

MATHEMATICAL METHODS (CAS)

Units 3 & 4 – Written examination 2



2010 Trial Examination

SOLUTIONS

SECTION 1: Multiple-choice questions (1 mark each)

Question 1

Answer: B

Explanation:

Use CAS:

solve($2x + 8z = 26$ and $4x - 4y - 14z = -38$ and $8x - 4y + 2z = 14$, x, y, z).

Change the parameter of calculator to p .

$$x = 13 - 4p, y = \frac{45 - 15p}{2}, z = p$$

or rref on calculator

Question 2

Answer: C

Explanation:

The equation of the horizontal asymptote is $y = 3$.

As $x \rightarrow \infty$, $y \rightarrow 3$.

The y – intercept has coordinates $(0, 1)$

The equation of the vertical asymptote is $x = 1$

The x – intercepts have coordinates $(0.18, 0)$ and $(1.82, 0)$

Question 3

Answer: E

Explanation:

$-1 - k^2$ is the y – intercept and there are values greater than $-1 - k^2$.

Question 4*Answer:* A*Explanation:* $f(x) = |2e^x - 2|$ can be written as a hybrid function

$$f(x) = \begin{cases} 2e^x - 2 & x \geq 0 \\ -(2e^x - 2) & x < 0 \end{cases}$$

Find the derivative of the two separate functions

$$f'(x) = \begin{cases} 2e^x & x > 0 \\ -2e^x & x < 0 \end{cases}$$

Question 5*Answer:* D*Explanation:*

$$\text{Average rate of change} = \frac{V(.25) - V(0)}{0.25} = \frac{0.213 - 0.211429}{0.25} = 0.0063 \text{ m}^3/\text{h}$$

Question 6*Answer:* A*Explanation:*

Use CAS Probability menu: inverse normal. Change the area to 0.75 because the calculators use $Z < c$ and not $Z > c$.

$$c = 0.6745$$

Question 7*Answer:* A*Explanation:*

$f'(x) = 0$ for $x = -1$ and $x = 1.25$ means that there are stationary points at $x = -1$ and $x = 1.25$. The gradient change from negative to positive at $x = 1.25$. $f'(x) < 0$ for all other real values of x means the gradient is the same before and after the stationary point, therefore $x = -1$ is a stationary point of inflection.

Question 8*Answer:* E*Explanation:*

$$\text{Use CAS: } f(x) = \int (-2x^2 + 9x - 4)dx = \frac{-2x^3}{3} + \frac{9x^2}{2} - 4x$$

Question 9*Answer:* C*Explanation:*

$$2m + m + 4m + 3m = 1, \quad m = \frac{1}{10}.$$

$$E(X) = \sum x \times \Pr(X = x) = \frac{2}{10} + \frac{2}{10} + \frac{12}{10} + \frac{12}{10} = \frac{28}{10} = \frac{14}{5}$$

x	1	2	3	4
$\Pr(X = x)$	$\frac{2}{10}$	$\frac{1}{10}$	$\frac{4}{10}$	$\frac{3}{10}$

Question 10*Answer:* E*Explanation:*

Use CAS: $1 - \text{Normal Cdf}(-\infty, 2.416, 0, 1), \quad \Pr((Z > 2.416)) = 0.0078$

Question 11*Answer:* C*Explanation:*

$$\int_0^4 2f(x) - 4dx = 2 \int_0^4 f(x)dx - \int_0^4 4dx = 18 - [4x]_0^4 = 18 - 16 = 2$$

Question 12*Answer:* B*Explanation:*

$$m = \sin\left(\frac{\pi x}{6}\right) + d = 0 \text{ at } x = 3.$$

$$d = -\sin\left(\frac{3\pi}{6}\right) = -1$$

$$m = \sin\left(\frac{\pi x}{6}\right) - 1$$

$$f(x) = \int \sin\left(\frac{\pi x}{6}\right) - 1 dx = \frac{-6}{\pi} \cos\left(\frac{\pi x}{6}\right) - x + c$$

$$2 = \frac{-6}{\pi} \cos\left(\frac{3\pi}{6}\right) - 3 + c$$

$$c = 5$$

$$f(x) = \frac{-6}{\pi} \cos\left(\frac{\pi x}{6}\right) - x + 5$$

Question 13

Answer: D

Explanation:

Use CAS: Binomial pdf $\left(30, \frac{1}{5}, 2, 2\right)$, $\Pr(X = 2) = 0.0337$.

Question 14

Answer: C

Explanation:

Use CAS: Inverse normal(0.53, 0, 1), $z = 0.07527$

$$z = \frac{x - \mu}{\sigma}$$

$$0.07527 = \frac{x - 10}{2}$$

$$x = 10.1505$$

or invNorm(0.53, 10, 2)

Question 15

Answer: B

Explanation:

Use CAS: norm Cdf($-\infty$, 27, 36, 3), $\Pr(X < 27) = 0.00135 = 0.0014$

Question 16

Answer: A

Explanation:

$f \circ g$ exists if $\text{ran } g \subseteq \text{dom } f$, therefore $D = \mathbb{R} \setminus (-2, 1)$.

Question 17

Answer: B

Explanation:

Use CAS: $\int 3x \times x^{\frac{1}{2}} dx = \int 3x^{\frac{3}{2}} dx = \frac{6\sqrt{x^5}}{5}$.

Question 18

Answer: D

Explanation

$$\therefore \frac{1}{k} \int_2^5 (x^3 - 4) dx = 1$$

$$\frac{1}{k} \times \frac{561}{4} = 1$$

$$k = \frac{561}{4}$$

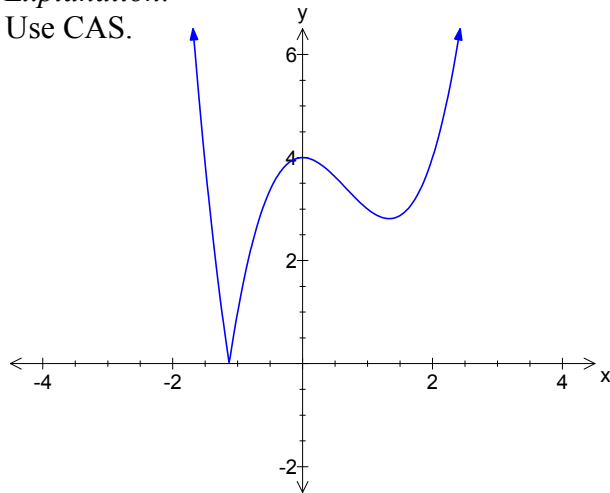
Question 19*Answer:* E*Explanation:*

$$l = \log_e(-x + 1) = \log_e(-(-4) + 1) = \log_e(5)$$

$$A = \log_e 5 + \log_e 4 + \log_e 3 + \log_e 2 = \log_e(5 \times 4 \times 3 \times 2) = \log_e 120$$

Question 20*Answer:* A*Explanation:*

Use CAS.



$$|x^3 - 2x^2 + 4|$$

Question 21*Answer:* C*Explanation:*

$$x = \frac{1}{3} \log_e(5y + 2) - 1$$

$$3(x + 1) = \log_e(5y + 2)$$

$$e^{3x+3} - 2 = 5y$$

$$\text{Equation of the inverse function: } \frac{1}{5}(e^{3x+3} - 2)$$

Question 22*Answer:* E*Explanation:*

The function $f: R \rightarrow R, f(x) = -5 \sin\left(\frac{\pi x}{4}\right) + 1$, has period and range respectively:

$$\text{period} = \frac{2\pi}{b} = \frac{2\pi}{1} \times \frac{4}{\pi} = 8$$

$$\text{maximum value} = -5(-1) = 6 \text{ and minimum value} = -5(1) = -4$$

$$\text{range: } [-4, 6]$$

SECTION 2: Analysis Questions**Question 1**

a. $f(x) = \sqrt{2-3x} = (2-3x)^{\frac{1}{2}}$

Use chain rule: $m = f'(x) = \frac{1}{2}(2-3x)^{-\frac{1}{2}} \times -3 = \frac{-3}{2\sqrt{2-3x}}$

M1+A1
2 marks

b. Use CAS: $m = \frac{-3}{2\sqrt{2-3x}} = -4$, $x = \frac{119}{192}$ and $\left(\frac{119}{192}, \frac{3}{8}\right)$

$$y - y_1 = m(x - x_1)$$

$$y - \frac{3}{8} = -4\left(x - \frac{119}{192}\right)$$

$$y = -4x + \frac{119}{48} + \frac{3}{8}$$

$$g(x) = -4x + \frac{137}{48}$$

M1+A1
2 marks

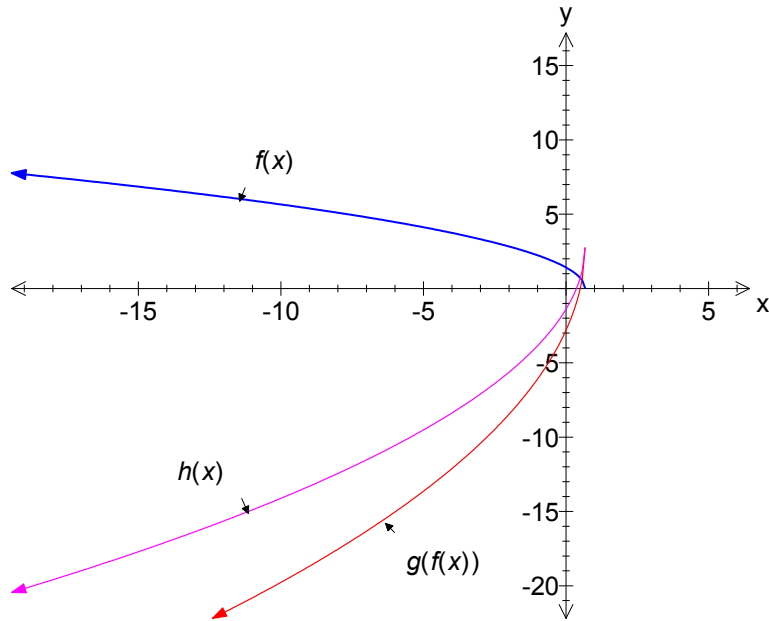
- c. Domain of f is $\left(-\infty, \frac{2}{3}\right]$ and the range of g is R . The range of g is R . The range of g is not a subset of the domain of f ($\text{ran } g \not\subseteq \text{dom } f$). So $f \circ g$ does not exist.

M1+A1
2 marks

- d. $g(f(x)) = g(\sqrt{2-3x}) = -4\sqrt{2-3x} + \frac{137}{48}$, the domain is the set of all values in the domain of f (the domain of the inner function). Domain: $x \in \left(-\infty, \frac{2}{3}\right]$.

M2+A1
3 marks

e. and f.



A2

A2

2 +2 marks

Question 2

a. $\Pr(X \geq 49.62) = 0.4$, $z = 0.253347$ (inv norm(0.6,0,1))

$\Pr(X < 51.37) = 0.75$, $z = 0.67449$ (inv norm(0.75,0,1))

$z = \frac{X - \mu}{\sigma}$, $0.253347 = \frac{49.62 - \mu}{\sigma}$, $0.67449 = \frac{51.37 - \mu}{\sigma}$ use CAS to solve simultaneously.

$\mu = 48.57$ mm and $\sigma = 4.16$ mm

M2+A1

3 marks

b. Use CAS: norm Cdf(47.15, 49.87, 48.57, 4.16) $\Pr(47.15 < X < 49.87) = 0.2562$

M1+A1

2 marks

c. Use CAS: norm Cdf($-\infty$, 51.42, 48.57, 4.16), $\Pr(X < 51.42) = 0.7534$

M1+A1

2 marks

d. $\Pr(47.15 < X < 49.87 | X < 51.42) = \frac{\Pr(47.15 < X < 49.87)}{\Pr(X < 51.42)} = \frac{0.256248}{0.753358} = 0.3401$

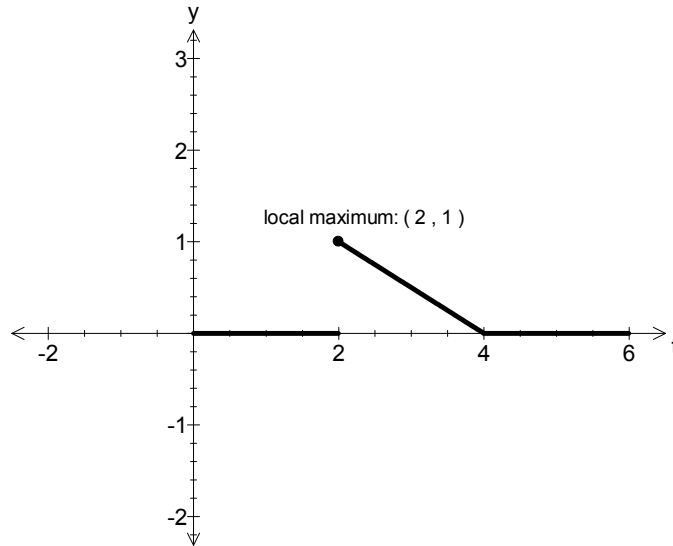
M2+A1

3 marks

e. $\int_2^a (-0.5t + 2) dt = 1, \quad a = 4$

A1
1 mark

f.



A2
2 marks

g. $\Pr\left(T > \frac{7}{3}\right) = \int_{\frac{7}{3}}^4 (-0.5t + 2) dt = 0.6944$

M1+A1
2 marks

h. $\mu = E(T) = \int_2^4 (t \times f(t)) dt = \int_2^4 (-0.5t + 2t) dt = 2.6667$

The average time taken to manufacture the large amount of candles is 2.6667 hours.

M1+A1
2 marks

i. $sd(T) = \sqrt{\text{var}(T)} = \sqrt{\int_2^4 t^2 \times (-0.5t + 2) dt - (2.6667)^2} = 0.4714$

$$\Pr(2.6667 - 2(0.4714) \leq T \leq 2.6667 + 2(0.4714))$$

$$= \int_2^{3.6095} (-0.5t + 2) dt = 0.9619 \text{ because } f(t) = 0, t < 2.$$

M2+A1
3 marks

Question 3

a. $h = \int \sin\left(\frac{\pi t}{6}\right) dx = \frac{-6}{\pi} \cos\left(\frac{\pi t}{6}\right) + c$

$$0 = \frac{-6}{\pi} \cos(0) + c, \quad c = \frac{6}{\pi}$$

$$h = \frac{-6}{\pi} \cos\left(\frac{\pi t}{6}\right) + \frac{6}{\pi}$$

M2+A1

3 marks

b. Sketch the graphs and find points of intersection or solve $\frac{-6}{\pi} \cos\left(\frac{\pi t}{6}\right) + \frac{6}{\pi} \leq 2$ on CAS.

between 6 a.m. – 9:05 a.m., between 2:55 p.m. – 9:05 p.m, between 2 a.m. – 6 a.m

M1+A1

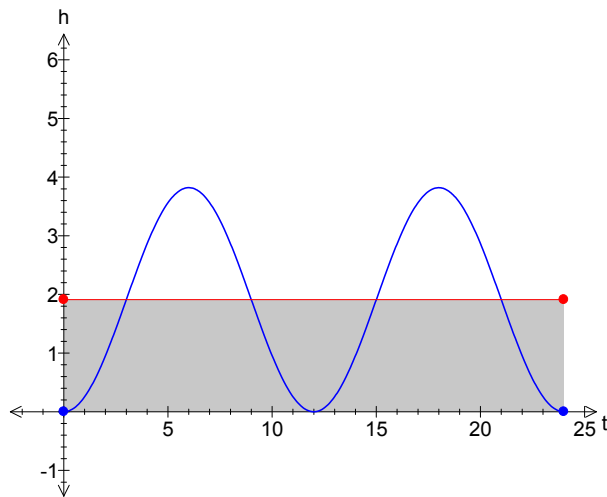
2 marks

c. average depth of $h = \frac{1}{24-0} \int_0^{24} \left(\frac{-6}{\pi} \cos\left(\frac{\pi t}{6}\right) + \frac{6}{\pi} \right) dt = \frac{6}{\pi} \text{ m.}$

M1+A1

2 marks

d.



A3

3 marks

e. Reflection in the x – axis, dilation by a factor of $\frac{6}{\pi}$ from the x – axis, dilation by a factor of $\frac{6}{\pi}$ from the y – axis, translation of $\frac{6}{\pi}$ units up.

M1+A1

2 marks

- f. Use CAS and solve $0 = \frac{-6}{\pi} \cos\left(\frac{\pi t}{6}\right) + \frac{6}{\pi}$. Substitute the parameter of the calculator with an appropriate variable.

$$t = 12n, n \in \mathbb{N}.$$

A1

1 mark

- g. $k(t + \pi) = \cos(2t + 2\pi) = \cos(2t) = k(2t)$. The two expressions are identical and so $k(t) = \cos(2t)$ is a solution of the functional equation.

M1+A1

2 marks

Question 4

a. $A = 4hx + x^2 = 675$

$$h = \frac{675 - x^2}{4x}$$

M1+A1
2 marks

b.

i. $V = x^2 h = x^2 \left(\frac{675 - x^2}{4x} \right) = \frac{675x - x^3}{4}$

To be a maximum: $\frac{dV}{dx} = 0$

$$\frac{dV}{dx} = \frac{675}{4} - \frac{3x^2}{4} = 0$$

$$3x^2 - 675 = 0$$

$$x^2 = 225$$

$$x = 15\text{cm} \quad \text{or use CAS – graph to find the maximum.}$$

$$h = \frac{675 - 15^2}{4 \times 15} = \frac{15}{2}\text{cm}$$

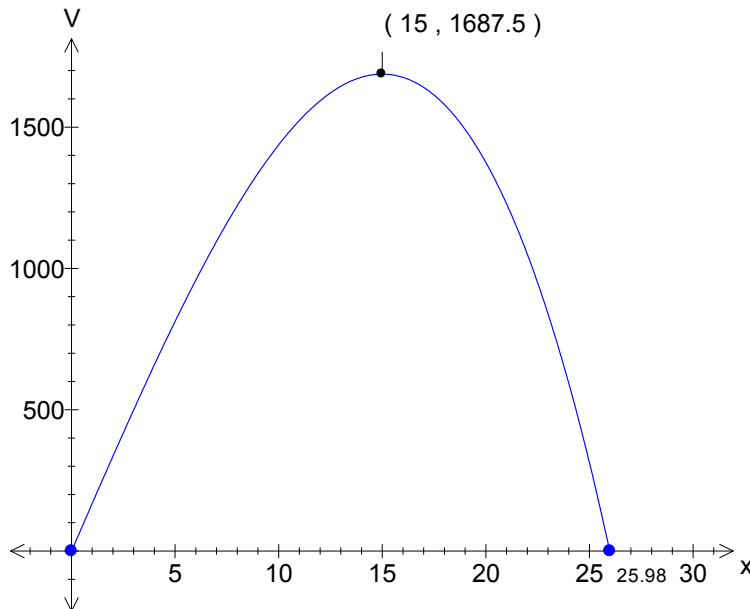
$$V = 225 \times \frac{15}{2} = 1687.5 \text{ cm}^3$$

M3+A1
4 marks

ii Domain: (0, 25.98)

M1+A1
2 marks

c.



A2
2 marks