MAV Trial Examination Paper 2010
Mathematical Methods CAS Examination 2
SOLUTIONS

Section 1 – Multiple Choice

ANSWERS

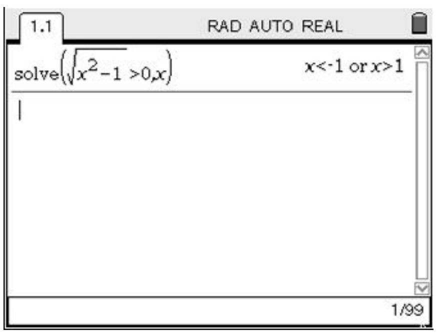
1. D	**2. C**	**3. D**	**4. E**	**5. E**	**6. A**	**7. E**
8. B	**9. A**	**10. B**	**11. A**	**12. B**	**13. A**	**14. E**
15. D	**16. C**	**17. B**	**18. C**	**19. D**	**20. D**	**21. E**
22. C						

SOLUTIONS

Question 1 **Answer D**

$\sqrt{x^2 - 1} > 0$

$x < -1$ or $x > 1$



Question 2 **Answer C**

Domain of f^{-1} = Range of $f = R$

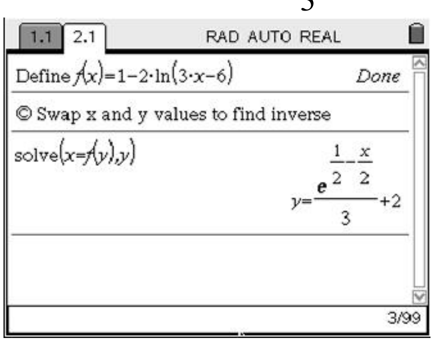
Rule of f^{-1}: interchange x and y values

$x = 1 - 2\log_e(3y - 6)$

Solve for y

$$y = \frac{e^{\left(\frac{1-x}{2}\right)}}{3} + 2$$

$$f^{-1} : R \to R,\ f^{-1}(x) = \frac{1}{3}e^{\left(\frac{1-x}{2}\right)} + 2$$



Question 3 **Answer D**

This can be solved by recognition, or using CAS.

$f(4x) = f(x) + 2f(2)$ may be recognised as a property of a logarithmic function.

If $f(x) = \log_e(x)$, then

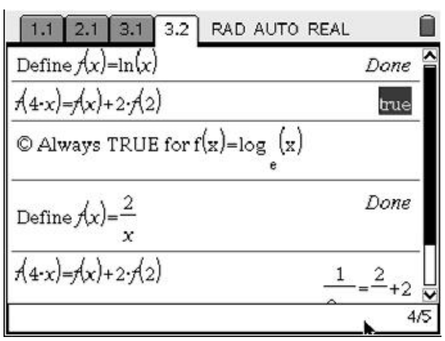
$$\begin{aligned} f(4x) &= \log_e(4x) \\ &= \log_e(x) + \log_e(4) \\ &= \log_e(x) + \log_e\left(2^2\right) \\ &= \log_e(x) + 2\log_e(2) \end{aligned}$$

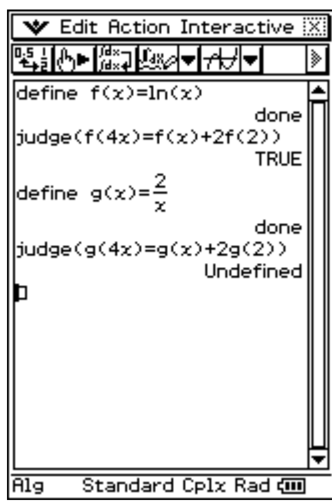
$f(4x) = f(x) + 2f(2)$, as required

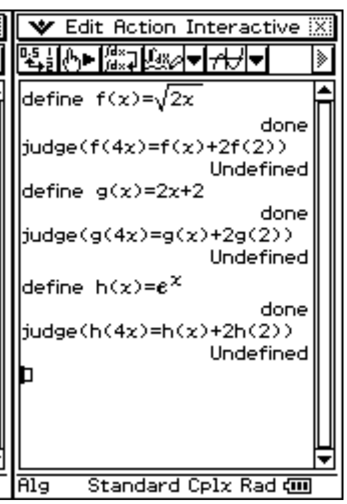
Alternatively, using CAS

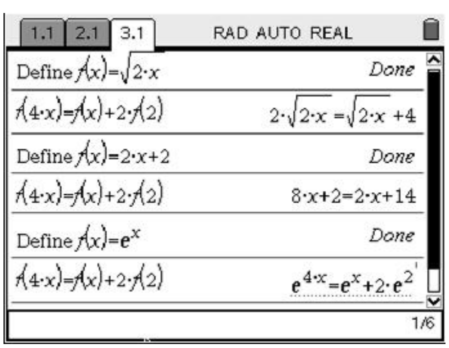
$f(x) = \log_e(x)$ is the only function that always satisfies the functional equation.

For a functional equation, a CAS output of ‘true’ indicates that it is *always* true. An output of ‘false’ indicates that is *never* true. Any other output indicates that it is *sometimes* true (often in the trivial case), but *not always*.









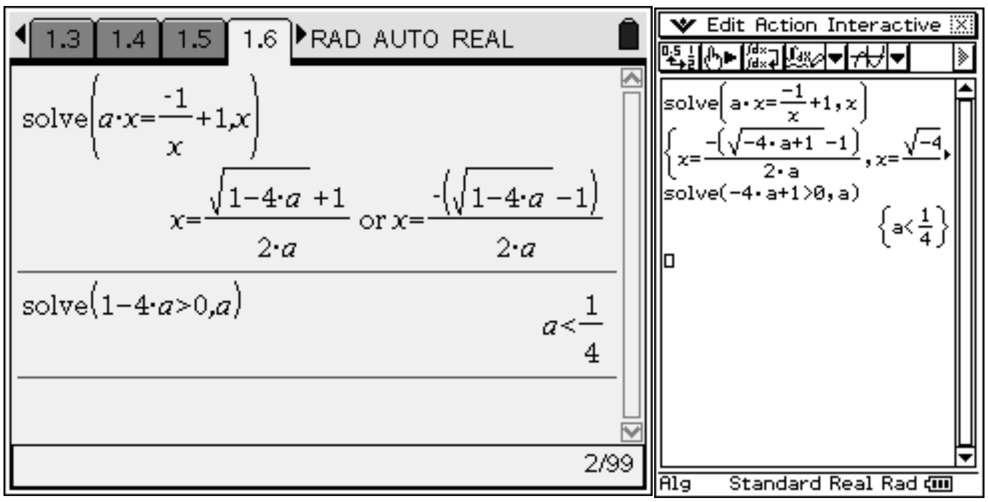
Question 4 **Answer E**

Solve $ax = -\frac{1}{x} + 1$ for x.

$$x = \frac{\sqrt{(1-4a)}+1}{2a} \text{ or } x = \frac{-\sqrt{(1-4a)}+1}{2a}$$

For two distinct solutions $1-4a>0$ and $a \neq 0$.

Hence $\{a: -\infty < a < 0\} \cup \{a: 0 < a < \frac{1}{4}\}$



Question 5 **Answer E**

$$\int_1^4 \left(f(x)+2\right)dx = \int_1^4 f(x)\,dx + \int_1^4 2\,dx$$

$$= 5 + [2x]_1^4$$

$$= 5 + [8-2]$$

$$= 11$$

Question 6 **Answer A**

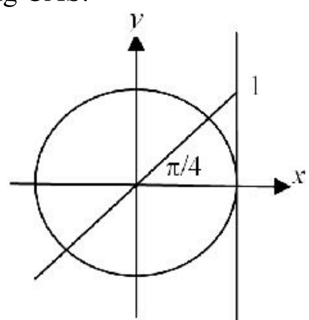
This can be done by hand or using CAS.

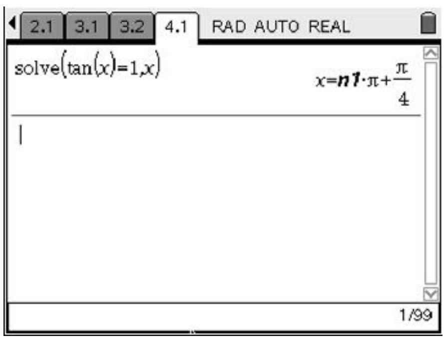
$2\tan(x)+1=3$

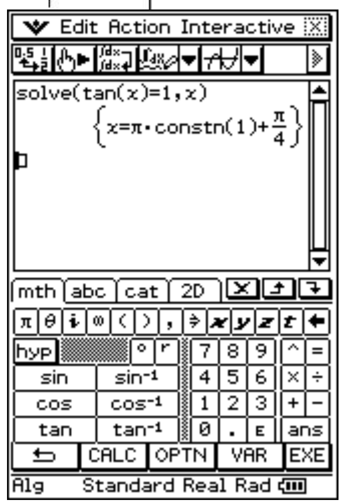
$\tan(x)=1$

$x=\dfrac{\pi}{4}+n\times\text{period}$

$x=\dfrac{\pi}{4}+n\pi$, where $n\in Z$







Question 7 **Answer E**

$g(x)=-2\cos\left(\dfrac{x}{3}-2\right)+1$

Period = $\dfrac{2\pi}{\frac{1}{3}}=6\pi$

Amplitude = $|-2|=2$

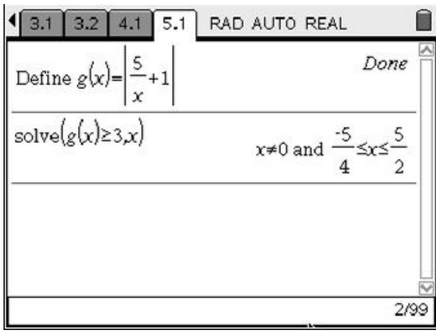
Range = $[-2+1,\ 2+1]=[-1,3]$

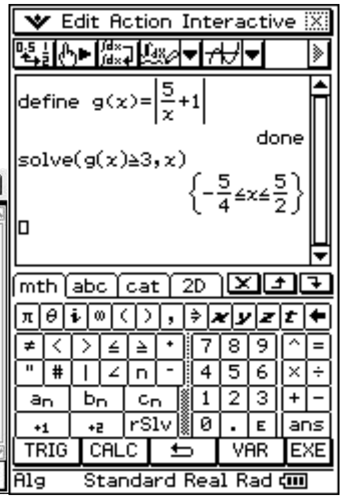
Question 8 **Answer B**

$\left|\dfrac{5}{x}+1\right|\geq 3$

$x\in\left[-\dfrac{5}{4},\dfrac{5}{2}\right]\backslash\{0\}$. Which is equivalent to

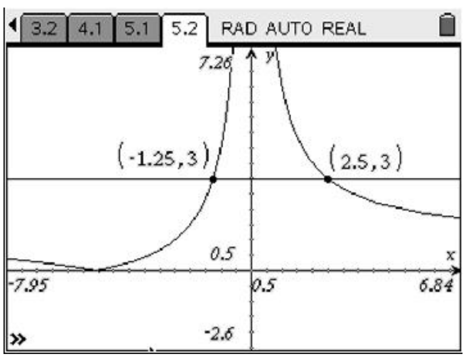
$\left\{x:-\dfrac{5}{4}\leq x<0\right\}\cup\left\{x:0<x\leq\dfrac{5}{2}\right\}$





Alternatively, a graphical approach may be used.

Find the points of intersection of $y = g(x)$ and $y = 3$



$\{x : g(x) \geq 3\} = \{x : -1.25 \leq x < 0\} \cup \{x : 0 < x \leq 2.5\}$

Question 9 **Answer A**

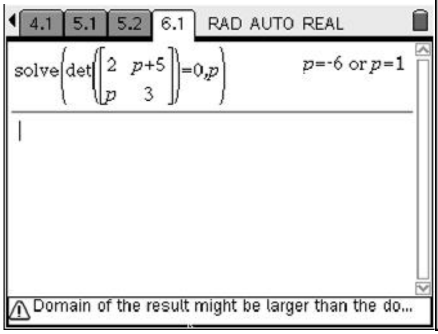
A system of linear equations will **not** have a unique solution when the **determinant** of the coefficient matrix is equal to zero.

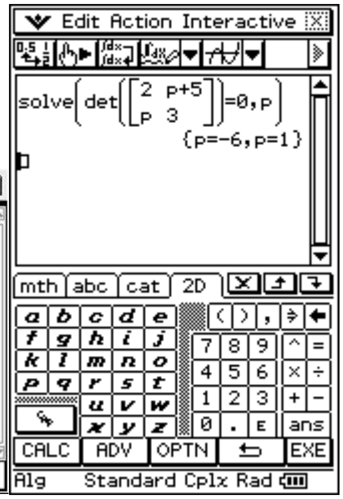
Solve $\begin{vmatrix} 2 & p+5 \\ p & 3 \end{vmatrix} = 0$ for p.

$p = -6$ or $p = 1$

Both of these values of p will give no solution.

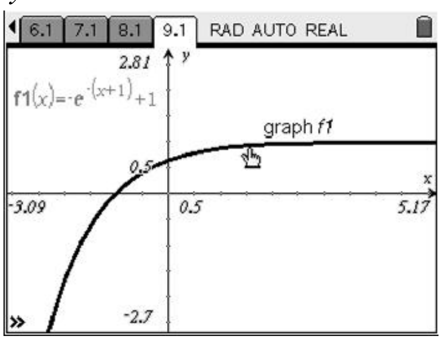
Note that it is impossible for the equations to have infinitely many solutions because $p = -6$ or $p = 1$ does not give rise to identical equations.





Question 10 **Answer B**

$y = -e^{-(x+1)} + 1$ meets the criteria. It is most like graph **B**.



Question 11 **Answer A**

A dilation of scale factor 2 from the y-axis: $h(x) \to h\left(\frac{x}{2}\right)$

A reflection in the x-axis: $h\left(\frac{x}{2}\right) \to -h\left(\frac{x}{2}\right)$

Translation 6 units right: $-h\left(\frac{x}{2}\right) \to -h\left(\frac{1}{2}(x-6)\right)$

Translation 1 unit up: $-h\left(\frac{1}{2}(x-6)\right) \to -h\left(\frac{1}{2}(x-6)\right)+1$

Question 12 **Answer B**

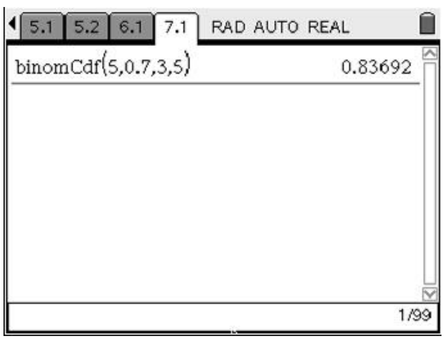
$$z=\frac{x-\mu}{\sigma}$$
$$=\frac{55-45}{8}$$
$$=1.25$$

$$\Pr(X>55)=\Pr(Z>1.25)$$
$$=1-\Pr(Z<1.25)$$

Question 13 **Answer A**

$X \sim Bi(5,0.7)$

$\Pr(X \geq 3)=0.8369$



Question 14 **Answer E**

$3r+0.2+.35+0.3=1$

$r=0.05$

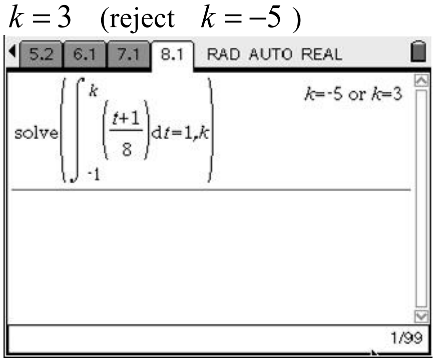
$E(X)=0+(0.2\times1)+(.35\times2)+(0.3\times3)+(0.1\times4)$

$E(X)=2.2$

Question 15 **Answer D**

Solve for k, $\int_{-1}^{k}\left(\frac{t+1}{8}\right)dt=1$

$k=3$ (reject $k=-5$)



Question 16 **Answer C**

If T is a 2×2 matrix, S_0 must be a 2×1 matrix. The columns in T must add to one

$$T=\begin{bmatrix}0.3 & 0.9\\0.7 & 0.1\end{bmatrix},\quad S_0=\begin{bmatrix}15\\30\end{bmatrix}$$

Question 17 **Answer B**

$$f(x)=A(x-B)^{\frac{1}{3}}+C$$

The graph has a vertical tangent at $x = 3$, passing through the point (3, 1).

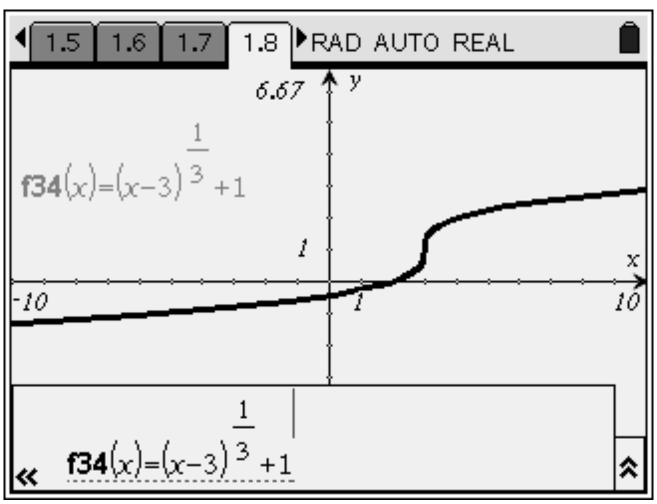
Hence $B = 3$ and $C = 1$.

$$f(x)=A(x-3)^{\frac{1}{3}}+1$$

The y-intercept is negative as $f(0)<0$.

A possible value for A is 1.

$$f(x)=(x-3)^{\frac{1}{3}}+1$$

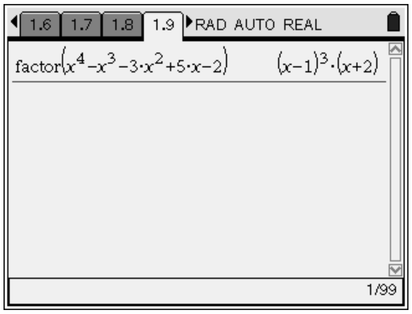


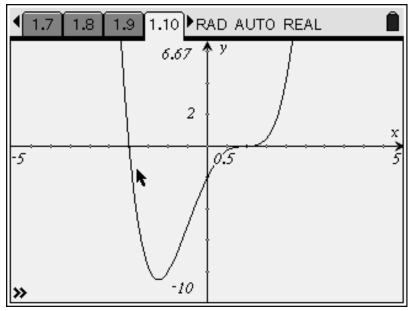
Question 18 **Answer C**

$$f(x)=x^4-x^3-3x^2+5x-2=(x-1)^3(x+2)$$

Hence the graph of f has a stationary point of inflection at (1, 0)

and a local minimum.





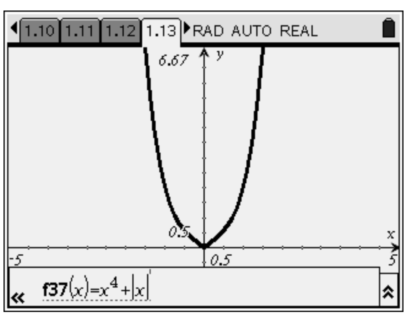
Question 19 **Answer D**

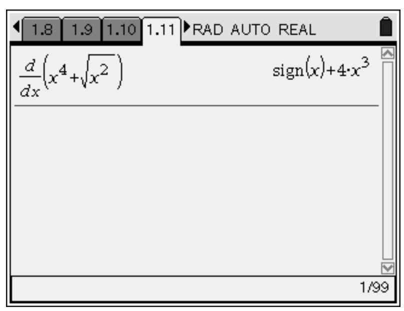
$\sqrt{x^2} = |x|$

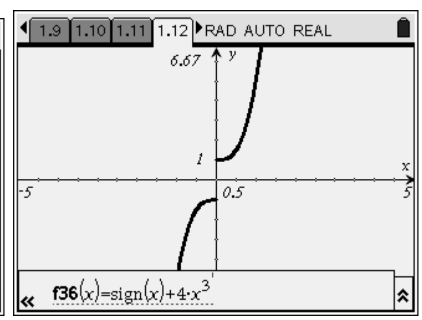
$f(g(x)) = x^4 + |x|$

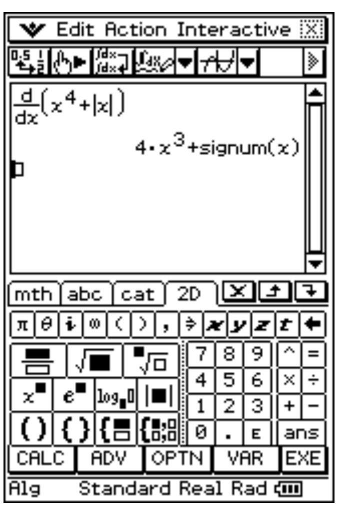
There is a cusp at $x = 0$.

$$\frac{d}{dx}\left(x^4 + |x|\right) = \begin{cases} 4x^3 + 1 \text{ for } x > 0 \\ 4x^3 - 1 \text{ for } x < 0 \end{cases}$$





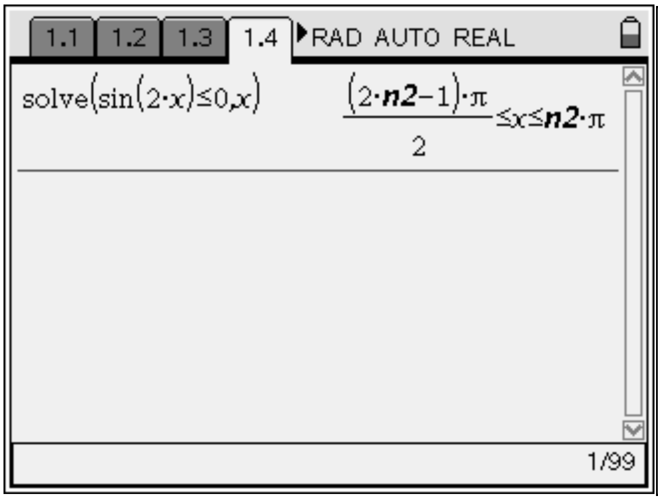


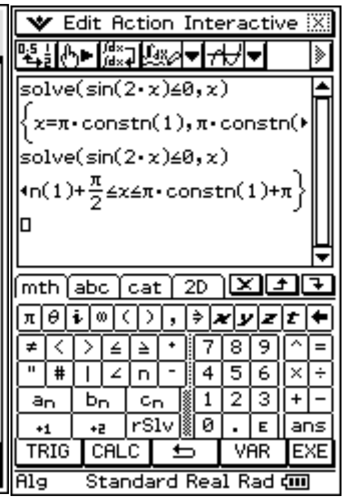


Question 20 **Answer D**

$\frac{d}{dx}\left(\log_e(\sin(2x))\right)$ is undefined when $\sin(2x) \le 0$

$\frac{(2k-1)\pi}{2} \le x \le \pi k, k \in Z$





Question 21 **Answer E**

For A, B and C, $y > 0$ for all x and as x increases, y increases.

The rectangles will always be under the curve.

For D and E $y < 0$ for all x.

For D as x increases y decreases and the area of the rectangles will under estimate the actual area.

For E as x increases y increases and the area of the rectangles will over estimate the actual area.

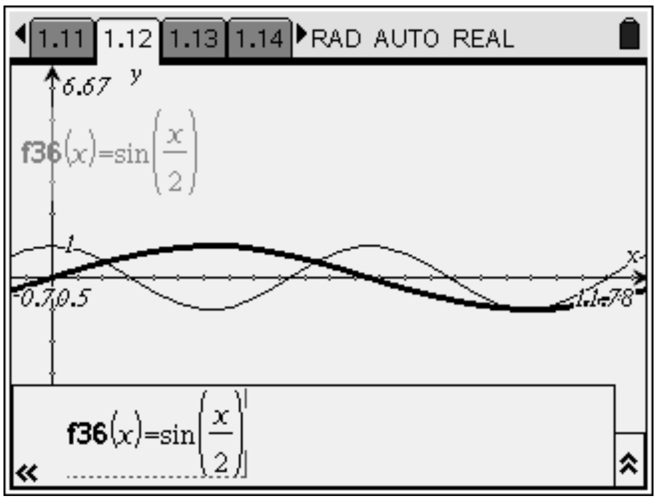
Question 22 **Answer C**

Solve $\sin\left(\frac{x}{2}\right)=\cos(x)$ for $0 \le x \le 4\pi$

$x=\frac{\pi}{3}$ or $x=\frac{5\pi}{3}$ or $x=3\pi$

The curve of *f* is above the curve of *g* for the first area and below for the second area.

$$\text{Area} = \int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} \left(f(x)-g(x)\right)dx + \int_{\frac{5\pi}{3}}^{3\pi} \left(g(x)-f(x)\right)dx$$

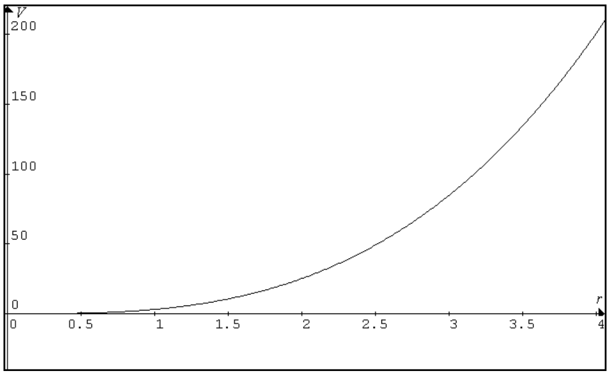


Section 2 - Extended Answer

Question 1

a. Shape **1A**

Open circle (0, 0) and closed circle $(4, 64\pi)$ **1A**



b. $V = \frac{1}{3}\pi r^2 h = \pi r^3$

$h = 3r = 3 \times 4 = 12$ cm **1A**

c. i. $V = \pi r^3$

$\frac{dV}{dr} = 3\pi r^2 = 3\pi \times 2^2 = 12\pi$ cm^3/cm **1A**

$V(2) = 8\pi$

$V(r) \approx V(2) + (r-2)V'(2)$

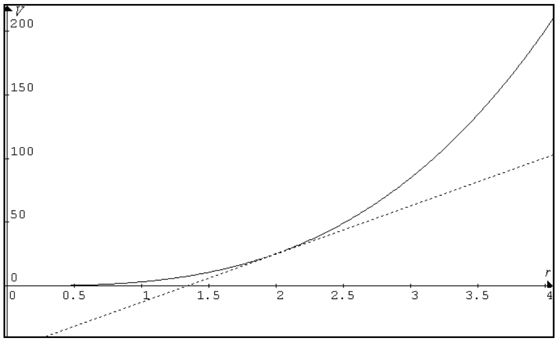
$= 8\pi + (r-2)12\pi$

$V(r) \approx 12\pi r - 16\pi$ as required **1M**

ii. $V(2.1) = 12\pi \times 2.1 - 16\pi = \frac{46\pi}{5}$ cm^3 **1A**

iii. Underestimate **1A**

The actual volume is being approximated by the tangent to the curve at $x = 2$. The tangent line is below the graph of V at $r = 2.1$.

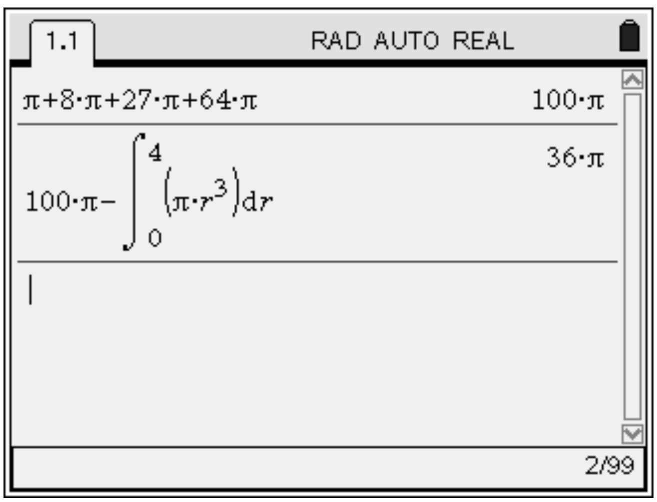


d. i. Area $\approx f(1)+f(2)+f(3)+f(4)$ **1A**

$= 100\pi \text{ cm}^2$ **1A**

ii. $100\pi - \int_0^4 V(r)dr$ **1A**

$= 36\pi \text{ cm}^2$ **1A**



iii. The average value $\approx \dfrac{\text{the approximate area under the curve}}{4-0} = \dfrac{100\pi}{4}$

$\dfrac{100\pi}{4} - \dfrac{1}{4}\int_0^4 V(r)dr$

$= 9\pi \text{ cm}$ **1A**

OR

$\dfrac{36\pi}{4} = 9\pi$ **1A**

Question 2

a. i.

Period of 1 cycle = 11 years

Number of cycles = $\dfrac{2008-1755}{11}=23$ **1A**

a. ii.

$N\in[10,110]$ **1A**

b.

$\text{Period}=\dfrac{2\pi}{n}$

$n=\dfrac{2\pi}{11}$, as required **1M**

c.

- The amplitude is 50, therefore a = 50. **1M**
- The cosine graph has been translated 60 units up (average value for a complete cycle is 60), therefore b = 60. **1M**

d.

$N(t)=60-50\cos\left(\dfrac{2\pi t}{11}\right)$

$N(2)=39$ **1A**

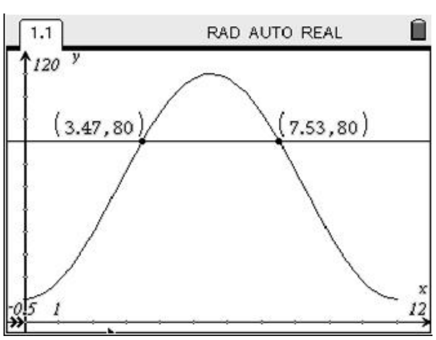
e.

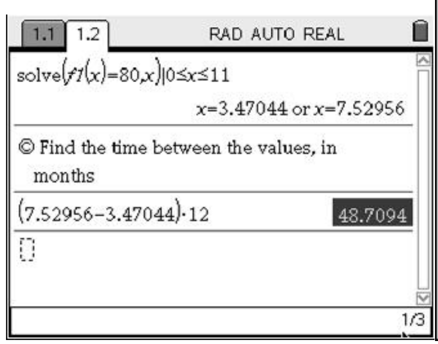
Using CAS, a graphical or algebraic method may be used to find where $N(t)=80$.

$N(t)=80$ for $t\approx 3.47$ or $t\approx 7.35$ **1M**

$\text{time}=(7.52956-3.47044)\times 12$

$\quad =49$ months **1A**

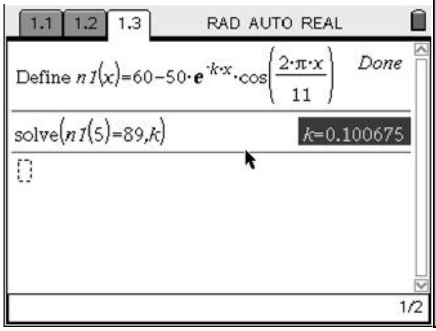


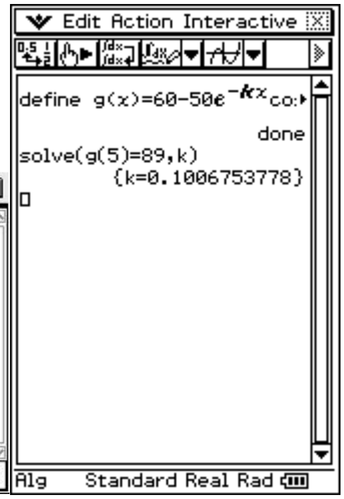


f.

Solve for *k*, $N_1(5) = 89$

$k = 0.10$, as required. **1M**





g.

Using CAS, a graphical or algebraic method may be used.

Solve for *t*, $N(t) = N_1(t)$,

$t \in \{0, 2.75, 8.75\}$ **1M**

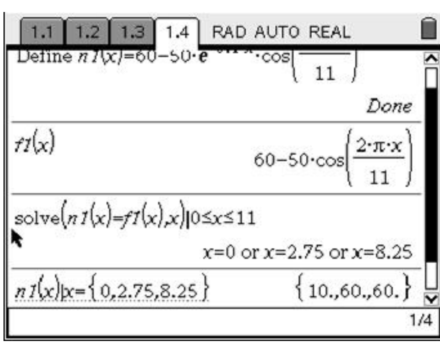
$N(0) = 10$

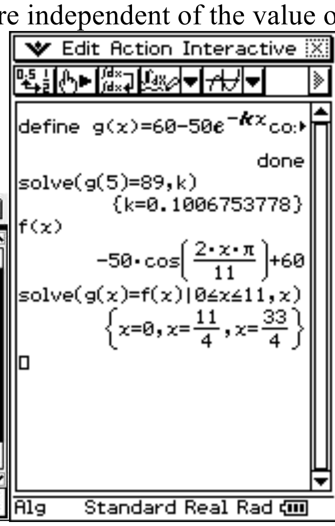
$N(2.75) = N(8.75) = 60$

Points of intersection are

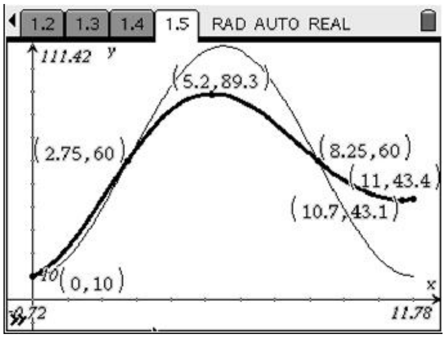
(0.0, 10.0), (2.8, 60.0), (8.8, 0.0) **1A**

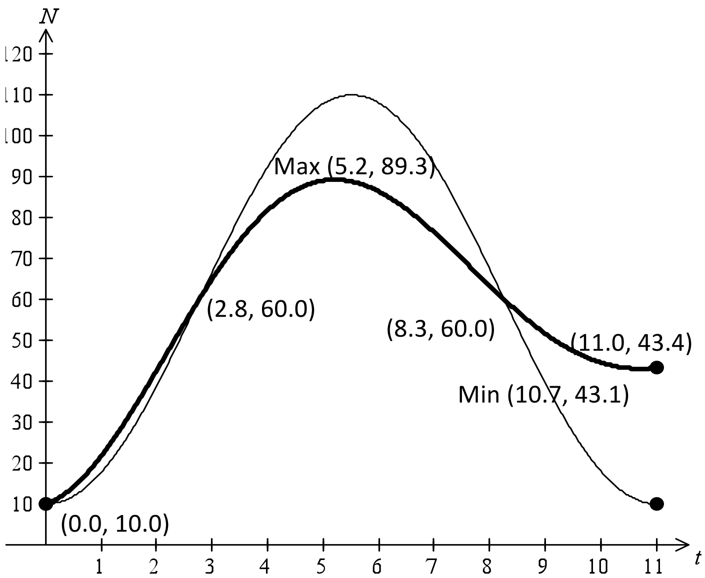
Note that the points of intersection are independent of the value of *k*, because *k* does not alter the period.





h.





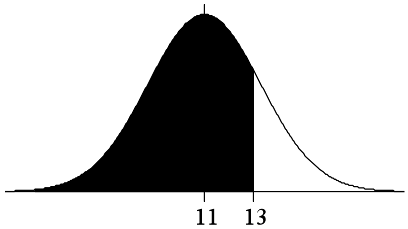
Correct shape	**1A**
Intersection points correctly placed and labelled	**1A**
Turning points and intersection points correctly labelled	**1A**
Endpoints with closed circles and correctly labelled	**1A**

Question 3

a. i.

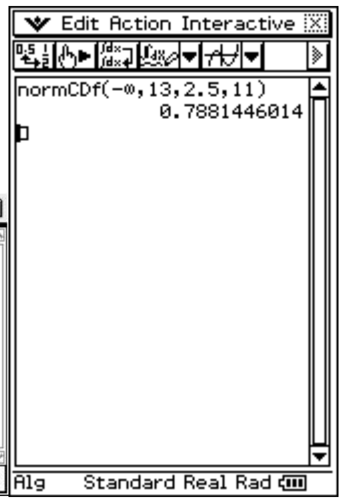
Let X be the healing time, in days.

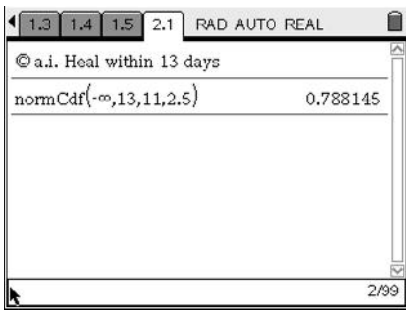
$X \sim N\left(11, 2.5^2\right)$



1M

$\Pr\left(X < 13\right) = 0.7881$ **1A**

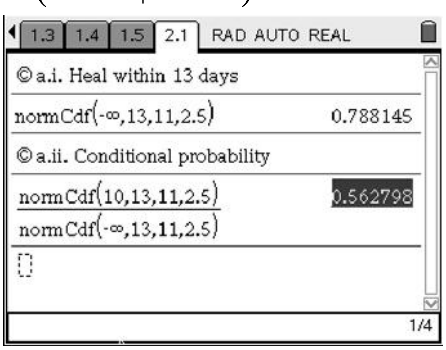




a. ii.

$$\Pr\left(X > 10 \mid X < 13\right) = \frac{\Pr\left(10 < X < 13\right)}{\Pr\left(X < 13\right)}$$ **1M**

$\Pr\left(X > 10 \mid X < 13\right) = 0.563$ **1A**

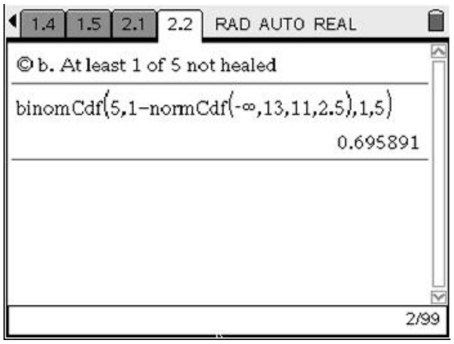


b.

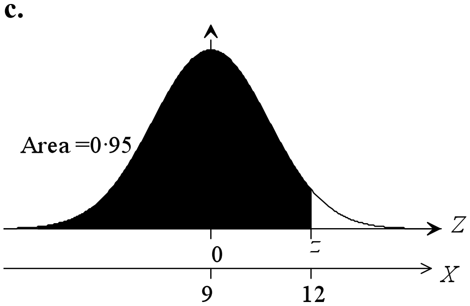
Let Y be the number of patients with incision **not** healed at the time of discharge.

$Y \sim Bi(5, 1-0.788145...)$ **1M**

$\Pr(Y \geq 1) = 0.696$ **1A**



c.



$\Pr(Z < z) = 0.95$ **1M**

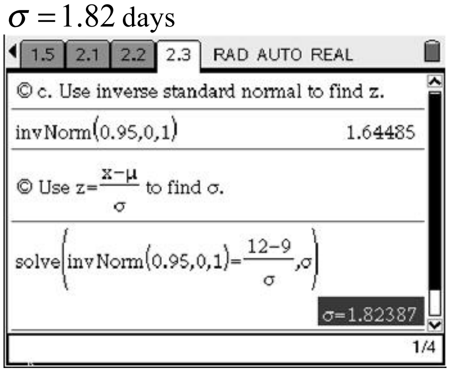
$\Pr(Z < z) = 1.64485$

$z = \dfrac{x-\mu}{\sigma}$

$1.64485 = \dfrac{12-9}{\sigma}$ **1M**

$\sigma = \dfrac{12-9}{1.64485}$

$\sigma = 1.82$ days **1A**



Alternative solutions: there are several other methods of solving this problem using the CAS device. A graphical approach is shown below.

Let W be the healing time using this technology

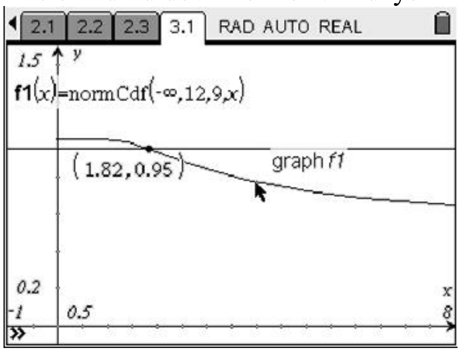
$W \sim N\left(9, x^2\right)$

Solve (graphically) for x,

$\Pr\left(W < 12\right) = 0.95$ **2M**

$x = 1.82$

The standard deviation is 1.82 days **1A**



d.

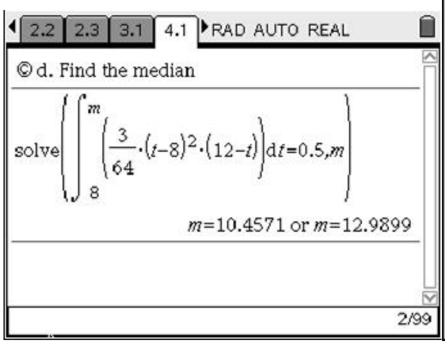
Let m be the median time in hospital

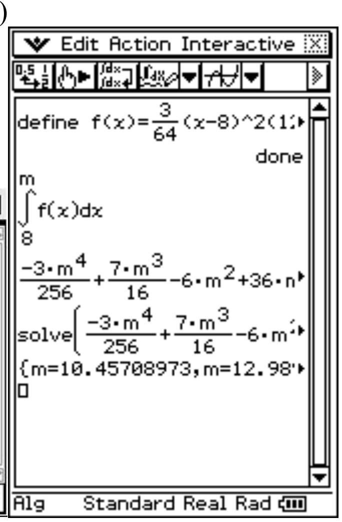
Solve for m,

$\int_8^m f\left(t\right)dt = 0.5$, or alternatively, $\int_m^{12} f\left(t\right)dt = 0.5$ **1M**

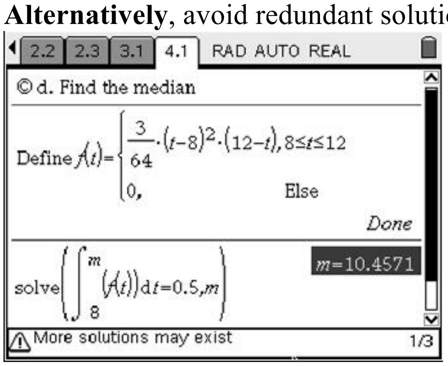
$t = 10.46$ days **1A**

(Reject solutions outside the domain)





Alternatively, avoid redundant solutions by defining (storing) the function with its domain.



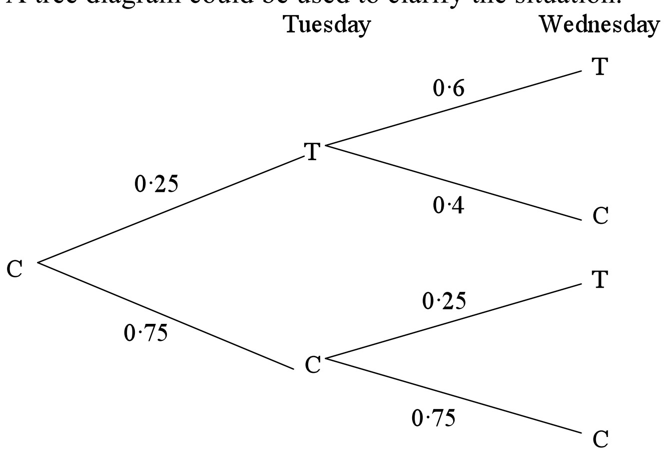
e.

	Canteen Today	*Tracebook* Today
Canteen Tomorrow	0.75	0.4
Tracebook Tomorrow	0.25	0.6

$\Pr(C,T)+\Pr(T,C)=(0.75\times 0.25)+(0.25\times 0.4)$ **1M**

$\Pr(C,T)+\Pr(T,C)=0.2875=\dfrac{23}{80}$ **1A**

A tree diagram could be used to clarify the situation.



f.

For a Markov chain with transition matrix T and initial state matrix S_0, the n^{th} state is given by $S_n = T^n \times S_0$.

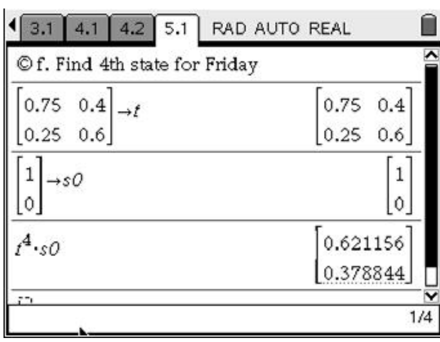
$T = \begin{bmatrix} 0.75 & 0.4 \\ 0.25 & 0.6 \end{bmatrix}$ and $S_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, or alternatively, $T = \begin{bmatrix} 0.6 & 0.25 \\ 0.4 & 0.75 \end{bmatrix}$ and $S_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ **1M**

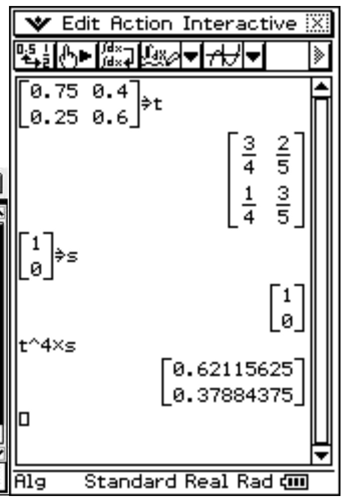
$S_4 = T^4 \times S_0$ **1M**

$S_4 = \begin{bmatrix} 0.62 \\ 0.38 \end{bmatrix} \begin{matrix} \leftarrow \textit{canteen} \\ \leftarrow \textit{Tracebook} \end{matrix}$

The probability of canteen on Friday is 0.62. **1A**

Note that the solution may also be obtained from T^4, without using S_0.





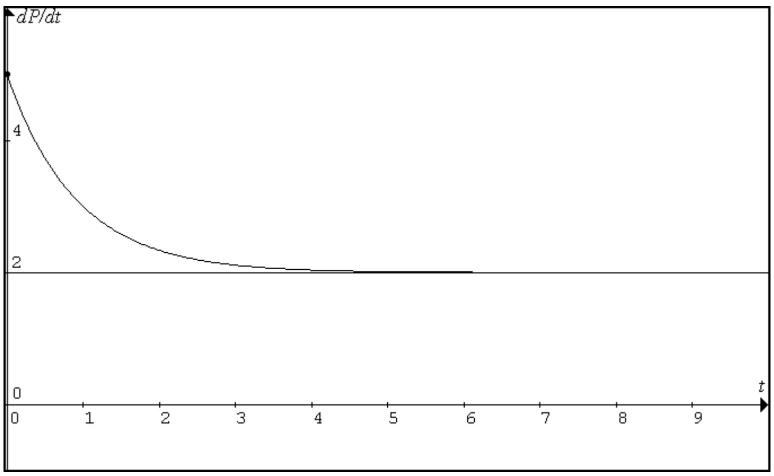
Question 4

a. i. (0, 5) **0.5A**

Asymptote $\frac{dP}{dt}=2$ **0.5A**

Shape **1A**

Round down



ii. As $t \to \infty, \frac{dP}{dt} \to 2$

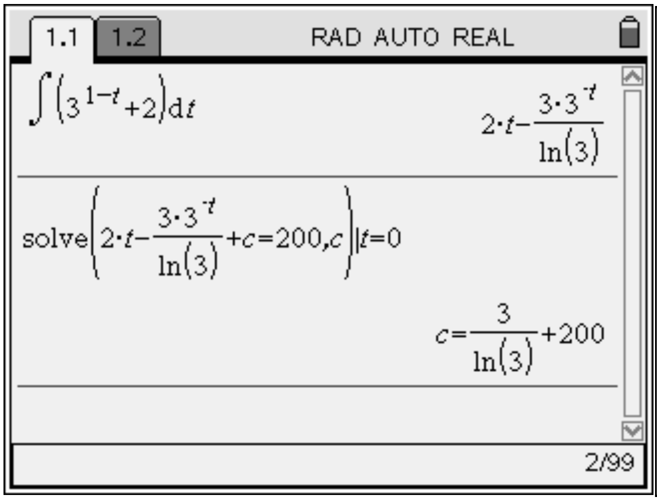
2000 insects per year **1A**

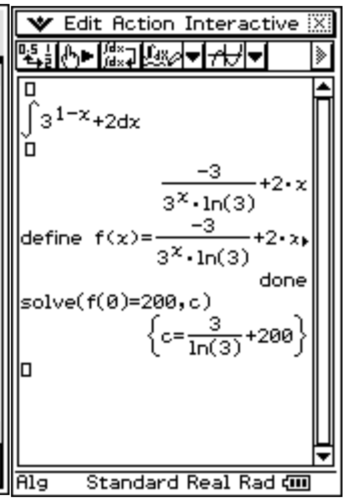
b. i. $P=\int\left(3^{1-t}+2\right)dt$

$$=2t-\frac{3\times3^{-t}}{\log_e(3)}+c$$ **1A**

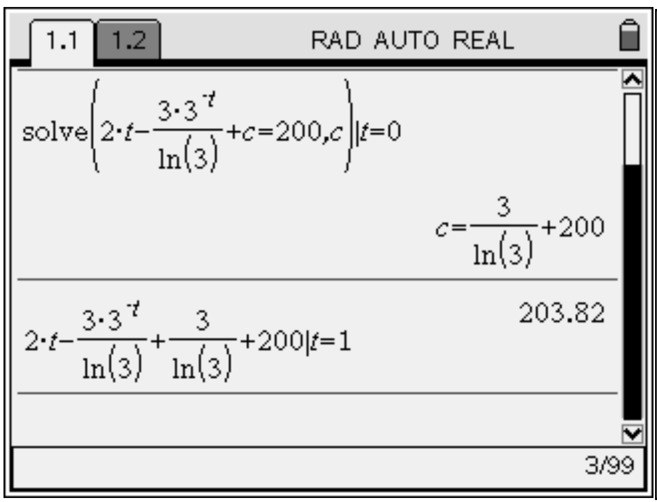
Solve $200=2t-\frac{3\times3^{-t}}{\log_e(3)}+c$ when $t=0$

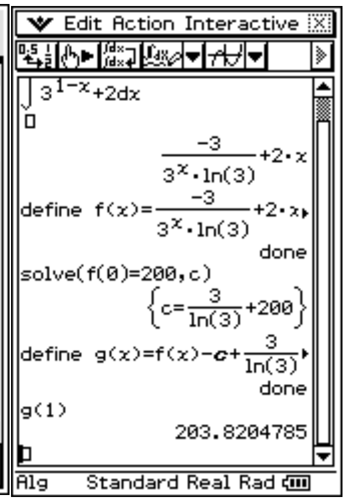
$$P=2t-\frac{3\times3^{-t}}{\log_e(3)}+200+\frac{3}{\log_e(3)}$$ **1A**





ii. $P(1) \approx 204\,000$ insects **1A**





c. i. Solve $\frac{dP}{dt} = t$ for t

$t = 2.25$ years **1A**

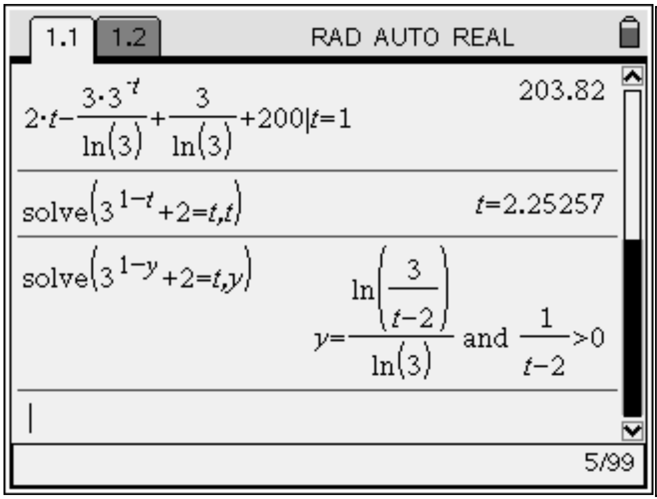
ii. Let $y = \frac{dP}{dt}$

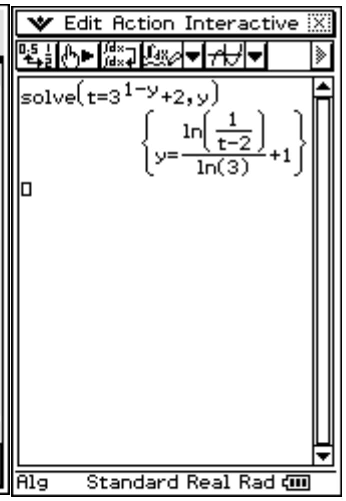
Inverse: swap t and y

Solve $t = 3^{1-y} + 2$ for y **1A**

$$\frac{dP}{dt} = \frac{\log_e\left(\frac{3}{t-2}\right)}{\log_e(3)} = 1 - \frac{\log_e(t-2)}{\log_3(3)}$$

Accept equivalent forms **1A**





iii. $t = 5$

January 1st 2014 **1A**

iv. $P = \int \left(\frac{\log_e\left(\frac{3}{t-2}\right)}{\log_e(3)} \right) dt$ **1H**

$$= \frac{(t-2)\log_e\left(\frac{1}{t-2}\right) + (\log_e(3)+1)t - 2}{\log_e(3)} + c$$

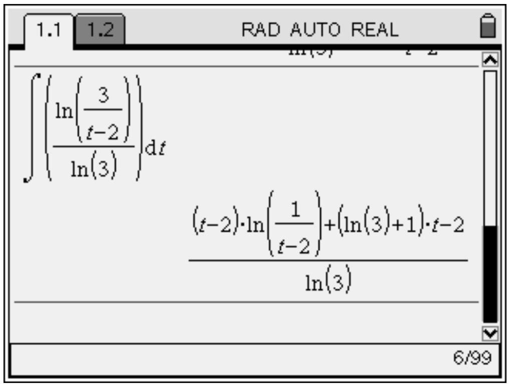
Solve $P = 2t - \frac{3 \times 3^{-t}}{\log_e(3)} + 200 + \frac{3}{\log_e(3)}$ for $t = 2.2525\ldots$

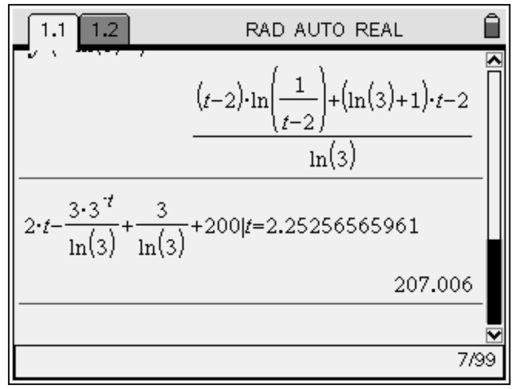
$P = 207.006\ldots$ **1M**

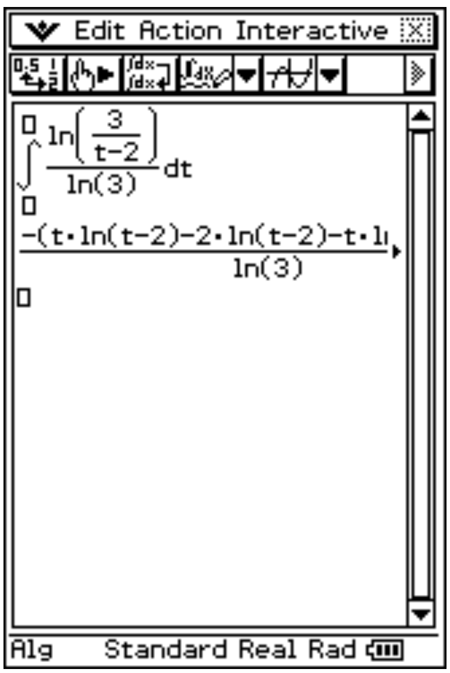
Solve $207.006\ldots = \frac{(t-2)\log_e\left(\frac{1}{t-2}\right) + (\log_e(3)+1)t - 2}{\log_e(3)} + c$ for c when $t = 2.2525\ldots$ **1M**

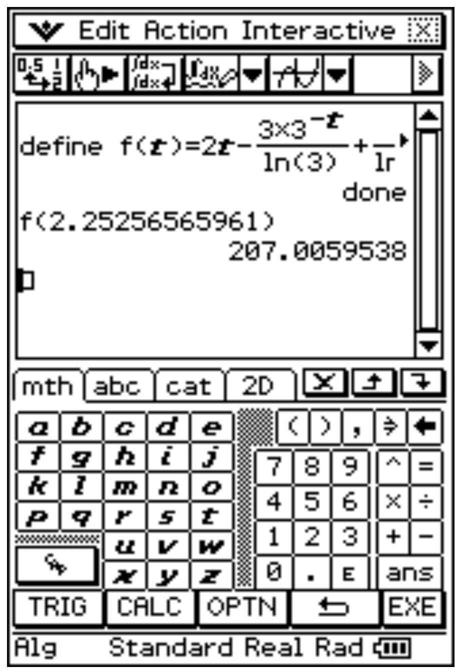
$c = 204.207\ldots$

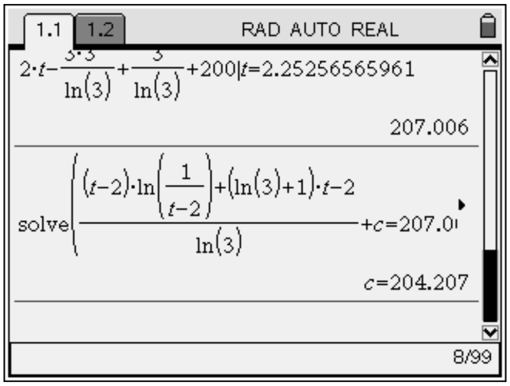
$P(5) = 208\,938$ insects **1A**

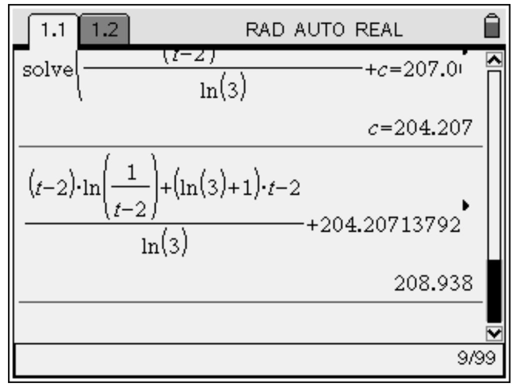

1.1 1.2 RAD AUTO REAL
$\int \left(\frac{\ln\left(\frac{3}{t-2}\right)}{\ln(3)} \right) dt$
$\frac{(t-2)\cdot\ln\left(\frac{1}{t-2}\right)+(\ln(3)+1)\cdot t-2}{\ln(3)}$
6/99

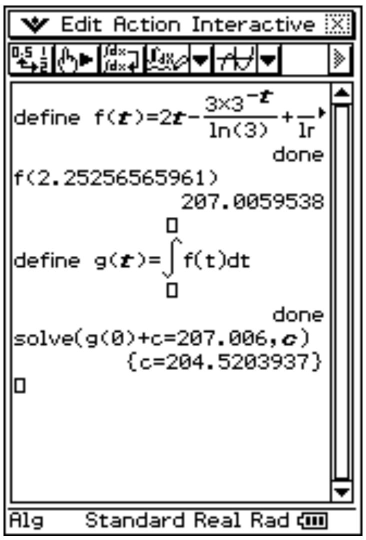

1.1 1.2 RAD AUTO REAL
$\frac{(t-2)\cdot\ln\left(\frac{1}{t-2}\right)+(\ln(3)+1)\cdot t-2}{\ln(3)}$
$2\cdot t-\frac{3\cdot 3^{-t}}{\ln(3)}+\frac{3}{\ln(3)}+200|t=2.25256565961$
207.006
7/99


Edit Action Interactive
$\int \frac{\ln\left(\frac{3}{t-2}\right)}{\ln(3)} dt$
$\frac{-(t\cdot\ln(t-2)-2\cdot\ln(t-2)-t\cdot\ln}{\ln(3)}$
Alg Standard Real Rad


Edit Action Interactive
define $f(t)=2t-\frac{3\times 3^{-t}}{\ln(3)}+\frac{}{\ln}$
done
f(2.25256565961)
207.0059538
mth abc cat 2D
TRIG CALC OPTN EXE
Alg Standard Real Rad







d. The population will start to decrease from January 1st 2014 until they become extinct. Hence the scientists are predicting climate change will not favour the insects. (anything suitable) **1A**