
SECTION 1 – Multiple-choice answers

- | | | |
|------|-------|-------|
| 1. D | 9. A | 17. A |
| 2. C | 10. C | 18. B |
| 3. C | 11. D | 19. D |
| 4. D | 12. B | 20. D |
| 5. B | 13. C | 21. E |
| 6. C | 14. C | 22. B |
| 7. A | 15. E | |
| 8. B | 16. A | |
-

SECTION 1 – Multiple-choice solutions

Question 1

$$y = \log_e(2x+1)$$

For this function to be defined,
 $2x+1 > 0$

$$x > -\frac{1}{2}$$

So maximal domain is $x \in \left(-\frac{1}{2}, \infty\right)$.

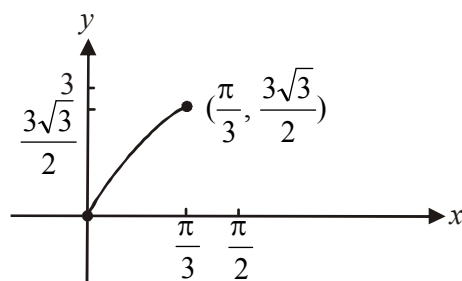
The answer is D.

Question 2

Sketch the graph of $y = f(x)$, taking particular note of the domain.

$$\begin{aligned}\text{Since } f\left(\frac{\pi}{3}\right) &= 3 \sin\left(\frac{\pi}{3}\right) \\ &= 3 \times \frac{\sqrt{3}}{2} \\ &= \frac{3\sqrt{3}}{2} \\ &(\approx 2.59...)\end{aligned}$$

$$\text{So } r_f = \left[0, \frac{3\sqrt{3}}{2}\right]$$



The answer is C.

Question 3

$$\begin{aligned}
 \text{Approximate area} &= f(2) \times 1 + f(3) \times 1 \\
 &= \log_e(2) + \log_e(3) \\
 &= \log_e(6) \text{ square units}
 \end{aligned}$$

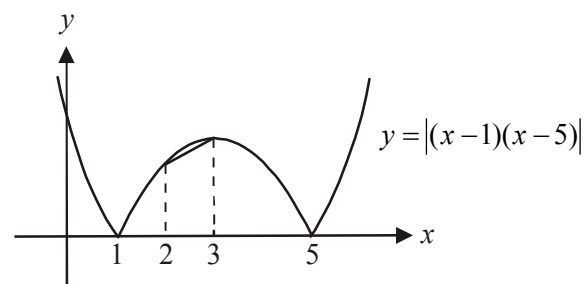
The answer is C.

Question 4

$$\begin{aligned}
 \text{Average rate of change} &= \frac{f(3) - f(2)}{3 - 2} \\
 &= \frac{|(3-1)(3-5)| - |(2-1)(2-5)|}{1} \\
 &= |2 \times -2| - |1 \times -3| \\
 &= |-4| - |-3| \\
 &= 4 - 3 \\
 &= 1
 \end{aligned}$$

A quick sketch tells you that the first two answers are not feasible. We are looking for the gradient of the line segment joining the points where $x = 2$ and $x = 3$ on the graph of

$$y = |(x-1)(x-5)|.$$



The answer is D.

Question 5

$$\begin{bmatrix} 2 & k-2 \\ k+1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

For no solutions or infinitely many solutions,

$$\det \begin{bmatrix} 2 & k-2 \\ k+1 & 2 \end{bmatrix} = 0$$

$$2 \times 2 - (k+1)(k-2) = 0$$

$$4 - (k^2 - k - 2) = 0$$

$$4 - k^2 + k + 2 = 0$$

$$k^2 - k - 6 = 0$$

$$(k-3)(k+2) = 0$$

If $k = 3$, we have

$$2x + y = 2 \quad - (1)$$

$$4x + 2y = -1 \quad - (2)$$

$$(2) \div 2 \quad 2x + y = -\frac{1}{2}$$

We have parallel lines so no solutions for $k = 3$.

If $k = -2$, we have

$$2x - 4y = 2 \quad - (1)$$

$$-x + 2y = -1 \quad - (2)$$

$$(1) \div 2 \quad x - 2y = 1$$

$$(2) \times -1 \quad x - 2y = 1$$

These equations represent the same line so infinite solutions for $k = -2$.

The answer is B.

Question 6

$$y = x - 3 \quad \text{and} \quad y = x^2 + kx - 1$$

$$x^2 + kx - 1 = x - 3$$

$$x^2 + kx - x - 1 + 3 = 0$$

$$x^2 + (k-1)x + 2 = 0 \quad (\text{equation of intersection})$$

For no solutions; and therefore no points of intersection,

$$\Delta < 0$$

$$b^2 - 4ac < 0$$

$$(k-1)^2 - 4 \times 1 \times 2 < 0$$

Method 1 – using CAS

Solve $(k-1)^2 - 8 < 0$ for k

$$1 - 2\sqrt{2} < k < 1 + 2\sqrt{2}$$

The answer is C.

Method 2 – by hand

Do a quick sketch of $y = (k-1)^2 - 8$.

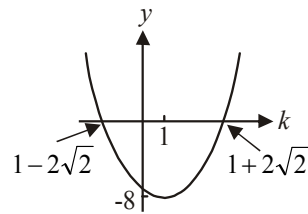
$$\text{If } (k-1)^2 - 8 = 0$$

$$(k-1)^2 = 8$$

$$k-1 = \pm\sqrt{8}$$

$$k-1 = \pm 2\sqrt{2}$$

$$k = 1 \pm 2\sqrt{2}$$



So, from the graph, $(k-1)^2 - 8 < 0$

$$\text{for } 1 - 2\sqrt{2} < k < 1 + 2\sqrt{2}$$

The answer is C.

Question 7

$$f: [0, \infty) \rightarrow \mathbb{R}, f(x) = x - 1$$

$$g(x) = f(f(x))$$

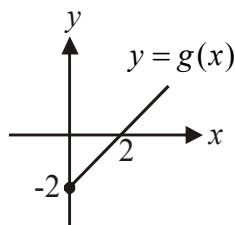
$$= f(x-1)$$

$$= (x-1) - 1$$

$$= x - 2$$

$$d_g = d_f = [0, \infty)$$

$$r_g = [-2, \infty)$$



So $d_{g^{-1}} = r_g = [-2, \infty)$

The answer is A.

Question 8

$$y = \frac{e^x - e^{3x}}{x^2}$$

Using CAS,

$$\frac{dy}{dx} = \frac{-e^x((3x-2)e^{2x} - x + 2)}{x^3}$$

$$\text{At } x=2, \frac{dy}{dx} = \frac{-e^6}{2}.$$

The answer is B.

Question 9

Method 1 – using CAS

$$\text{Let } y = \sqrt{x-2}$$

Swap x and y for inverse to obtain $x = \sqrt{y-2}$ and solve for y .

$$y = x^2 + 2 \text{ and } x \geq 0$$

$$\text{So } d_{f^{-1}} = [0, \infty)$$

$$f^{-1} : [0, \infty) \rightarrow R, f^{-1}(x) = x^2 + 2$$

The answer is A.

Method 2 – by hand

$$f : [2, \infty) \rightarrow R, f(x) = \sqrt{x-2}$$

$$\text{Let } y = \sqrt{x-2}$$

Swap x and y for inverse.

$$x = \sqrt{y-2}$$

$$x^2 = y - 2$$

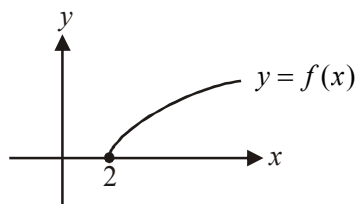
$$y = x^2 + 2$$

$$d_f = [2, \infty) \quad r_f = [0, \infty)$$

$$\text{So } d_{f^{-1}} = r_f = [0, \infty)$$

$$f^{-1} : [0, \infty) \rightarrow R, f^{-1}(x) = x^2 + 2$$

The answer is A.

**Question 10**

$$\begin{aligned} \text{Average value} &= \frac{1}{4-0} \int_0^4 2x \log_e(x^2 + 2) dx \\ &= 8.6601... \end{aligned}$$

The closest answer is 8.66.

The answer is C.

Question 11

This is a binomial distribution with $n = 10$ and $p = \frac{1}{6}$.

$$\Pr(X < 3) = \Pr(X = 0) + \Pr(X = 1) + \Pr(X = 2)$$

$$\text{binom cdf}\left(10, \frac{1}{6}, 0, 2\right) = 0.775227\dots$$

So $\Pr(X < 3) = 0.775227\dots$

The closest answer is 0.7752.

The answer is D.

Question 12

$$\text{normal cdf}(-\infty, 20, 23, 2) = 0.066807\dots$$

So 6.68...% of people completing the survey did so in less than 20 minutes.

The closest answer is 6.7%.

The answer is B.

Question 13Method 1

$$\begin{aligned}\text{Var}(X) &= E(X^2) - (E(X))^2 \\ &= 0^2 \times 0.4 + 2^2 \times 0.1 + 5^2 \times 0.5 - 2.7^2 \\ &= 5.61\end{aligned}$$

The answer is C.

Method 2

$$\begin{aligned}\text{Var}(X) &= \sum (x - \mu)^2 p(x) && \text{(from formula sheet)} \\ &= (0 - 2.7)^2 \times 0.4 + (2 - 2.7)^2 \times 0.1 + (5 - 2.7)^2 \times 0.5 \\ &= 2.916 + 0.049 + 2.645 \\ &= 5.61\end{aligned}$$

The answer is C.

Question 14Method 1 – using CAS

$$\text{Solve } \int_0^a \frac{3\sqrt{x}}{2} dx = 0.5 \text{ for } a.$$

$$a = 0.6299$$

The answer is C.

Method 2 – by hand

$$\int_0^a \frac{3\sqrt{x}}{2} dx = 0.5$$

$$\frac{3}{2} \int_0^a x^{\frac{1}{2}} dx = 0.5$$

$$\frac{3}{2} \left[\frac{2x^{\frac{3}{2}}}{\frac{3}{2}} \right]_0^a = 0.5$$

$$\left[x^{\frac{3}{2}} \right]_0^a = 0.5$$

$$a^{\frac{3}{2}} - 0 = 0.5$$

$$\left(a^{\frac{3}{2}} \right)^{\frac{2}{3}} = 0.5^{\frac{2}{3}}$$

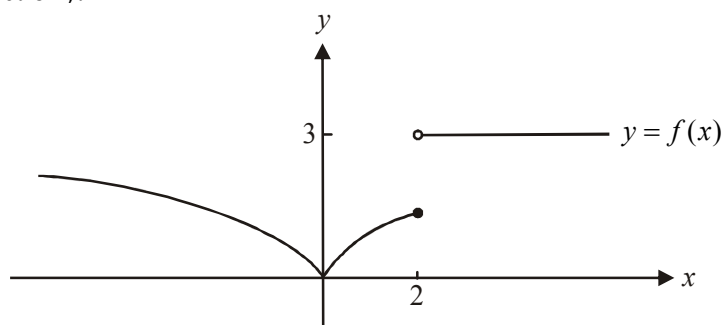
$$a = 0.5^{\frac{2}{3}}$$

$$a = 0.6299...$$

The answer is C.

Question 15

Sketch the function f .



The derivative function f' is defined for $x \in \mathbb{R} \setminus \{0, 2\}$.

The answer is E.

Question 16

Method 1 - using CAS

The equation of the tangent to the curve $y = x^{\frac{1}{3}}$ at the point where $x = 8$ is given by

$$y = \frac{x}{12} + \frac{4}{3}.$$

The answer is A.

Method 2 – by hand

$$y = x^{\frac{1}{3}}$$

$$\frac{dy}{dx} = \frac{1}{3} x^{-\frac{2}{3}}$$

$$= \frac{1}{3(\sqrt[3]{x})^2}$$

At $x = 8$

$$\frac{dy}{dx} = \frac{1}{3 \times (\sqrt[3]{8})^2}$$

$$= \frac{1}{3 \times 2^2}$$

$$= \frac{1}{12}$$

So $m = \frac{1}{12}$ and $(x_1, y_1) = (8, 2)$

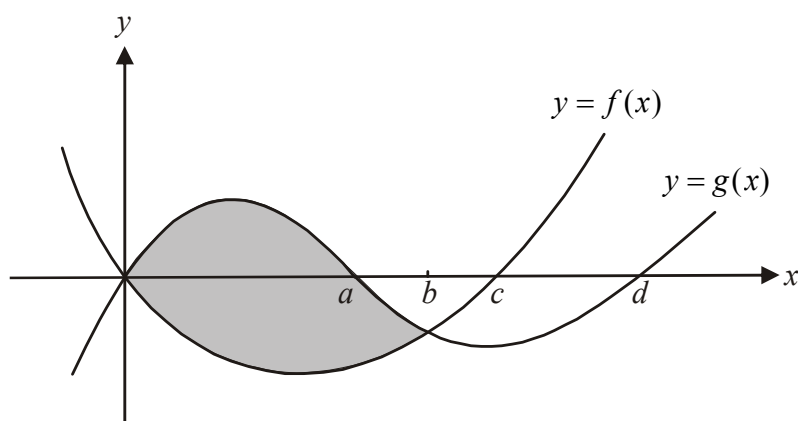
$$y - y_1 = m(x - x_1)$$

$$y - 2 = \frac{1}{12}(x - 8)$$

$$y = \frac{x}{12} - \frac{8}{12} + 2$$

$$y = \frac{x}{12} + \frac{4}{3}$$

The answer is A.

Question 17

The area required is shaded.

The graphs of $y = f(x)$ and $y = g(x)$ intersect at the points where $x = 0$ and $x = b$. Over this interval $x \in (0, b)$, $g(x) > f(x)$.

So area required $= \int_0^b (g(x) - f(x)) dx$.

The answer is A.

Question 18

$$y = 4e^{x+1}$$

The graph of $y = e^x$ has been dilated from the x -axis by a factor of 4, and translated 1 unit to the left.

The rule could therefore be

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

The answer is B.

Question 19

Stationary points occur where $x = -1, 0$ and 4 .

The gradient is positive for $x < -1$ and $0 < x < 4$.

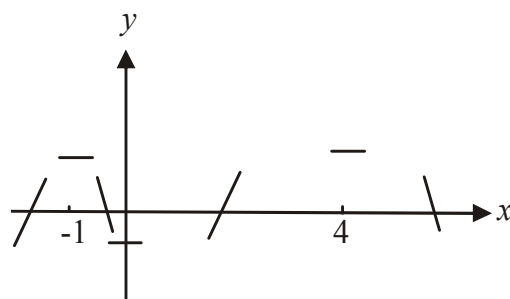
The gradient is negative for $-1 < x < 0$ and $x > 4$.

So, there is a local maximum where

$x = -1$ and $x = 4$ and a local minimum at $x = 0$.

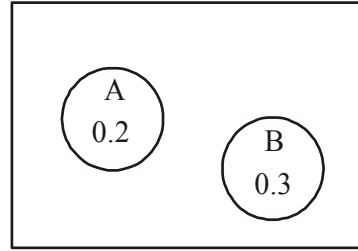
Only option D is true.

The answer is D.



Question 20

For mutually exclusive events $\Pr(A \cap B) = 0$
 so option D is not true.
 The other options can be verified using a
 Venn diagram.
 The answer is D.

**Question 21**

The graph of $y = f(x)$ has been

- reflected in the x -axis
- dilated by a factor of $\frac{1}{2}$ from the y -axis
- translated 1 unit to the right
- translated 3 units up

to become the graph of
 $y = 3 - f(2x - 2)$

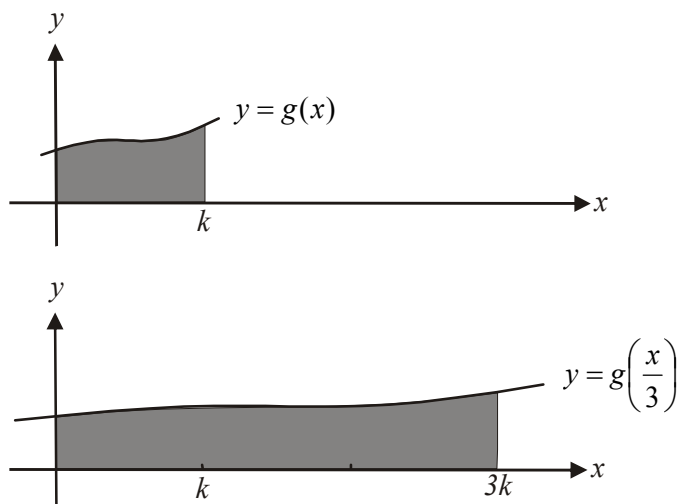
$$= 3 - f(2(x - 1))$$

Only graph E shows this.
 The answer is E.

Question 22

$$\begin{aligned} 4 \int_0^{3k} \left(g\left(\frac{x}{3}\right) - 1 \right) dx &= 4 \int_0^{3k} g\left(\frac{x}{3}\right) dx - 4 \int_0^{3k} 1 dx \\ &= 4 \times 3 \times 2k - 4[x]_0^{3k} \\ &= 24k - 4(3k - 0) \\ &= 12k \end{aligned}$$

Note that the graph of $y = g(x)$ has been dilated by a factor of 3 from the y -axis to become
 the graph of $y = g\left(\frac{x}{3}\right)$ so if $\int_0^k g(x) dx = 2k$ then $\int_0^{3k} g\left(\frac{x}{3}\right) dx = 3 \times 2k$.



The answer is B.

SECTION 2

Question 1

a. i. Method 1 – using CAS

$$\text{Solve } 2 \cos\left(\frac{x}{4}\right) + 1 = 0 \text{ for } x \quad (1 \text{ mark})$$

$$x = \dots - \frac{8\pi}{3}, \frac{8\pi}{3}, \frac{16\pi}{3}, \dots$$

From the graph, we are looking for the second smallest positive value of x ; that is, not the first positive x -intercept.

$$\text{So } d = \frac{16\pi}{3}.$$

(1 mark)

Method 2 – by hand

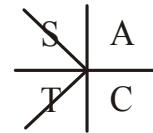
$$2 \cos\left(\frac{x}{4}\right) + 1 = 0 \quad (1 \text{ mark})$$

$$2 \cos\left(\frac{x}{4}\right) = -1$$

$$\cos\left(\frac{x}{4}\right) = -\frac{1}{2}$$

$$\frac{x}{4} = \dots - \frac{2\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \dots$$

$$x = \dots - \frac{8\pi}{3}, \frac{8\pi}{3}, \frac{16\pi}{3}, \dots$$

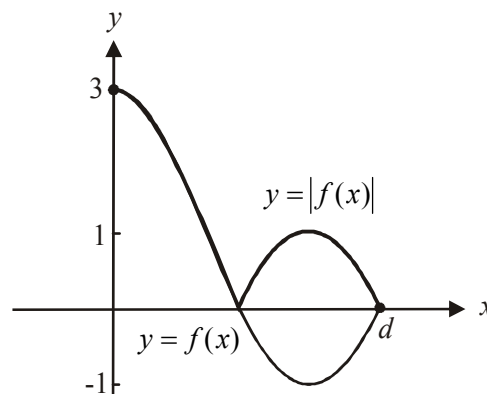


From the graph, we are looking for the second smallest positive value of x ; that is, not the first positive x -intercept.

$$\text{So } d = \frac{16\pi}{3}.$$

(1 mark)

ii.

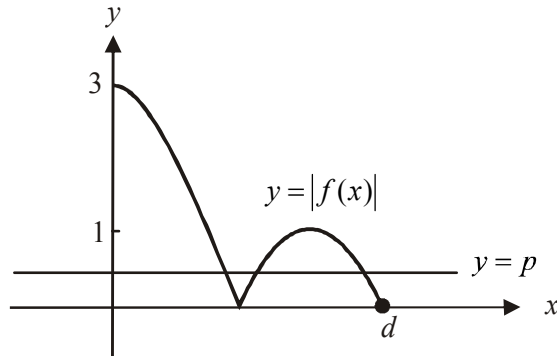


(1 mark)

iii. From the graph, the maximum value of $|f(x)|$ is 3.

(1 mark)

- iv. Again from the graph, if $p < 1$ and $p > 0$ then there are three solutions to the equation $|f(x)| = p$.



If $p = 1$, there are two solutions and if $p = 0$ there are 2 solutions.
So $p \in (0, 1)$ or $\{p : 0 < p < 1\}$.

(1 mark) – correct values
(1 mark) – endpoints excluded

- b. Method 1 – using CAS

Solve $2\cos\left(\frac{x}{4}\right) + 1 = \sqrt{3} + 1$ for x .

(1 mark)

$$x = \frac{2\pi(12n+1)}{3} \quad \text{or} \quad x = \frac{2\pi(12n-1)}{3}, \quad n \in \mathbb{Z}$$

(1 mark) **(1 mark)**

Method 2 – by hand

$$2\cos\left(\frac{x}{4}\right) + 1 = \sqrt{3} + 1$$

(1 mark)

$$\cos\left(\frac{x}{4}\right) = \frac{\sqrt{3}}{2}$$

$$\frac{x}{4} = \frac{-\pi}{6} + 2n\pi \quad \text{or} \quad \frac{x}{4} = \frac{\pi}{6} + 2n\pi, \quad n \in \mathbb{Z}$$

$$x = 4\left(\frac{-\pi}{6} + 2n\pi\right) \quad \text{or} \quad x = 4\left(\frac{\pi}{6} + 2n\pi\right)$$

$$x = -\frac{2\pi}{3} + 8n\pi \quad \text{or} \quad x = \frac{2\pi}{3} + 8n\pi$$

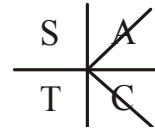
$$x = \pm \frac{2\pi}{3} + 8n\pi \quad n \in \mathbb{Z}$$

(1 mark) – first answer

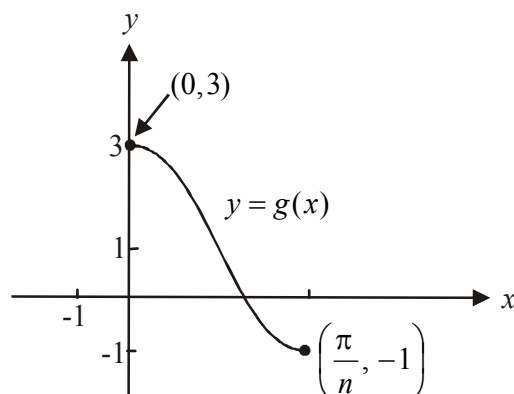
1 mark) – second answer

$$\begin{aligned} \text{(Note that } x = \frac{-2\pi}{3} + 8n\pi \quad \text{and} \quad x = \frac{2\pi}{3} + 8n\pi \\ = \frac{-2\pi + 24n\pi}{3} \quad \quad \quad = \frac{2\pi + 24n\pi}{3} \\ = \frac{2\pi(-1 + 12n)}{3} \quad \quad \quad = \frac{2\pi(1 + 12n)}{3} \end{aligned}$$

which confirms the CAS answer.)

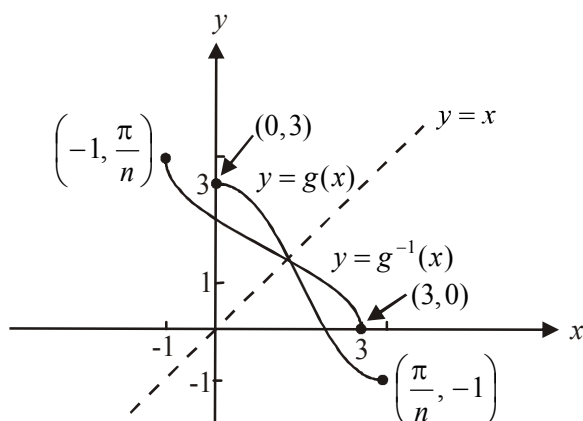


- c. i. For $y = 2 \cos(nx) + 1$,
 period $= \frac{2\pi}{n}$
 For g^{-1} to exist, g must be 1:1
 so $q = \frac{2\pi}{n} \div 2$
 $= \frac{\pi}{n}$



(1 mark) – correct value of q
 (1 mark) – correct endpoints and shape

ii.



Note that the points $\left(\frac{\pi}{n}, -1\right)$ and $\left(-1, \frac{\pi}{n}\right)$ can be anywhere along the x -axis and y -axis respectively as long as the two axes are consistent.

(1 mark) – correct endpoints
 (1 mark) – correct shape

iii. $g(x) = 2\cos(nx) + 1$
 $y = 2\cos(nx) + 1$
 Since $T(t, 1 - \sqrt{3})$ lies on g ,
 $1 - \sqrt{3} = 2\cos(nt) + 1$
 $-\sqrt{3} = 2\cos(nt)$
 $-\frac{\sqrt{3}}{2} = \cos(nt)$
 $nt = \frac{5\pi}{6}$
 $n = \frac{5\pi}{6t}$
 as required.

(1 mark)



(1 mark)

iv. $g(x) = 2\cos(nx) + 1$
 $g'(x) = -2n\sin(nx)$
 At $T(t, 1 - \sqrt{3})$
 $g'(x) = -2n\sin(nt)$
 $= -2 \times \frac{5\pi}{6t} \times \sin\left(\frac{5\pi}{6t} \times t\right)$
 $= \frac{-10\pi}{6t} \times \sin\left(\frac{5\pi}{6}\right)$
 $= \frac{-5\pi}{3t} \times \frac{1}{2}$
 $= \frac{-5\pi}{6t}$

(1 mark)

So the gradient of the normal to the graph of $y = g(x)$ at the point

$T(t, 1 - \sqrt{3})$ is $\frac{6t}{5\pi}$. (1 mark)

$$y - y_1 = m(x - x_1)$$

$$y - (1 - \sqrt{3}) = \frac{6t}{5\pi}(x - t)$$

$$y = \frac{6t}{5\pi}(x - t) + 1 - \sqrt{3}$$

(1 mark)

Total 18 marks

Question 2

- a. Create a transition matrix.

this serve

$$\begin{array}{cc} & \begin{array}{cc} F & B \end{array} \\ \begin{array}{c} F \\ B \end{array} & \begin{bmatrix} 0.4 & 0.7 \\ 0.6 & 0.3 \end{bmatrix} \end{array} \begin{array}{c} F \\ B \end{array} \text{ next serve}$$

$$\begin{aligned} \Pr(F F F F) &= 0.4 \times 0.4 \times 0.4 \times 0.4 \\ &= 0.0256 \end{aligned}$$

(1 mark)

- b. Method 1 – use the transition matrix

this serve

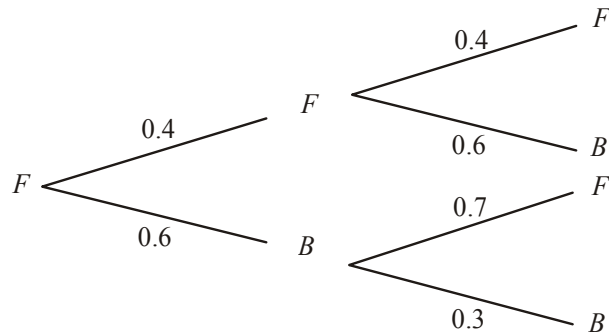
$$\begin{array}{cc} & \begin{array}{cc} F & B \end{array} \\ \begin{array}{c} F \\ B \end{array} & \begin{bmatrix} 0.4 & 0.7 \\ 0.6 & 0.3 \end{bmatrix}^2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.58 \\ 0.42 \end{bmatrix} \end{array}$$

(1 mark)

The probability that the third serve was to her forehand is 0.58.

(1 mark)

Method 2 – use a tree diagram



$$\begin{aligned} &\Pr(FFF) + \Pr(FBF) \\ &= 1 \times 0.4 \times 0.4 + 1 \times 0.6 \times 0.7 \\ &= 0.16 + 0.42 \\ &= 0.58 \end{aligned}$$

(1 mark)**(1 mark)**

- c. $\Pr(FFF) = 1 \times 0.4 \times 0.4 = 0.16$
 $\Pr(FBF) = 1 \times 0.6 \times 0.7 = 0.42$
 $\Pr(FFB) = 1 \times 0.4 \times 0.6 = 0.24$
 $\Pr(FBB) = 1 \times 0.6 \times 0.3 = 0.18$

Set up a distribution table.

x	0	1	2
$\Pr(X=x)$	0.16	$0.42 + 0.24 = 0.66$	0.18

(2 marks)

$$\begin{aligned} &(\text{Check } 0.16 + 0.66 + 0.18 = 1) \\ &E(X) = 0 \times 0.16 + 1 \times 0.66 + 2 \times 0.18 \\ &= 1.02 \end{aligned}$$

(1 mark)**(1 mark)**

d. Method 1

Find the steady state.

$$\begin{bmatrix} 0.4 & 0.7 \\ 0.6 & 0.3 \end{bmatrix}^{20} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.5384... \\ 0.4615... \end{bmatrix}$$

$$\begin{bmatrix} 0.4 & 0.7 \\ 0.6 & 0.3 \end{bmatrix}^{21} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.5384... \\ 0.4615... \end{bmatrix}$$

Steady state has been reached after 20 serves (maybe earlier).

In the long term 46%(to the nearest whole percent) of serves Sam received were to her backhand.

(1 mark)**Method 2**Let $X_n = 0$ represent Sam receiving a forehand serve.Let $X_n = 1$ represent Sam receiving a backhand serve.

$$T = \begin{bmatrix} 1-a & b \\ a & 1-b \end{bmatrix} = \begin{bmatrix} 0.4 & 0.7 \\ 0.6 & 0.3 \end{bmatrix}$$

So $a = 0.6$ and $b = 0.7$.

$$\text{For steady state, } \Pr(X_n = 1) = \frac{a}{a+b} = \frac{0.6}{0.6+0.7} = \frac{0.6}{1.3} = 0.46$$

In the long term 46% of serves Sam received were to her backhand.

(1 mark)**e. Method 1**

The transition matrix for Wednesday is

this serve

F	B
-----	-----

$$\begin{bmatrix} p & 1-2p \\ 1-p & 2p \end{bmatrix} \begin{matrix} F \\ B \end{matrix} \text{ next serve}$$

Solve

$$\begin{bmatrix} p & 1-2p \\ 1-p & 2p \end{bmatrix}^2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.73 \\ 0.27 \end{bmatrix} \text{ for } p$$

(1 mark)

$$p = 0.1 \text{ or } p = 0.9$$

(1 mark)

but from the transition matrix since

$$\Pr(B \text{ this serve} | B \text{ last serve}) = 2p,$$

$$p \neq 0.9$$

$$\text{So } p = 0.1$$

(1 mark)

Method 2

$$\Pr(FFF) + \Pr(FBF) = 0.73$$

$$1 \times p \times p + 1 \times (1-p)(1-2p) = 0.73$$

$$p^2 + 1 - 2p - p + 2p^2 = 0.73$$

$$3p^2 - 3p + 1 = 0.73$$

(1 mark)

$$3p^2 - 3p + 0.27 = 0$$

$$3(p^2 - p + 0.09) = 0$$

$$3(p - 0.1)(p - 0.9) = 0$$

$$p = 0.1 \text{ or } p = 0.9$$

(1 mark)

but from the transition matrix since

$$\Pr(B \text{ this serve} | B \text{ last serve}) = 2p,$$

$$p \neq 0.9$$

$$\text{So } p = 0.1$$

(1 mark)

f. i.
$$\Pr(X < 4) = \int_2^4 \frac{29-4x}{45} dx$$

$$= \frac{34}{45}$$

(1 mark)

ii.
$$\Pr(X < 3 | X < 4) = \frac{\Pr(X < 3 \cap X < 4)}{\Pr(X < 4)}$$

$$= \frac{\Pr(X < 3)}{\Pr(X < 4)}$$

(1 mark)

$$\Pr(X < 3) = \int_2^3 \frac{29-4x}{45} dx$$

$$= \frac{19}{45}$$

$$\text{So } \frac{\Pr(X < 3)}{\Pr(X < 4)} = \frac{19}{45} \div \frac{34}{45}$$

$$= \frac{19}{34}$$

(1 mark)**Total 14 marks**

Question 3

a.
$$C(0) = \frac{500}{100 - 0}$$
$$= 5 \text{ mg/m}^3$$
 (1 mark)

b. Since the function is continuous,

$$C(p) = \frac{500}{100 - p} \text{ and } C(p) = m.$$

 So
$$m = \frac{500}{100 - p}$$

c. From the graph and part a., the minimum value of m is 5 which occurs when $p = 0$. (1 mark)
 From the graph and part b., the maximum value of m must occur when $p = 90$ (since the function is continuous). (1 mark)

$$\text{So } m = \frac{500}{100 - 90}$$
$$= 50$$

(1 mark)

d. Method 1 – using CAS

$$C(t) = \frac{500}{100 - t} \text{ for } 0 < t < p$$
$$C'(t) = \frac{500}{(t - 100)^2}$$

(1 mark)

Method 2 – by hand

$$\text{Let } y = \frac{500}{100 - t}$$
$$= \frac{500}{u} \quad \text{where } u = 100 - t$$
$$= 500u^{-1} \quad \frac{du}{dt} = -1$$

$$\frac{dy}{du} = -500u^{-2}$$
$$= \frac{-500}{u^2}$$

$$\frac{dy}{dt} = \frac{dy}{du} \cdot \frac{du}{dt} \quad (\text{chain rule})$$
$$= \frac{-500}{u^2} \times -1$$

$$\text{So } \frac{dy}{dt} = \frac{500}{(100 - t)^2}$$

$$\text{So } C'(t) = \frac{500}{(100 - t)^2}$$
 (1 mark)

$$\text{Note } \frac{500}{(100 - t)^2} = \frac{500}{(-1(t - 100))^2} = \frac{500}{(t - 100)^2} \text{ confirming the CAS answer above.}$$

e. Solve $C'(t) = 1$

i.e. $\frac{500}{(t-100)^2} = 1$ for t .

(1 mark)

$$t = 77.6393... \text{ or } t = 122.361...$$

but $t < 90$

So $t = 77.64$ minutes (correct to 2 decimal places)

(1 mark)

f. average concentration $= \frac{1}{10-0} \int_0^{10} \frac{500}{100-t} dt$

$$= 5.26803...$$

(1 mark)

$$= 5.27 \text{ mg/m}^3 \text{ (correct to 2 decimal places)}$$

(1 mark)

g. For $0 \leq t \leq 90$ the maximum concentration is $m \text{ mg/m}^3$.

From part b., $m = \frac{500}{100-p}$.

For Victoria to survive, we require that

$$m < 6,$$

Solve $\frac{500}{100-p} < 6$ for p .

(1 mark)

$$p < \frac{50}{3} \quad (\text{since } 0 \leq p < 90, \text{ reject } p > 100)$$

So in order for Victoria to survive, $p \in \left[0, \frac{50}{3}\right)$.

(1 mark)

Total 11 marks

Question 4

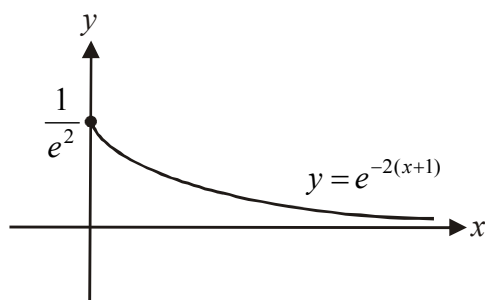
a. i. $f(g(x)) = f(2(x+1))$
 $= e^{-2(x+1)}$

(1 mark)

ii. $d_{f \circ g} = d_g$
 $= [0, \infty)$

(1 mark)

To find $r_{f \circ g}$ sketch the graph of $y = e^{-2(x+1)}$ and restrict the domain to $x \in [0, \infty)$.



y-intercept occurs when $x = 0$

$$y = e^{-2(0+1)}$$

$$= e^{-2}$$

$$= \frac{1}{e^2}$$

$$\text{So } r_{f \circ g} = \left[0, \frac{1}{e^2} \right]$$

(1 mark) – correct left endpoint and bracket**(1 mark)** – correct right endpoint and bracket

iii. $\text{Area} = \int_0^1 e^{-2(x+1)} dx$
 $= \frac{e^{-4}(e^2 - 1)}{2} \text{ square units}$

(1 mark)**(1 mark)**

iv. The graph of $y = e^{-x}$ is

- dilated by a factor of $\frac{1}{2}$ from the y-axis and
- translated 1 unit left

to become the graph of $y = e^{-2(x+1)}$

(1 mark) – description of dilation**(1 mark)** – description of translation

- b. i.** Stationary point occurs when $h'(x) = 0$. **(1 mark)**

$$\text{Solve } h'(x) = (2ae^{-a^2} - 2xe^{-a^2})e^{2ax-x^2} = 0 \text{ for } x.$$

$$\text{so } x = a$$

Since $h(a) = 1$, stationary point occurs at $(a, 1)$.

(1 mark)

- ii.** $h(x)$ is strictly increasing for $x \in (-\infty, a]$.

(1 mark)

- iii.** $A = \text{length} \times \text{width}$
 $= MN \times NP$
 $= 2(x-a) \times h(x)$
 $= 2(x-a) \times e^{-(x-a)^2}$
 as required.

(1 mark)

- iv.** $A = 2(x-a) \times e^{-(x-a)^2}$
 Maximum occurs when $\frac{dA}{dx} = 0$.

$$\text{Solve } \frac{dA}{dx} = 0 \text{ for } x.$$

(1 mark)

$$x = \frac{2a - \sqrt{2}}{2} \text{ or } x = \frac{2a + \sqrt{2}}{2}$$

$$x = a - \frac{\sqrt{2}}{2} \text{ or } x = a + \frac{\sqrt{2}}{2}$$

$$\text{Since } x > a, \text{ (from the diagram) } x = a + \frac{\sqrt{2}}{2}$$

(1 mark)

$$\text{Substitute } x = a + \frac{\sqrt{2}}{2} \text{ into } A = 2(x-a)e^{-(x-a)^2}.$$

$$A = \sqrt{\frac{2}{e}}$$

$$\text{Maximum area is } \sqrt{\frac{2}{e}} \text{ square units.}$$

(1 mark)

Total 15 marks