A \left(1.6584, \frac{16}{9} \right) \quad B \left(4, \frac{16}{9} \right) \quad C \left(8, \frac{16}{9} \right)$$

the length of $AD = 2(4 - 1.65836) + 4$ or alternatively $= 2(6 - 1.65836)$

length $ABCD$ is 8.683 metres

The screenshot shows a CAS interface with the following commands and results:

- $\text{nSolve} \left(\frac{16}{9} = f(x), x \right)$ for $0 < x < 2$. Result: 1.65836
- $\text{nSolve} \left(\frac{16}{9} = g(x), x \right)$ for $2 < x < 10$. Result: 4.00000
- $\text{nSolve} \left(\frac{16}{9} = g(x), x \right)$ for $6 < x < 10$. Result: 8.00000
- Calculation of the length of AD : $2(4 - 1.6583592135001) + 4$. Result: 8.68328

A1

Question 2

$$\begin{array}{c} C \quad S \\ \text{a.i.} \quad C \begin{bmatrix} 0.65 & 0.55 \\ 0.35 & 0.45 \end{bmatrix} \quad \text{Let } A = \begin{bmatrix} 0.65 & 0.55 \\ 0.35 & 0.35 \end{bmatrix} \\ C \rightarrow C \ 0.65 \quad C \rightarrow S \ 0.35 \quad S \rightarrow C \ 0.55 \quad S \rightarrow S \ 0.45 \end{array}$$

$$\Pr(\text{Coles once}) = \Pr(\text{CSSS}) = 0.35 \times 0.45^2 = 0.0709 \quad \text{A1}$$

$$\begin{aligned} \text{ii.} \quad \Pr(\text{Coles twice}) &= \Pr(\text{CCSS}) + \Pr(\text{CSCS}) + \Pr(\text{CSSC}) & \text{M1} \\ &= 0.65 \times 0.35 \times 0.45 + 0.35 \times 0.55 \times 0.35 + 0.35 \times 0.45 \times 0.55 \\ &= 0.2564 & \text{A1} \end{aligned}$$

$$\begin{aligned} \text{iii.} \quad \Pr(\text{Coles 3 times}) &= \Pr(\text{CCCS}) + \Pr(\text{CCSC}) + \Pr(\text{CSCC}) \\ &= 0.65^2 \times 0.35 + 0.65 \times 0.35 \times 0.55 + 0.35 \times 0.55 \times 0.65 \\ &= 0.3918 & \text{A1} \end{aligned}$$

$$\begin{aligned} \Pr(\text{Coles 4 times}) &= \Pr(\text{CCCC}) \\ &= 0.65^3 \\ &= 0.2746 & \text{A1} \end{aligned}$$

Number of times at Coles	1	2	3	4
Probability	$\frac{567}{8000} = 0.0709$	$\frac{2051}{8000} = 0.2564$	$\frac{637}{1600} = 0.3918$	$\frac{2197}{8000} = 0.2746$

Expected number of times at Coles

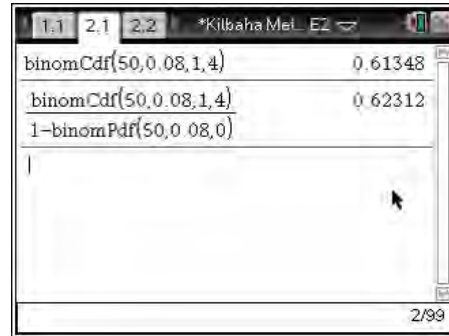
$$\begin{aligned} E(C) &= 1 \times \frac{567}{8000} + 2 \times \frac{2051}{8000} + 3 \times \frac{637}{1600} + 4 \times \frac{2197}{8000} \\ E(C) &= \frac{5753}{2000} & \text{A1} \end{aligned}$$

$$\text{b.} \quad \text{Now as } n \rightarrow \infty \quad A^n \rightarrow \begin{bmatrix} 0.6\dot{1} & 0.6\dot{1} \\ 0.3\dot{8} & 0.3\dot{8} \end{bmatrix} \quad \text{or} \quad \frac{0.55}{0.55 + 0.35} = 0.6\dot{1} = \frac{11}{18}$$

so the steady state probability that they go to Coles is $\frac{11}{18}$ A1

c.i. $X \sim \text{Bi}(n = 50, p = 0.08)$

$$\begin{aligned} \Pr(X < 5 | X \geq 1) &= \frac{\Pr(1 \leq X \leq 4)}{\Pr(X \geq 1)} \\ &= \frac{0.61348}{1 - 0.01547} \\ &= 0.6231 \end{aligned}$$

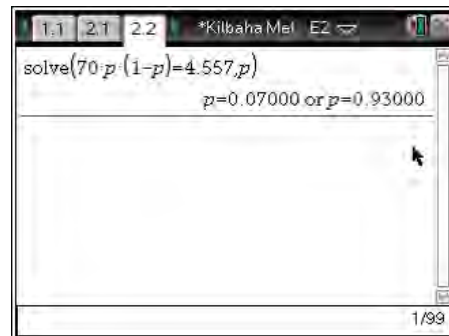


M1

A1

ii. $X \sim \text{Bi}(n = 70, p = ?)$

$$\begin{aligned} \text{var}(X) &= npq = 70p(1-p) = 4.557 \\ \text{solving for } p &\text{ since} \\ 0 < p < 1 &\text{ gives } p = 0.07 \text{ or } 0.93 \\ \text{Since } E(X) < 5 & \\ p &= 0.07 \end{aligned}$$

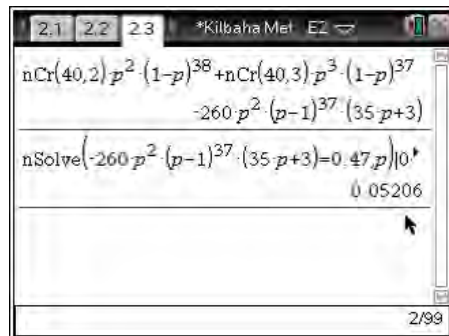


M1

A1

iii. $X \sim \text{Bi}(n = 40, p = ?)$

$$\begin{aligned} \Pr(X = 2) + \Pr(X = 3) &= 0.47 \\ \binom{40}{2} p^2 (1-p)^{38} + \binom{40}{3} p^3 (1-p)^{37} &= 0.47 \\ \Rightarrow -260 p^2 (p-1)^{37} (35p+3) &= 0.47 \\ \text{solving numerically using CAS,} & \\ \text{with } 0 < p < 1 & \\ \Rightarrow p &= 0.0521 \end{aligned}$$



M1

A1

d. X is the time in minutes spent shopping, $X \sim N(\mu = ?, \sigma^2 = ?)$

$$(1) \quad \Pr(X > 50) = 0.2$$

M1

$$(2) \quad \Pr(X < 36) = 0.37$$

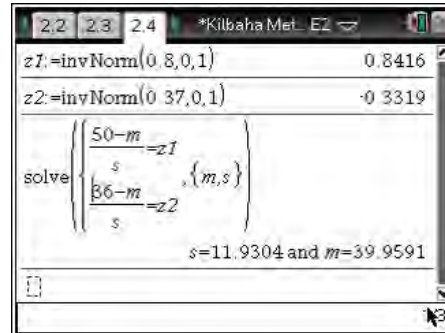
$$(1) \Rightarrow \frac{50 - \mu}{\sigma} = 0.842$$

$$(2) \Rightarrow \frac{36 - \mu}{\sigma} = -0.332$$

$$(1) \quad 50 - \mu = 0.842 \sigma$$

$$(2) \quad 36 - \mu = -0.332 \sigma$$

M1



now subtract equations (1) – (2)

$$14 = 1.174 \sigma$$

$$\sigma = 12 \text{ minutes}$$

A1

substituting gives

$$\mu = 40 \text{ minutes}$$

A1

Question 3

a. arc length $l = r\theta$ but $l = 2\pi r$ circumference of base circle of the cone

$$2\pi r = 12\theta \Rightarrow (1) \quad r = \frac{6\theta}{\pi}$$

A1

$$\text{Pythagoras} \Rightarrow (2) \quad h^2 + r^2 = 12^2 = 144$$

$$(2) \quad h^2 = 144 - r^2 = 144 - \left(\frac{6\theta}{\pi}\right)^2 = 144 - \frac{36\theta^2}{\pi^2}$$

M1

$$h^2 = \frac{36}{\pi^2}(4\pi^2 - \theta^2) \quad \text{so that} \quad h = \frac{6}{\pi}\sqrt{4\pi^2 - \theta^2} \quad \text{since } h > 0$$

$$\text{Now volume of cone } V = \frac{1}{3}\pi r^2 h$$

$$V = \frac{\pi}{3} \left(\frac{36\theta^2}{\pi^2} \right) \frac{6}{\pi} \sqrt{4\pi^2 - \theta^2}$$

M1

$$V = V(\theta) = \frac{72\theta^2}{\pi^2} \sqrt{4\pi^2 - \theta^2} \quad \text{shown}$$

b. $\frac{dV}{d\theta} = \frac{72\theta(8\pi^2 - 3\theta^2)}{\pi^2\sqrt{4\pi^2 - \theta^2}}$ by CAS A1

for max/min $\frac{dV}{d\theta} = 0$ solving, since $\theta > 0 \Rightarrow \theta = \frac{2\pi\sqrt{6}}{3}$ by CAS A1

$V_{\max} = V\left(\frac{2\pi\sqrt{6}}{3}\right) = 128\pi\sqrt{3} \text{ cm}^3$ by CAS A1

Define $v(\theta) = \frac{72\theta^2\sqrt{4\pi^2 - \theta^2}}{\pi^2}$ Done

factor(comDenom($\frac{d}{d\theta}(v(\theta))$))

$\frac{-72\theta(3\theta^2 - 8\pi^2)}{\sqrt{4\pi^2 - \theta^2}\pi^2}$

2/99

solve($\frac{d}{d\theta}(v(\theta)) = 0, \theta > 0$) $\theta = \frac{2\pi\sqrt{6}}{3}$

$v\left(\frac{2\pi\sqrt{6}}{3}\right)$ $128\pi\sqrt{3}$

nSolve($v(\theta) = 64\pi\sqrt{3}, \theta$) 2.93074

3/99

c. Numerically solving $V(\theta) = \frac{1}{2}V_{\max} \Rightarrow 64\pi\sqrt{3} = \frac{72\theta^2}{\pi^2}\sqrt{4\pi^2 - \theta^2}$ A1

for θ since $0 < \theta < \pi \Rightarrow \theta = 2.93$ A1

d. r and h are now the radius and height respectively of the mouse in the cone,
given that $\frac{dV}{dt} = -0.5 \text{ cm}^3/\text{sec}$ find $\frac{dh}{dt}$ when $h = 4 \text{ cm}$

$V = \frac{1}{3}\pi r^2 h$ and $\tan(\alpha) = \frac{r}{h} = \sqrt{2} \Rightarrow r = \sqrt{2}h$ substituting A1

$V = \frac{2\pi h^3}{3} \Rightarrow \frac{dV}{dh} = 2\pi h^2$ M1

$\frac{dh}{dt} = \frac{dh}{dV} \frac{dV}{dt} = \frac{-0.5}{2\pi h^2}$

when $h = 4$ $\frac{dh}{dt} = -\frac{1}{64\pi} \text{ cm/sec}$ or falling at a rate of $\frac{1}{64\pi} \text{ cm/sec}$ A1

Question 4

a. $y = f(x) = \frac{1}{x} \quad R\left(r, \frac{1}{r}\right)$

$$\frac{dy}{dx} = f'(x) = -\frac{1}{x^2} \quad f'(r) = -\frac{1}{r^2} \quad \text{A1}$$

the equation of the tangent at R is

$$y - \frac{1}{r} = -\frac{1}{r^2}(x - r) = -\frac{x}{r^2} + \frac{1}{r}$$

$$y = -\frac{x}{r^2} + \frac{2}{r}$$

$$m = -\frac{1}{r^2} \quad \text{and} \quad c = \frac{2}{r} \quad \text{A1}$$

b. $P\left(p, \frac{1}{p}\right), Q\left(q, \frac{1}{q}\right) \quad R\left(r, \frac{1}{r}\right)$

Since M is the midpoint of PQ $M\left(\frac{1}{2}(p+q), \frac{1}{2}\left(\frac{1}{p} + \frac{1}{q}\right)\right)$ A1

$$\text{gradient } OM = \frac{\frac{1}{2}\left(\frac{1}{p} + \frac{1}{q}\right)}{\frac{1}{2}(p+q)} = \frac{\frac{p+q}{pq}}{p+q} = \frac{1}{pq} \quad \text{A1}$$

$$\text{gradient } OR = \frac{\frac{1}{r}}{r} = \frac{1}{r^2} = \text{gradient } OM = \frac{1}{pq}$$

so that $r^2 = pq$ shown A1

c. $\text{gradient } PQ = \frac{\frac{1}{q} - \frac{1}{p}}{q - p} = \frac{\frac{p-q}{pq}}{q-p} = -\frac{1}{pq}$ A1

$$f'(r) = -\frac{1}{r^2} = -\frac{1}{pq} \quad \text{from a. and b.} \quad \text{A1}$$

so the tangent to the curve at R, is parallel to the line segment joining P and Q.

d. $A = \int_p^q \frac{1}{x} dx$

$$A = [\log_e |x|]_p^q = \log_e(q) - \log_e(p) \quad \text{since } q > p > 0$$

$$A = \log_e\left(\frac{q}{p}\right) \quad \text{A1}$$

e. $\text{Area} = \int_p^r \frac{1}{x} dx$

$$\text{Area} = [\log_e |x|]_p^r = \log_e(r) - \log_e(p) \quad \text{since } q > r > p > 0 \quad \text{A1}$$

$$\text{Area} = \log_e\left(\frac{r}{p}\right) \quad \text{now from b. since } r = \sqrt{pq}$$

$$\text{Area} = \log_e\left(\frac{\sqrt{pq}}{p}\right) = \log_e\left(\frac{\sqrt{q}}{\sqrt{p}}\right) = \log_e\left(\frac{q}{p}\right)^{\frac{1}{2}} \quad \text{A1}$$

$$\text{Area} = \frac{1}{2} \log_e\left(\frac{q}{p}\right) = \frac{1}{2} A \quad \text{A1}$$

f. The line OP is $y = \frac{x}{p^2}$ for $0 \leq x \leq p$, the line OQ is $y = \frac{x}{q^2}$ for $0 \leq x \leq q$ A1

the area between the curves is

$$B = \int_0^p \left(\frac{x}{p^2} - \frac{x}{q^2}\right) dx + \int_p^q \left(\frac{1}{x} - \frac{x}{q^2}\right) dx$$

$$g(x) = \frac{x}{p^2} - \frac{x}{q^2} \quad \text{and} \quad h(x) = \frac{1}{x} - \frac{x}{q^2} \quad \text{A1}$$

g. $B = \left[\frac{x^2}{2p^2} - \frac{x^2}{2q^2} \right]_0^p + \left[\log_e |x| - \frac{x^2}{2q^2} \right]_p^q$

$$B = \frac{p^2}{2p^2} - \frac{p^2}{2q^2} + \left[\left(\log_e(q) - \frac{q^2}{2q^2} \right) - \left(\log_e(p) - \frac{p^2}{2q^2} \right) \right] \quad \text{M1}$$

$$B = \frac{1}{2} - \frac{p^2}{2q^2} + \log_e\left(\frac{q}{p}\right) - \frac{1}{2} + \frac{p^2}{2q^2}$$

$$B = \log_e\left(\frac{q}{p}\right) = A \quad \text{A1}$$

END OF SECTION 2 SUGGESTED ANSWERS