Neap:

Trial Examination 2010

VCE Mathematical Methods (CAS) Units 3 & 4

Written Examination 1

Suggested Solutions

Question 1

a. y-intercept ($x = 0$)

$$f(0) = \sin^2\left(-\frac{\pi}{2}\right) - 1$$

$$= (-1)^2 - 1$$

$$= 0$$ A1

b. x-intercepts ($y = 0$)

$$\sin^2\left(2x - \frac{\pi}{2}\right) - 1 = 0$$

$$\sin^2\left(2x - \frac{\pi}{2}\right) = 1 \qquad x \in [0, 2\pi]$$
$$\therefore 2x \in [0, 4\pi]$$

$$\sin\left(2x - \frac{\pi}{2}\right) = \pm 1 \qquad \therefore 2x - \frac{\pi}{2} \in \left[-\frac{\pi}{2}, \frac{7\pi}{2}\right]$$ M1

$$\sin\left(2x - \frac{\pi}{2}\right) = -1 \qquad \text{OR} \qquad \sin\left(2x - \frac{\pi}{2}\right) = 1$$ M1

$$2x - \frac{\pi}{2} = -\frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{2} \qquad 2x - \frac{\pi}{2} = \frac{\pi}{2}, \frac{5\pi}{2}$$

$$2x = 0, 2\pi, 4\pi \qquad 2x = \pi, 3\pi$$

$$x = 0, \pi, 2\pi \qquad x = \frac{\pi}{2}, \frac{3\pi}{2}$$

$$x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$$ A1

Question 2

a. $y = x^2\log_e\left(\frac{1}{x}\right) = x^2\log_e(x^{-1}) = -x^2\log_e(x)$,

thus $\frac{dy}{dx} = -2x \cdot \log_e(x) - x^2 \cdot \frac{1}{x} = -2x\log_e(x) - x$ A1

(Alternatively, using the product rule directly gives $\frac{dy}{dx} = 2x\log_e\left(\frac{1}{x}\right) + x^2\left(\frac{-1}{x}\right)$, which is equivalent to the answer above.)

b. For stationary points we require $\frac{dy}{dx} = 0$.

$-x(1 + 2\log_e(x)) = 0$ gives $x = 0,\ \log_e(x) = -\frac{1}{2}$ M1

Clearly $x > 0$ so a stationary point exists if $\log_e(x) = -\frac{1}{2} \Rightarrow x = e^{-\frac{1}{2}}$.

$$\text{Now } y = \left(e^{-\frac{1}{2}}\right)^2 \times \log_e\left(\frac{1}{e^{-\frac{1}{2}}}\right)$$

$$= e^{-1} \times \frac{1}{2}$$

$$= \frac{1}{2e}$$

Thus the stationary point occurs at $\left(e^{-\frac{1}{2}}, \frac{1}{2e}\right)$. A1

Sign of the derivative test: Use suitable values such as for $x < e^{-\frac{1}{2}}$, choose $x = e^{-1}$ and for $x > e^{-\frac{1}{2}}$, choose $x = 1$.

$x = e^{-1}$, $\frac{dy}{dx} = -e^{-1} \times [1 + 2\log_e e^{-1}] = -e^{-1} \times (1 - 2) > 0$ M1

$x = 1$, $\frac{dy}{dx} = -1 \times [1 + 2\log_e 1] = -1 \times 1 < 0$

$\therefore$ a maximum turning point occurs at $\left(e^{-\frac{1}{2}}, \frac{1}{2e}\right)$. A1

Question 3

a. f is a parabola with turning point (2, 4).

$\therefore$ maximum value of a is 2, so f is one-to-one. A1

b. inverse $x = (y - 2)^2 + 4$ M1

$x - 4 = (y - 2)^2$

$y - 2 = \pm\sqrt{x - 4}$

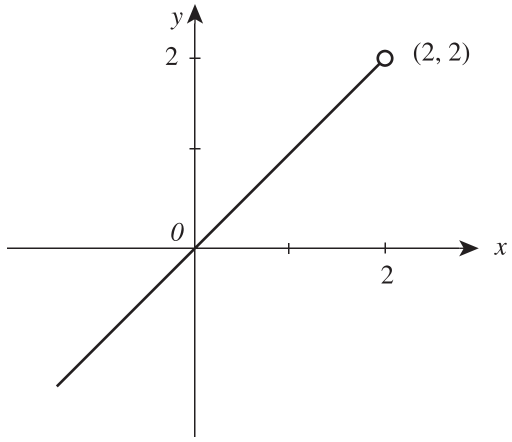
$y = 2 - \sqrt{x - 4}$ (as $y \in (-\infty, 2)$ i.e. domain of f)

$\therefore f^{-1}(x) = 2 - \sqrt{x - 4}$ A1

c. rule of $f^{-1}(f(x)) = x$

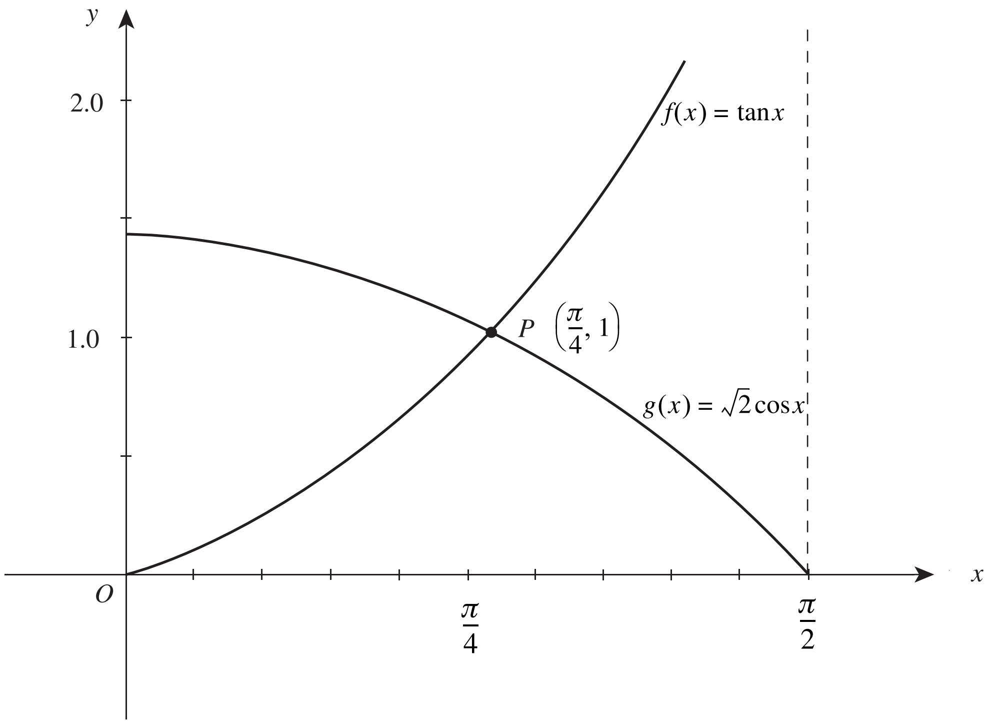
domain of $f^{-1}(f(x))$ = domain $f = (-\infty, 2)$

$\therefore$ graph is: A1



Question 4

a.



correct identification of f and g A1

coordinates of P A1

b. Given $y = \log_e(\cos(x))$, then $\dfrac{dy}{dx} = \dfrac{-\sin(x)}{\cos(x)} = -\tan(x)$. M1

Thus $\displaystyle\int \tan(x)dx = -\log_e(\cos(x))$. A1

c. Area $= \int_0^{\frac{\pi}{4}} (\sqrt{2}\cos(x) - \tan(x))\,dx$ A1

$= [\sqrt{2}\sin(x) + \log_e(\cos(x))]_0^{\frac{\pi}{4}}$

$= \left[\sqrt{2}\sin\left(\frac{\pi}{4}\right) + \log_e\left(\cos\left(\frac{\pi}{4}\right)\right)\right] - [\sqrt{2}\sin 0 + \log_e \cos 0]$ M1

$= \left[\sqrt{2} \times \frac{1}{\sqrt{2}} + \log_e\left(\frac{1}{\sqrt{2}}\right)\right] - [\sqrt{2} \times 0 + \log_e(1)]$

$= 1 - \log_e(\sqrt{2})$ (or equivalent: $1 + \log_e\left(\frac{1}{\sqrt{2}}\right)$ or $1 - \frac{1}{2}\log_e(2)$) A1

Question 5

As events A and B are independent, $\Pr(A|B) = \Pr(A) = \frac{3}{4} \Rightarrow \Pr(A') = \frac{1}{4}$. A1

Also, events A' and B' will be independent so we know $\Pr(A' \cap B') = \Pr(A') \times \Pr(B')$.

From the question, $\Pr(B) = \Pr(A' \cap B')$.

Thus $\Pr(B) = \frac{1}{4}(1 - \Pr(B))$, giving $4\Pr(B) = (1 - \Pr(B)) \Rightarrow 5\Pr(B) = 1$ M1

$\Pr(B) = \frac{1}{5}$ A1

Question 6

$e^x - 4 = 5e^{-x}$

Multiplying by e^x gives: M1

$(e^x)^2 - 4e^x = 5$

$(e^x)^2 - 4e^x - 5 = 0$

$(e^x - 5)(e^x + 1) = 0$

$e^x = 5$ or -1, $e^x \neq -1$ A1

$x = \log_e 5$

Question 7

We require $\Pr(T \le 6 | T \ge 3) = \dfrac{\Pr(3 \le T \le 6)}{\Pr(T \ge 3)}$ A1

$$= \frac{\int_3^6 \frac{-t}{50} + \frac{1}{5} dt}{1 - \Pr(T < 3)} = \frac{\left[\frac{-t^2}{100} + \frac{t}{5}\right]_3^6}{1 - \frac{51}{100}} = \frac{\left(\frac{-36}{100} + \frac{6}{5}\right) - \left(\frac{-9}{100} + \frac{3}{5}\right)}{\frac{49}{100}}$$ M1

$$= \frac{\frac{-27}{100} + \frac{3}{5}}{\frac{49}{100}} = \frac{33}{100} \times \frac{100}{49} = \frac{33}{49}$$ A1

Question 8

$f(a) = a^2 + k$

$f\,'(x) = 2x$

$f\,'(a) = 2a$ M1

Equation of tangent $y - (a^2 + k) = 2a(x - a)$

$y = 2ax - a^2 + k$ A1

Hence to pass through (0, 0): $-a^2 + k = 0$

$k = a^2$

$\therefore k \ge 0$ A1

Question 9

a. $X' = TX + B$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} -4 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ -3 \end{bmatrix} + \begin{bmatrix} -4 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$\therefore$ image is (0, –1). A1

b.
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -4 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} - \begin{bmatrix} -4 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\frac{1}{-2}\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x'+4 \\ y'-2 \end{bmatrix} = \frac{1}{-2}\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{-2}\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x'+4 \\ y'-2 \end{bmatrix}$$ M1

$$\therefore x = \frac{x'+4}{2}$$
$$y = -y'+2$$ M1

$$\therefore y = 2x - 4 \text{ becomes } -y'+2 = 2\left(\frac{x'+4}{2}\right) - 4$$
$$y' = 2 - x'$$ A1

$\therefore$ required image is $y = 2 - x$.

Alternatively, a more elegant solution which avoids the need to find an inverse matrix is shown below:

$$TX + B = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ 2x+4 \end{bmatrix} + \begin{bmatrix} -4 \\ 2 \end{bmatrix}$$
$$= \begin{bmatrix} 2x \\ -2x+4 \end{bmatrix} + \begin{bmatrix} -4 \\ 2 \end{bmatrix}$$ M1

$$\therefore X' = \begin{bmatrix} 2x-4 \\ -2x+6 \end{bmatrix}$$

$$\Rightarrow x' = 2x - 4$$ M1
$$y' = -2x + 6$$

Adding gives $x' + y' = 2$.

$\therefore$ required image is $x + y = 2$. A1

c. A dilation of factor 2 from the y-axis and a reflection in the y-axis A1

followed by translations of 4 units to the left and 2 upwards. A1

Note: the order of the first two transformations could be exchanged and similarly, the order of the translations is not significant, however the translations must follow the dilation and reflection.

Question 10

a. Approximate change $= V(x+h) - V(x)$ M1

$= hV'(x)$

$V = 6x^3$, so $V'(x) = 18x^2$

$x = 3$, $h = 0.02$: $hV'(x) = 0.02V'(3)$

$= 0.02 \times 18 \times 3^2$

$= 3.24$

$= 3.24 \text{ cm}^3$ A1

b. The gradient function at $x = 3$ is positive and increasing. Therefore the approximation of the change is less than the exact change. A1