2011 Mathematical Methods (CAS) Trial Exam 2 Solutions

SECTION 1

1	2	3	4	5	6	7	8	9	10	11
C	E	A	C	D	B	D	A	B	A	E

12	13	14	15	16	17	18	19	20	21	22
D	E	D	B	A	C	B	A	C	C	D

Q1 $\tan\theta = \frac{x}{x-1}$, $\tan\phi = 1-\frac{1}{x} = \frac{x-1}{x}$, .: $\tan\phi = \frac{1}{\tan\theta}$

.: $\theta+\phi = \frac{\pi}{2}$ C

Q2 $f(x) = e^{\log_{\frac{1}{e}}\sqrt{x}} = e^{\frac{\log_e\sqrt{x}}{\log_e\frac{1}{e}}} = e^{-\log_e\sqrt{x}} = e^{\log_e x^{-\frac{1}{2}}} = x^{-\frac{1}{2}}$ E

Q3 $x=0$, $nx=\pm\pi$, $nx=\pm\frac{\pi}{2}$

.: $x=0$, $x=\pm\frac{\pi}{n}$, $x=\pm\frac{\pi}{2n}$ A

Q4 $\int_2^{10} f(x)dx = 10\times10-28-4\times2 = 64$ C

Q5 x-intercepts: $ax^2-1=0$, $x=\pm\frac{1}{\sqrt{a}}$

Area $= -\int_{-\frac{1}{\sqrt{a}}}^{\frac{1}{\sqrt{a}}}\left(ax^2-1\right)dx = \int_{-\frac{1}{\sqrt{a}}}^{\frac{1}{\sqrt{a}}}\left(1-ax^2\right)dx$ D

Q6 $x^2-a^2\ge0$, $a^2-x^2\ge0$ and $x+a\ne0$, .: $x=a$ B

Q7 D

Q8 A

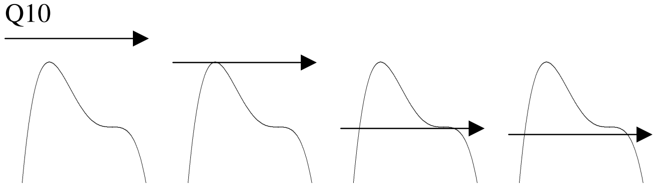
Q9 $\sin x+\cos 2x=0$, $\sin x+1-2\sin^2 x=0$

$2\sin^2 x-\sin x-1=0$, $(2\sin x+1)(\sin x-1)=0$

$\sin x=-\frac{1}{2}$, 1 and $x\in[-\pi,\pi]$

.: $x=-\frac{5\pi}{6}$, $-\frac{\pi}{6}$ or $\frac{\pi}{2}$, .: sum $=-\frac{\pi}{2}$ B

Q10 A



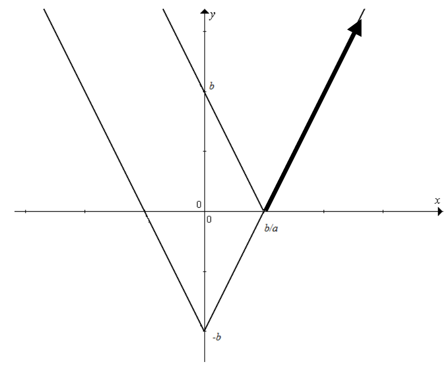
Q11 $f(x)=-x$, $f(y)=-y$, $f(x)f(y)=xy$

$f(xy)=-xy$, .: $f(xy)\ne f(x)f(y)$ E

Q12 $\lim_{h\to0}\frac{f(h)-f(0)}{h}$ = gradient of the tangent at $x=0$

≈ 1.2 D

Q13 E



Q14 $\int f(x)dx = 1-2x-\frac{1}{4}\log_e(1-2x)$

$f(x)=\frac{d}{dx}\left(1-2x-\frac{1}{4}\log_e(1-2x)\right) = -2+\frac{1}{2}\times\frac{1}{1-2x}$ D

Q15 $\int_0^{\frac{\pi}{8}} g(x)dx = \int_0^{\frac{\pi}{8}}\frac{2}{\cos^2(2x)}dx = \int_0^{\frac{\pi}{8}} 2\sec^2(2x)dx$

$=[\tan(2x)]_0^{\frac{\pi}{8}} = \tan\frac{\pi}{4}-\tan 0 = 1$

Average value $=\frac{\int_0^{\frac{\pi}{8}} g(x)dx}{\frac{\pi}{8}-0} = \frac{1}{\frac{\pi}{8}} = \frac{8}{\pi}$ B

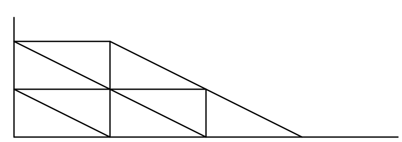
Q16 f is a many-to-one function, .: f^{-1} does not exist. A

Q17 The last draw is equally likely to be blue, green or red,

.: $\Pr(last.draw.is.red)=\frac{1}{3}$ C

Q18 B

Q19



$\Pr(Home.between.8.and.9)$

$=\Pr(left.between.6.and.9)-\Pr(left.between.6.and.8)$

$=1-\frac{7}{8}=\frac{1}{8}=0.125$ A

Q20

Transition matrix $\begin{bmatrix}0.85 & 0.10\\0.15 & 0.90\end{bmatrix}$, state matrix $\begin{bmatrix}0.65\\0.35\end{bmatrix}$.

State matrix after 2 weeks: $\begin{bmatrix}0.85 & 0.10\\0.15 & 0.90\end{bmatrix}^2\begin{bmatrix}0.65\\0.35\end{bmatrix}=\begin{bmatrix}0.54\\0.46\end{bmatrix}$

C

Q21 $\sum p(x)=1$

.: $p(0.01)+p(0.10)+p(0.30)+p(0.50)+p(0.59)=1$

.: $5a+1.009=1$, .: $a=-0.0018$

$\overline{X}=\sum xp(x)$

$=0.01p(0.01)+0.10p(0.10)+0.30p(0.30)+0.50p(0.50)+0.59p(0.59)$

≈ 0.30

C

Q22 $\mu=12.3$, $\sigma=\sqrt{1.69}=1.3$

$\Pr(9.1\le X<b)=\Pr(X<b)-\Pr(X<9.1)=0.95$

$\Pr(X<b)=0.95+0.006917\approx 0.9569$, .: $b\approx 14.53$

D

SECTION 2

Q1a $y=f(x)=e^x-2x$

Let $f'(x)=e^x-2=0$, .: $e^x=2$, $x=\log_e 2$

.: $y=2-2\log_e 2=2(1-\log_e 2)$

f has a stationary point, $(\log_e 2\,,2(1-\log_e 2))$.

Q1b

x	0	$\log_e 2$	1
$f'(x)$	negative	zero	positive

$(\log_e 2\,,2(1-\log_e 2))$ is a minimum.

Q1c Since $(\log_e 2\,,2(1-\log_e 2))$ is a minimum and $2(1-\log_e 2)>0$, .: $e^x-2x>0$ for $x\in R$

.: $e^x>2x$ for $x\in R$.

Q1d When $x=0$, $e^x=1$ and $x^2+1=1$

.: $e^x=x^2+1$ at $x=0$, i.e. both functions have a common point at $x=0$.

$\frac{d}{dx}e^x=e^x$ and $\frac{d}{dx}\left(x^2+1\right)=2x$

Since $e^x>2x$ for $x\in R$, .: $e^x>2x$ for $x>0$, i.e. the rate of increase of e^x is greater than the rate of increase of x^2+1.

.: $e^x\ge x^2+1$ for $x\ge 0$

Q1e $(x-1)^2\ge 0$, $x^2-2x+1\ge 0$, .: $x^2+1\ge 2x$

Q1f $\frac{d}{dx}\log_e\left(x^2+1\right)=\frac{d}{du}\log_e(u)\times\frac{d}{dx}\left(x^2+1\right)=\frac{2x}{x^2+1}$

Q1h $\int_0^x \frac{2t}{t^2+1}dt=\left[\log_e\left(t^2+1\right)\right]_0^x=\log_e\left(x^2+1\right)$

Q1i From part e, $t^2+1\ge 2t$, .: $1\ge\frac{2t}{t^2+1}$

.: $\int_0^x 1dt\ge\int_0^x\frac{2t}{t^2+1}dt$ for $x\ge 0$

.: $[t]_0^x\ge\log_e\left(x^2+1\right)$, .: $x\ge\log_e\left(x^2+1\right)$

Hence $e^x\ge x^2+1$ for $x\ge 0$

Q2a $P(x)=x^3+6ax^2+6bx+4c$

$P'(x)=3x^2+12ax+6b$

Q2b The turning point is on the *x*-axis, .: $P'(x)=0$ and $P(x)=0$

$P'(x)=0$.: $3x^2+12ax+6b=0$, $x^2+4ax+2b=0$,

.: $x^3+4ax^2+2bx=0$ for $x\ne 0$ .... (1)

$ax^2+4a^2x+2ab=0$ for $a\ne 0$ .... (2)

$P(x)=0$, .: $x^3+6ax^2+6bx+4c=0$ ...... (3)

(3) – (1): $2ax^2+4bx+4c=0$, .: $ax^2+2bx+2c=0$ .... (4)

(2) – (4): $4a^2x-2bx+2ab-2c=0$

.: $\left(2a^2-b\right)x+ab-c=0$

.: $x=\frac{c-ab}{2a^2-b}$

.: the turning point on the *x*-axis is $\left(\frac{c-ab}{2a^2-b},0\right)$.

Q2ci Compare $Q(x)=x^3+0.4x^2-3.36x-2.88$ with $P(x)=x^3+6ax^2+6bx+4c$

$a=\frac{1}{15}$, $b=-0.56$ and $c=-0.72$

.: $x=\frac{c-ab}{2a^2-b}=-1.2$

.: the turning point on the *x*-axis is $(-1.2,0)$.

Q2cii $Q(x)=x^3+0.4x^2-3.36x-2.88=(x+1.2)^2(x-p)$

.: $1.2^2p=2.88$, $p=2$

.: the *x*-coordinate of the other *x*-intercept is 2

$Q'(x)=3x^2+0.8x-3.36=3(x+1.2)(x-q)=0$

.: $3.6q=3.36$, $q=\frac{14}{15}$

.: the *x*-coordinate of the other turning point is $\frac{14}{15}$.

Q2d Area $=-\int_{-1.2}^{2}Q(x)dx\approx 8.74$ by CAS/graphics calc.

Q3a $y = a\log_e x + b$

$(1,0)$, $0 = a\log_e 1 + b$, $b = 0$

$(e,1)$, $1 = a\log_e e + 0$, $a = 1$

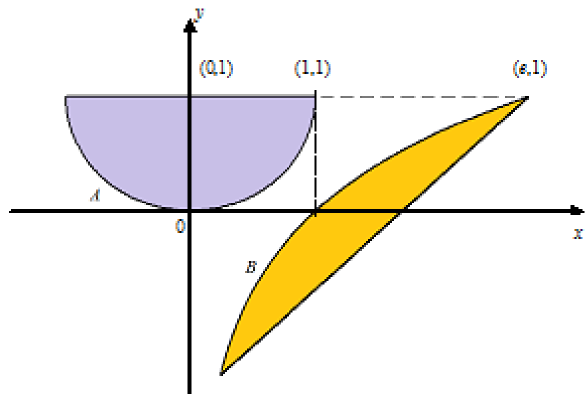
.: curve B has the equation $y = \log_e x$.

Q3b $x^2 + (y-1)^2 = 1$, where $x \in [-1,1]$ and $y \in [0,1]$

$(y-1)^2 = 1 - x^2$, $y - 1 = -\sqrt{1-x^2}$

.: curve A has the equation $y = 1 - \sqrt{1-x^2}$.

Q3c



Q3d Area of the ground space

$= \text{rectangle } (e \times 1) - \text{quarter circle } \left(\frac{\pi \times 1^2}{4}\right) - \int_1^e \log_e x\,dx$

$= e - \frac{\pi}{4} - 1$

Q3ei Curve B: $y = \log_e x$, at $x = p$, $y = \log_e p$

$\frac{dy}{dx} = \frac{1}{x}$, at $x = p$, $\frac{dy}{dx} = \frac{1}{p}$, .: gradient of the normal $= -p$

Equation of the normal at $x = p$:

$y - \log_e p = -p(x - p)$

$y = -px + p^2 + \log_e p$

Q3eii $y = -px + p^2 + \log_e p$

Centre $(0,1)$, $1 = p^2 + \log_e p$, .: $p = 1$

.: equation of the normal at $x = 1$ is $y = -x + 1$.

Q3eiii The normal is perpendicular to both curves, .: the distance between the two curves along the normal is the shortest.

Shortest distance = distance between $(0,1)$ and $(1,0)$ – radius

$= \sqrt{(1-0)^2 + (0-1)^2} - 1 = \sqrt{2} - 1$

Q3fi Curve B: $\frac{dy}{dx} = \frac{1}{x}$, curve A: $\frac{dy}{dx} = \frac{x}{\sqrt{1-x^2}}$

At $y = c$, $x = e^c$ for curve B, and $x = \sqrt{1-(c-1)^2}$ for curve A.

.: $\frac{1}{e^c} = \frac{\sqrt{1-(c-1)^2}}{\sqrt{(c-1)^2}}$, $c = 0.22007 \approx 0.22$ by CAS/graphics calc.

Q3fii At $y = c = 0.22007$,

$x = e^{0.22007} \approx 0.62587$ for curve A,

and $x = \sqrt{1-(0.22007-1)^2} \approx 1.24616$ for curve B.

Magnitude of translation $= 1.24616 - 0.62587 \approx 0.62$

Q4ai $\Pr(\mu - w < X < \mu + w) = 0.8000$,

.: $\Pr(X < \mu - w) = \frac{1-0.8000}{2} = 0.1000$, $\mu - w = 1.9468$

$w = \mu - 1.9468 = 1.9500 - 1.9468 = 0.0032$

Q4aii $(1 - 0.8000) \times 1000 = 200$

Q4aiii $\Pr(\mu - 2\sigma < X < \mu + 2\sigma \mid X < \mu - w \cup X > \mu + w)$

$= \frac{\Pr((\mu - 2\sigma < X < \mu + 2\sigma) \cap (X < \mu - w \cup X > \mu + w))}{\Pr(X < \mu - w \cup X > \mu + w)}$

$= \frac{0.9545 - 0.8000}{1 - 0.8000} \approx 0.7725$

Q4bi $\mu = 1.9500 \times 4 = 7.8000$

$\sigma^2 = 0.0250^2 \times 4 = 0.0025$, .: $\sigma = 0.0500$

Q4bii $\Pr(X > 7.8750) \approx 0.0668$

Q4biii Binomial distribution: $n = 26$, $p = 0.0668$

$\Pr(X > 2) = 1 - \Pr(X \le 2) \approx 1 - 0.7500 = 0.2500$

Q4biv No extra postage is required if the letter is $7.8750 - 0.1500 = 7.7250$ grams or less.

$\Pr(X < 7.7250) = 0.0668$

Q4ci $\int_{1.5000}^{2.2000} k e^{\frac{x}{2}} \sin(x)\,dx = 1$, $\int_{1.5000}^{2.2000} e^{\frac{x}{2}} \sin(x)\,dx = \frac{1}{k}$,

$k = 0.602036 \approx 0.6020$

Q4cii $\mu = \int_{1.5000}^{2.2000} x \times 0.602036 e^{\frac{x}{2}} \sin(x)\,dx = 1.8582344 \approx 1.8582$

Q4ciii Let m grams be the median weight.

$\int_{1.5000}^{m} 0.602036 e^{\frac{x}{2}} \sin(x)\,dx = 0.5$, $m = 1.86224 \approx 1.8622$ by CAS/graphics calc.

Q4civ $\Pr(X > m \mid X > \mu) = \frac{\Pr(X > m \cap X > \mu)}{\Pr(X > \mu)}$

$= \frac{\Pr(X > m)}{\Pr(X > \mu)} = \frac{0.5}{\int_{1.8582344}^{2.2000} 0.602036 e^{\frac{x}{2}} \sin(x)\,dx} \approx 0.9884$

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