MAV Trial Examination Papers 2010
Mathematical Methods (CAS) Examination 1
SOLUTIONS

Question 1

a. $\Pr(A' \mid B) = \dfrac{\Pr(A' \cap B)}{\Pr(B)}$ **1M**

$= \dfrac{\Pr(A') \times \Pr(B)}{\Pr(B)}$, as the events are independent

$= \Pr(A')$

$= 0.7$ **1A**

b. Let b represent black and w white

$\Pr(bb) + \Pr(ww) = \dfrac{2}{6} \times \dfrac{1}{5} + \dfrac{4}{6} \times \dfrac{3}{5}$ **1M**

$= \dfrac{14}{30} = \dfrac{7}{15}$ **1A**

Question 2

a. $f(x) = \left(\sin(2x) + 1\right)^2$

Using the chain rule, **1M**

$f'(x) = 2\left(\sin(2x) + 1\right) \times 2\cos(2x)$

$= 4\left(\sin(2x) + 1\right)\cos(2x)$ **1A**

Therefore

$f'\left(\dfrac{\pi}{2}\right) = 4\left(\sin(\pi) + 1\right)\cos(\pi)$

$= 4(0 + 1) \times -1$

$f'\left(\dfrac{\pi}{2}\right) = -4$ **1A**

b. i. $y = (x^2 - 2x)e^x$

Using the product rule,

Let $u = (x^2 - 2x)$, therefore $\dfrac{du}{dx} = 2x - 2$

$v = e^x$, therefore $\dfrac{dv}{dx} = e^x$

$\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$ **1M**

$\dfrac{dy}{dx} = (x^2 - 2x)e^x + (2x - 2)e^x$

$\dfrac{dy}{dx} = (x^2 - 2)e^x$ **1A**

b. ii. From the previous answer,

$$\frac{d\left((x^2-2x)e^x \right)}{dx} = x^2e^x - 2e^x$$

Take the integral of both sides, with respect to *x*.

$(x^2-2x)e^{2x} + c = \int \left(x^2e^x \right) dx - 2e^x$, where *c* is a constant. **1M**

Rearrange to make $\int \left(x^2e^x \right) dx$ the subject

$\int \left(x^2e^x \right) dx = (x^2 - 2x + 2)e^x + c$ **1A**

Note that any value of *c* (including zero) is acceptable, as *an* antiderivative is asked for.

Question 3

a. i. Area $= \int_1^7 \left(1 + x^{-2} \right) dx$ **1M**

$= \left[x - \frac{1}{x} \right]_1^7$

$= \left[\left(7 - \frac{1}{7} \right) - (1-1) \right]$ **1A**

Area $= 6\frac{6}{7} = \frac{48}{7}$, as required

a. ii. Average value $= \frac{1}{7-1}\int_1^7 f(x)dx$

Average value $= \frac{1}{6} \times \frac{48}{7} = \frac{8}{7}$ **1A**

b.

	Domain	Range
f	$(0,\infty)$	$(1,\infty)$
f^{-1}	$(1,\infty)$	$(0,\infty)$

Domain of f^{-1} is $(1,\infty)$. **1A**

To find the rule of f^{-1}, interchange the *x* and *y* values and make *y* the subject.

$x = 1 + \frac{1}{y^2}$ **1M**

$y = \pm\frac{1}{\sqrt{x-1}}$. However, since the domain of f^{-1} is $(1,\infty)$, reject the negative case.

$f^{-1}(x) = \frac{1}{\sqrt{x-1}}$ **1A**

Question 4

Given that: $\dfrac{dV}{dt} = 32$ m^3/hour … eqn(1)

Know that: $V = \pi r^2 h$

When $r = 4$ m, $V = 16\pi h$,

$\dfrac{dV}{dh} = 16\pi$ … eqn(2) **1M**

Need: $\dfrac{dh}{dt} = \dfrac{dh}{dV} \times \dfrac{dV}{dt}$

Substitute equations (1) and (2) **1M**

$\dfrac{dh}{dt} = \dfrac{1}{16\pi} \times 32$

$\dfrac{dh}{dt} = \dfrac{2}{\pi}$ m/hour **1A**

Question 5

$np = 3$ and variance, $npq = \dfrac{3}{4}$ **1M**

Hence, $3q = \dfrac{3}{4}$, $q = \dfrac{1}{4}$

$p = \dfrac{3}{4}$ and $n = 4$

$\Pr(X = 2) = {}^4C_2 \left(\dfrac{3}{4}\right)^2 \left(\dfrac{1}{4}\right)^2$ **1H**

$= 6 \times \dfrac{9}{16} \times \dfrac{1}{16} = \dfrac{27}{128}$ **1A**

Question 6

a. $f(a) = f(b)$

$2a = 30 - 3b$

$b = \dfrac{30 - 2a}{3} = 10 - \dfrac{2}{3}a$ **1A**

b. Width $= 2a$

Length $= \left(10 - \dfrac{2}{3}a\right) - a$

$= 10 - \dfrac{5}{3}a$ **1M**

Area $=$ Length $\times$ width

$A = 2a\left(10 - \dfrac{5}{3}a\right)$ **1A**

$A = 20a - \dfrac{10a^2}{3}$, as required

c. For the maximum area, $\dfrac{dA}{da} = 0$.

$20-\frac{20a}{3}=0$

$a=3$

The area of the inscribed rectangle will be a maximum when $a=3$. **1A**

Maximum area = $A(3)$

$A(3)=20\times3-\frac{10\times3^2}{3}$

$=30$

The maximum area is 30 units^2. **1A**

Question 7

$T\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=\begin{bmatrix}2&0\\0&-1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}+\begin{bmatrix}-\pi\\3\end{bmatrix}$

Let (x',y') be the image of (x,y) under *T*.

$x'=2x-\pi$, hence $x=\frac{x'+\pi}{2}$ … eqn(1)

$y'=-y+3$, hence $y=3-y'$ … eqn(2) **1M**

Substitute equations (1) and (2) in $y=\cos(x)$.

$3-y'=\cos\left(\frac{x'+\pi}{2}\right)$ **1M**

$y'=3-\cos\left(\frac{1}{2}(x'+\pi)\right)$

Hence, $h(x)=-\cos\left(\frac{1}{2}(x+\pi)\right)+3$ **1A**

Alternative solution

The solution is of the form $h(x)=a\cos(n(x-h))+k$, where *a*, *n*, *h* and *k* are real constants.

By recognition, *T* involves the following sequence of transformations:

- A dilation of factor 2 from the *y*-axis, hence $n=\frac{1}{2}$
- A reflection in the *x*-axis, hence $a=-1$ **1M**
- Translations π units left and 3 units up, hence $h=-\pi$ and $k=3$. **1M**

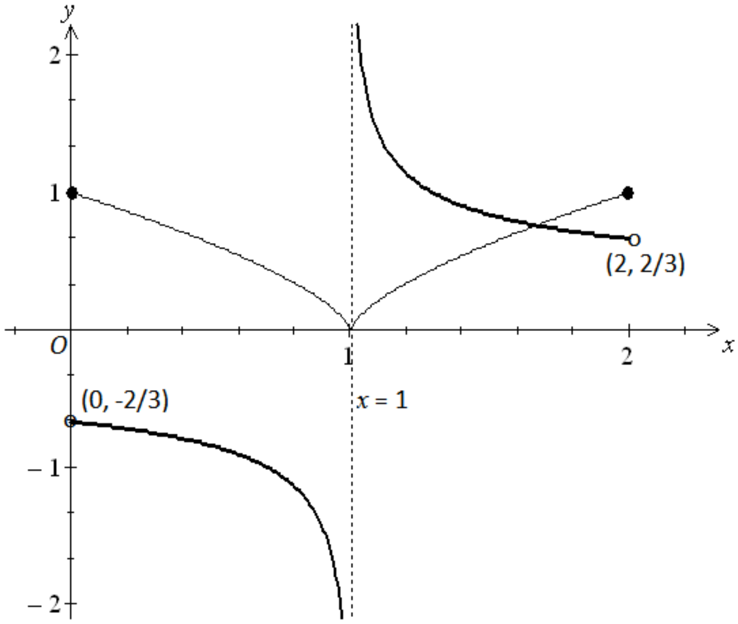
Hence, $h(x)=-\cos\left(\frac{1}{2}(x+\pi)\right)+3$ **1A**

Question 8

a. $f'(x)=\dfrac{2}{3(x-1)^{\frac{1}{3}}}, x\neq 1$ **1A**

b. Shape and Asymptote at $x = 1$ **1A**

Open circles for endpoints $\left(0,-\dfrac{2}{3}\right)$ and $\left(2,\dfrac{2}{3}\right)$ **1A**



Question 9

$$\begin{bmatrix}0.2 & 0.9\\0.8 & 0.1\end{bmatrix}\begin{bmatrix}x\\1-x\end{bmatrix}=\begin{bmatrix}x\\1-x\end{bmatrix}$$ **1M**

$0.2x+0.9(1-x)=x$

$-1.7x=-0.9$

$x=\dfrac{9}{17}$ **1A**

Alternatively, for the general case with transition matrix $\begin{bmatrix}1-a & b\\a & 1-b\end{bmatrix}$, the long-term state matrix is given by $\begin{bmatrix}\dfrac{b}{a+b}\\1-\dfrac{b}{a+b}\end{bmatrix}$ **1M**

In this case, $a=0.8$ and $b=0.9$.

Hence $x=\dfrac{0.9}{0.8+0.9}=\dfrac{9}{17}$ **1A**

Question 10

$g: R^+ \to R,\ g(x) = x - \log_e(x)$

The equation of a **tangent** to the graph of g is $y = -\dfrac{x}{2} + k$.

The gradient of the tangent, $g'(x) = -\dfrac{1}{2}$.

Therefore, $1 - \dfrac{1}{x} = -\dfrac{1}{2}$ **1M**

$\dfrac{1}{x} = \dfrac{3}{2}$

$x = \dfrac{2}{3}$

The tangent intersects the graph of g at the point with coordinates

$\left(\dfrac{2}{3}, \dfrac{2}{3} - \log_e\left(\dfrac{2}{3}\right)\right)$ **1A**

The equation of the tangent is of the form

$y - y_1 = m(x - x_1)$, hence

$y - \left(\dfrac{2}{3} - \log_e\left(\dfrac{2}{3}\right)\right) = -\dfrac{1}{2}\left(x - \dfrac{2}{3}\right)$ **1M**

$y = -\dfrac{x}{2} + \dfrac{1}{3} + \left(\dfrac{2}{3} - \log_e\left(\dfrac{2}{3}\right)\right)$

$y = -\dfrac{x}{2} + 1 - \log_e\left(\dfrac{2}{3}\right)$

Therefore, $k = 1 - \log_e\left(\dfrac{2}{3}\right)$ **1A**