

Year 2011
VCE
Mathematical Methods
CAS
Solutions
Trial Examination 1



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Question 1

$$\Delta = \begin{vmatrix} -2 & k \\ k+1 & -6 \end{vmatrix}$$

$$\Delta = 12 - k(k+1) = -k^2 - k + 12 \quad \text{M1}$$

$$\Delta = -(k^2 + k - 12) = -(k+4)(k-3)$$

i. for a unique solution $\Delta \neq 0 \Rightarrow k \in \mathbb{R} \setminus \{-4, 3\}$ A1

ii. if $k = 3$ $(1) -2x + 3y = 5$
 $(2) 4x - 6y = -10$

These equations represent the same line and are consistent,
 so for infinitely many solutions $k = 3$ A1

iii. if $k = -4$ $(1) -2x - 4y = -2$
 $(2) -3x - 6y = -10$

These lines are parallel, they have the same gradient, but different y-intercepts,
 the equations are inconsistent, so for no solution $k = -4$ A1

Question 2

$f: y = \frac{1}{1-2x}$ swapping x and y M1

$$f^{-1} x = \frac{1}{1-2y} \Rightarrow 1-2y = \frac{1}{x} \Rightarrow 2y = 1 - \frac{1}{x}$$

$$y = f^{-1}(x) = \frac{1}{2} \left(1 - \frac{1}{x} \right) \quad \text{A1}$$

domain $f^{-1} = \text{range } f = \mathbb{R} \setminus \{0\}$ A1

domain $f = \text{range } f^{-1} = \mathbb{R} \setminus \{\frac{1}{2}\}$

Question 3

a. $y = x \sin(2x)$ using the product rule

$$\frac{dy}{dx} = \sin(2x) + 2x \cos(2x) \quad \text{A1}$$

b. Since $\frac{d}{dx}(x \sin(2x)) = \sin(2x) + 2x \cos(2x)$ it follows that

$$2 \int x \cos(2x) dx = x \sin(2x) - \int \sin(2x) dx$$

$$2 \int x \cos(2x) dx = x \sin(2x) + \frac{1}{2} \cos(2x) \quad \text{M1}$$

$$\int x \cos(2x) dx = \frac{1}{2} x \sin(2x) + \frac{1}{4} \cos(2x) + c \quad \text{A1}$$

c.i. $\int_0^b 2 \cos(2x) dx = 1$

$$[\sin(2x)]_0^b = \sin(2b) - \sin(0) = 1$$

$$\sin(2b) = 1$$

$$2b = \frac{\pi}{2} \quad \text{M1}$$

$$b = \frac{\pi}{4}$$

ii. $E(X) = \int_0^{\frac{\pi}{4}} 2x \cos(2x) dx$

$$E(X) = \left[x \sin(2x) + \frac{1}{2} \cos(2x) \right]_0^{\frac{\pi}{4}} \quad \text{M1}$$

$$E(X) = \left(\left(\frac{\pi}{4} \sin\left(\frac{\pi}{2}\right) + \frac{1}{2} \cos\left(\frac{\pi}{2}\right) \right) - \left(0 \times \sin(0) + \frac{1}{2} \cos(0) \right) \right)$$

$$E(X) = \frac{\pi}{4} - \frac{1}{2} = \frac{\pi - 2}{4} \quad \text{A1}$$

iii. $\int_0^m 2 \cos(2x) dx = \frac{1}{2} \quad \text{median } m \quad \text{A1}$

$$[\sin(2x)]_0^m = \sin(2m) - \sin(0) = \frac{1}{2}$$

$$\sin(2m) = \frac{1}{2}$$

$$2m = \frac{\pi}{6}$$

$$m = \frac{\pi}{12} \quad \text{A1}$$

Question 4

$$\text{Let } y = \log_e \left(\frac{3x^2 + 4}{4x^2 + 3} \right) = \log_e (3x^2 + 4) - \log_e (4x^2 + 3)$$

$$\frac{dy}{dx} = \frac{6x}{3x^2 + 4} - \frac{8x}{4x^2 + 3}$$

$$\frac{dy}{dx} = \frac{6x(4x^2 + 3) - 8x(3x^2 + 4)}{(3x^2 + 4)(4x^2 + 3)} \quad \text{M1}$$

$$\frac{dy}{dx} = \frac{-14x}{12x^4 + 25x^2 + 12}$$

$$b = -14 \quad \text{A1}$$

$$g(x) = 12x^4 + 25x^2 + 12 \quad \text{A1}$$

Question 5

$$s = \int_0^2 \frac{72}{(3t+2)^2} dt \quad \text{A1}$$

$$s = \left[\frac{-72}{3(3t+2)} \right]_0^2 = \left[\frac{-24}{3t+2} \right]_0^2 \quad \text{M1}$$

$$s = \frac{-24}{8} + \frac{24}{2} = -3 + 12$$

$$s = 9 \text{ metres} \quad \text{A1}$$

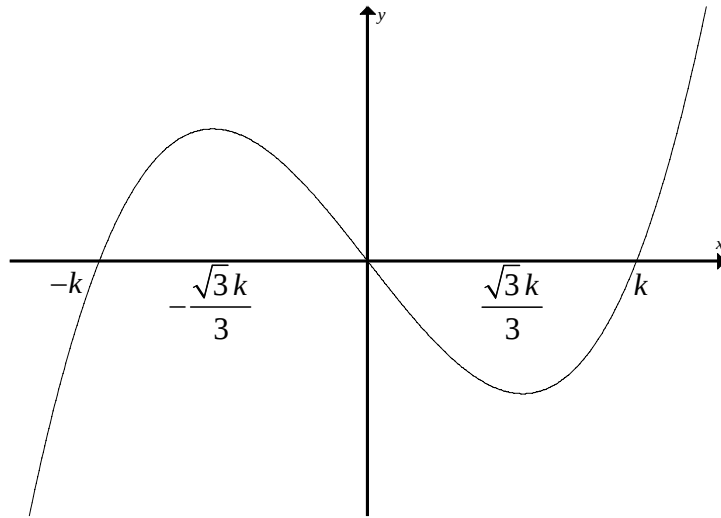
Question 6

a. $f(x) = x^3 - k^2x = x(x^2 - k^2) = x(x+k)(x-k)$

the graph crosses the x-axis at $(k, 0)$ and $(-k, 0)$

$$f'(x) = 3x^2 - k^2 = 0 \text{ for stationary points}$$

$$x^2 = \frac{k^2}{2} \Rightarrow x = \pm \frac{k}{\sqrt{2}} = \pm \frac{\sqrt{3}k}{3} \quad \text{M1}$$



$$f'(x) \geq 0 \Rightarrow \left(-\infty, -\frac{\sqrt{3}k}{3}\right] \cup \left[\frac{\sqrt{3}k}{3}, \infty\right) \quad \text{A1}$$

- b. The area $A = \int_0^k (x^3 - k^2x) dx$ is below the x -axis so is negative

$$A = -\int_0^k (x^3 - k^2x) dx = 64 \quad \text{M1}$$

$$-\left[\frac{1}{4}x^4 - \frac{k^2x^2}{2}\right]_0^k = \left(\frac{k^4}{2} - \frac{k^4}{4}\right) = \frac{k^4}{4} = 64$$

$$k^4 = 4 \times 64, \text{ since } k > 0$$

$$k = 4 \quad \text{A1}$$

Question 7

a. $f(\alpha) = g(\alpha) \Rightarrow \tan(\alpha) = \cos(\alpha)$

$$\frac{\sin(\alpha)}{\cos(\alpha)} = \cos(\alpha) \quad \text{M1}$$

$$\sin(\alpha) = \cos^2(\alpha) = 1 - \sin^2(\alpha)$$

$$\sin^2(\alpha) + \sin(\alpha) - 1 = 0 \text{ solving using quadratic formula} \quad \text{M1}$$

$$\sin(\alpha) = \frac{-1 \pm \sqrt{5}}{2} \text{ but } \sin(\alpha) > 0 \text{ since } \alpha \in \left(0, \frac{\pi}{2}\right) \quad \text{M1}$$

$$\sin(\alpha) = \frac{1}{2}(\sqrt{5} - 1) \text{ shown}$$

$$\mathbf{b.} \quad f'(x) = \frac{1}{\cos^2(x)} \quad g'(x) = -\sin(x)$$

$$f'(\alpha) = \frac{1}{\cos^2(\alpha)} \quad g'(\alpha) = -\sin(\alpha) \quad \text{A1}$$

$$\begin{aligned} \text{Now } f'(\alpha)g'(\alpha) &= \frac{1}{\cos^2(\alpha)} \times -\sin(\alpha) \\ &= -\frac{\sin(\alpha)}{\cos(\alpha)} \times \frac{1}{\cos(\alpha)} \quad \text{M1} \\ &= -\frac{f(\alpha)}{g(\alpha)} = -1 \quad \text{since } f(\alpha) = g(\alpha) \end{aligned}$$

Since the products of the gradients is -1 ,
the functions intersect at right angles. A1

Question 8

$$X \sim \text{Bi}(n=6, p="p")$$

$$\Pr(X=3) = \binom{6}{3} p^3 (1-p)^3$$

$$\Pr(X=4) = \binom{6}{4} p^4 (1-p)^2 \quad \text{M1}$$

$$\Pr(X=3) = \Pr(X=4)$$

$$\frac{6 \times 5 \times 4}{3 \times 2} p^3 (1-p)^3 = \frac{6 \times 5}{2} p^4 (1-p)^2$$

$$20p^3(1-p)^3 - 15p^4(1-p)^2 = 0 \quad \text{A1}$$

$$5p^3(1-p)^2[4(1-p)-3p] = 0$$

$$5p^3(1-p)^2(4-7p) = 0 \quad \text{since } 0 < p < 1$$

$$p = \frac{4}{7} \quad \text{A1}$$

Question 9

a. $\int_1^4 \frac{4}{\sqrt{x}} dx$

$$= 4 \int_1^4 x^{-\frac{1}{2}} dx = 8 \left[\sqrt{x} \right]_1^4 = 8(\sqrt{4} - \sqrt{1})$$

$$= 8$$

A1

b. i. $f: [-4, -1] \rightarrow \mathbb{R} \quad f(x) = \frac{-4}{\sqrt{-x}}$

both domain and rule are required

A1

ii. Total area $8 \times 2 + 4 \times 8$ area of rectangle and symmetry from a.
 $= 48 \text{ units}^2$

A1

Question 10

$$y' = -2 \cos \left(\frac{\pi}{3} \left(x' + \frac{3}{\pi} \right) \right) + 4$$

mapping $y = \cos(x)$ into $\frac{y' - 4}{-2} = \cos \left(\frac{\pi}{3} \left(x' + \frac{3}{\pi} \right) \right)$

$$y = \frac{y' - 4}{-2} \quad x = \frac{\pi}{3} \left(x' + \frac{3}{\pi} \right)$$

M1

$$y' = -2y + 4 \quad x' = \frac{3x}{\pi} - \frac{3}{\pi}$$

$$T \left(\begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} \frac{3}{\pi} & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -\frac{3}{\pi} \\ 4 \end{bmatrix}$$

$$a = \frac{3}{\pi}, \quad b = 0, \quad c = 0, \quad d = -2$$

A1

$$h = -\frac{3}{\pi} \quad k = 4$$

A1

END OF SUGGESTED SOLUTIONS