

Year 2010
VCE
Mathematical Methods
CAS
Solutions
Trial Examination 2



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SECTION 1

ANSWERS

1		A		B		C		D		E
2		A		B		C		D		E
3		A		B		C		D		E
4		A		B		C		D		E
5		A		B		C		D		E
6		A		B		C		D		E
7		A		B		C		D		E
8		A		B		C		D		E
9		A		B		C		D		E
10		A		B		C		D		E
11		A		B		C		D		E
12		A		B		C		D		E
13		A		B		C		D		E
14		A		B		C		D		E
15		A		B		C		D		E
16		A		B		C		D		E
17		A		B		C		D		E
18		A		B		C		D		E
19		A		B		C		D		E
20		A		B		C		D		E
21		A		B		C		D		E
22		A		B		C		D		E

SECTION 1

Question 1

Answer D

$$f(x) = 1 + 3g(x) \text{ for } 0 \leq x \leq 4$$

$$\begin{aligned} & \int_0^4 (f(x) - g(x)) dx \\ &= \int_0^4 (1 + 3g(x) - g(x)) dx \\ &= \int_0^4 (1 + 2g(x)) dx \\ &= [x]_0^4 + 2 \int_0^4 (g(x)) dx \\ &= 4 + 2 \int_0^4 (g(x)) dx \end{aligned}$$

Question 2

Answer E

All of **A.** **B.** **C.** and **D.** are true, the graph is not defined at $x = -a$,
and the graph is not continuous at $x = -a$
E. is false.

Question 3

Answer E

$$f(x+h) \approx f(x) + hf'(x)$$

$$\text{with } f(x) = \cos(x) \quad f'(x) = -\sin(x) \quad x = \frac{\pi}{6} \quad (30^\circ) \quad h = -\frac{\pi}{180} \quad (-1^\circ)$$

$$\cos(29^\circ) = f\left(\frac{\pi}{6}\right) + \left(-\frac{\pi}{180}\right) f'\left(\frac{\pi}{6}\right)$$

$$\cos(29^\circ) = \cos\left(\frac{\pi}{6}\right) + \left(-\frac{\pi}{180}\right) \left(-\sin\left(\frac{\pi}{6}\right)\right)$$

$$\cos(29^\circ) = \frac{\sqrt{3}}{2} + \frac{\pi}{360}$$

Question 4

Answer B

$$f(x) = \frac{\sqrt{x^2+16}}{g(x)} \quad \text{using the quotient rule} \quad g(3)=3 \quad \text{and} \quad g'(3)=-1$$

$$f'(x) = \frac{\frac{1}{2} \frac{2x}{\sqrt{x^2+16}} g(x) - g'(x) \sqrt{x^2+16}}{[g(x)]^2}, \quad \text{if } x=3 \quad \sqrt{x^2+16}=5$$

$$f'(3) = \frac{\frac{3}{5} \times 3 + 1 \times 5}{9} = \frac{1}{9} \left(\frac{9}{5} + 5 \right) = \frac{34}{45}$$

Question 5

Answer C

$$\frac{dy}{dx} = \frac{1}{2x-9} \Rightarrow y = \int \frac{1}{2x-9} dx$$

$$y = \frac{1}{2} \log_e |2x-9| + C, \quad \text{now when } x = \frac{5}{2} \quad y = 0$$

$$0 = \frac{1}{2} \log_e |5-9| + C = \frac{1}{2} \log_e |-4| + C = \frac{1}{2} \log_e (4) + C \Rightarrow C = -\frac{1}{2} \log_e (4)$$

$$y = \frac{1}{2} \log_e |2x-9| - \frac{1}{2} \log_e (4) = \frac{1}{2} \log_e \left(\frac{|2x-9|}{4} \right)$$

$$\text{when } x=0 \quad y = \frac{1}{2} \log_e \left(\frac{|-9|}{4} \right) = \frac{1}{2} \log_e \left(\frac{9}{4} \right) = \log_e \left(\sqrt{\frac{9}{4}} \right) = \log_e \left(\frac{3}{2} \right)$$

Question 6

Answer C

$$\text{Let } f(x) = x^5 - bx + c \quad f(-1) = 1$$

$$f(-1) = (-1)^5 + b + c = -1 + b + c = 1 \Rightarrow b + c = 2$$

$$f'(x) = 5x^4 - b \quad f'(-1) = 0 \quad \text{since it is a stationary point.}$$

$$f'(-1) = 5(-1)^4 - b = 5 - b = 0 \Rightarrow b = 5 \quad \text{and} \quad c = -3$$

Question 7

Answer A

$$V = L^3 \Rightarrow \frac{dV}{dL} = 3L^2 \quad \text{given that} \quad \frac{dV}{dt} = p$$

$$\frac{dL}{dt} = \frac{dL}{dV} \cdot \frac{dV}{dt} = \frac{p}{3L^2}$$

Question 8

Answer E

None of **A. B. C.** and **D.** are true, however since

$$y = -3\log_e\left(\frac{x}{2}\right) = -3(\log_e(x) - \log_e(2)) = -3\log_e(x) + 3\log_e(2) = -3\log_e(x) + \log_e(8)$$

From the graph of $y = \log_e(x)$, a reflection in the x -axis, gives $y = -\log_e(x)$ then a dilation by a scale factor of 3 parallel to the y -axis, gives $y = -3\log_e(x)$, followed by a translation of $\log_e(8)$ units up and parallel to the y -axis, gives $y = -3\log_e(x) + \log_e(8)$.

Question 9

Answer C

$$f: \quad y = \frac{1}{x+a} \quad \text{dom } f = R \setminus \{-a\} = \text{ran } f^{-1}$$

$$f^{-1} \quad x = \frac{1}{y+a} \quad \text{transposing}$$

$$y+a = \frac{1}{x} \quad y = \frac{1}{x} - a = \frac{1-ax}{x} \quad \text{but } \text{ran } f = \text{dom } f^{-1} = R \setminus \{0\}, \text{ so that}$$

$$f^{-1}: R \setminus \{0\} \rightarrow R, \quad f^{-1}(x) = \frac{1-ax}{x}$$

Question 10

Answer D

All of **A. B. C.** and **E.** are true, however if $a < 0$ and n is even, the point (h, k) is a local maximum.

Question 11

Answer A

at $x = \alpha$ $f'(\alpha) = 0$	at $x = \beta$ $f'(\beta) = 0$
if $x < \alpha$ $f'(x) > 0$	if $x < \beta$ $f'(x) < 0$
if $x > \alpha$ $f'(x) < 0$	if $x > \beta$ $f'(x) > 0$
local maximum at $x = \alpha$	local minimum at $x = \beta$

Question 12

Answer A

For $y = 2 \cos(2x)$, the average rate of change

$$\frac{y\left(\frac{\pi}{8}\right) - y(0)}{\frac{\pi}{8} - 0} = \frac{2 \cos\left(\frac{\pi}{4}\right) - 2 \cos(0)}{\frac{\pi}{8}} = \frac{8 \left(2 \left(\frac{\sqrt{2}}{2} \right) - 2 \right)}{\pi} = \frac{8(\sqrt{2} - 2)}{\pi}$$

Question 13

Answer C

$$v(t) = \frac{72}{(3t+2)^2} \quad \text{initial speed } v(0) = \frac{72}{4} = 18 \text{ m/s}$$

$$s = \int_0^2 \frac{72}{(3t+2)^2} dt$$

$$s = \left[\frac{72}{-3(3t+2)} \right]_0^2 = -24 \left(\frac{1}{8} - \frac{1}{2} \right) = 9$$

Question 14

Answer B

$$g(x) = x^2 \quad g'(x) = 2x \quad f(x) = \cos(2x) \quad f'(x) = -2 \sin(2x)$$

$$\frac{d}{dx} [g(f(x))] = \frac{d}{dx} [g(\cos(2x))]$$

$$= \frac{d}{dx} [\cos^2(2x)]$$

$$= -4 \sin(2x) \cos(2x) = 2(-2 \sin(2x)) \cos(2x) = 2 f'(x) f(x)$$

$$\text{Alternatively } \frac{d}{dx} [g(f(x))] = f'(x) g'(f(x)) = -2 \sin(2x) 2(\cos(2x)) = 2 f(x) f'(x)$$

Question 15

Answer E

$$y = \frac{1}{x} \text{ into } y = \frac{4}{8-2x} - 2 \Rightarrow y+2 = \frac{-2}{x-4} \Rightarrow \frac{y+2}{-2} = \frac{1}{x-4}$$

$$y' = \frac{y+2}{-2} \text{ and } x' = x-4 \text{ become}$$

$$x = x' + 4 \text{ and } y = -2y' - 2 \text{ in matrix form}$$

$$T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} + \begin{bmatrix} 4 \\ -2 \end{bmatrix}$$

Question 16

Answer A

$$\text{Let } y_1 = x^2 + mx + m \text{ and } y_2 = 2mx + 2m \quad y_1 = y_2$$

$$x^2 + mx + m = 2mx + 2m$$

$$x^2 - mx - m = 0$$

$$\Delta = b^2 - 4ac = (-m)^2 - 4(1)(-m) = m^2 + 4m = m(m + 4)$$

For the graphs to intersect at more than one point, we require

$$\Delta > 0 \Rightarrow m > 0 \text{ or } m < -4 \text{ or } m \in (-\infty, -4) \cup (0, \infty)$$

Question 17

Answer D

$$X \stackrel{d}{=} \text{Bi}(n=7, p=0.25)$$

$$\Pr(\text{more than one}) = \Pr(X > 1)$$

$$\Pr(X > 1) = 1 - [\Pr(X = 0) + \Pr(X = 1)]$$

$$\Pr(X > 1) = 1 - (0.75^7 + 7(0.25)(0.75)^6)$$

$$\text{Now } \Pr(X = 0) = q^n \text{ and } \Pr(X = 1) = npq^{n-1}$$

$$n = 7, q = 0.75 \text{ and } p = 0.25$$

Question 18

Answer D

$$\Pr(A \cap B) = \frac{1}{5} \neq \Pr(A)\Pr(B) = \frac{8}{15} \times \frac{1}{3} \text{ not independent}$$

$$\Pr(A \cap B) \neq 0 \text{ not mutually exclusive}$$

$$\Pr(A' \cap B') = \frac{1}{3}$$

$$\Pr(A' \cup B') = \frac{7}{15} + \frac{2}{3} - \frac{1}{3} = \frac{4}{5}$$

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$$

$$\Pr(A \cup B) = \frac{8}{15} + \frac{1}{3} - \frac{1}{5} = \frac{2}{3}$$

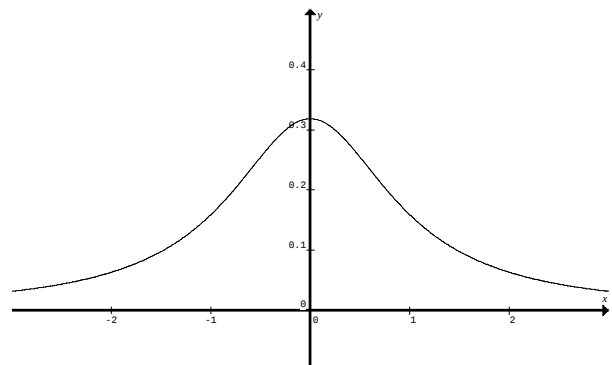
	A	A'	
B	$\frac{1}{5}$	$\frac{2}{15}$	$\frac{1}{3}$
B'	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{3}$
	$\frac{8}{15}$	$\frac{7}{15}$	

Question 19

Answer A

$$E(X) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x}{1+x^2} dx, \text{ this is undefined and}$$

does not exist, the mode is $x = 0$



Question 20

Answer B

$$\text{Let } A_1 = \int_{-1}^0 f(x) dx < 0 \quad A_2 = \int_0^3 f(x) dx \quad A_3 = \int_3^5 f(x) dx < 0$$

The required area is $A = -A_1 + A_2 - A_3$

$$A = -\int_{-1}^0 f(x) dx + \int_0^3 f(x) dx - \int_3^5 f(x) dx$$

$$A = \int_0^{-1} f(x) dx + \int_0^3 f(x) dx + \int_5^3 f(x) dx$$

Question 21

Answer B

$$X \stackrel{d}{=} N(\mu, \sigma^2), \text{ the normal curve is } f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \text{ for } x \in R$$

$$\text{So that } f(x) = \frac{1}{\sqrt{10\pi}} e^{-\frac{(x-5)^2}{10}} \text{ for } x \in R \Rightarrow \sqrt{10\pi} = \sigma\sqrt{2\pi}$$

$$\mu = 5 \text{ and } \sigma = \sqrt{5}$$

Question 22

Answer C

X	0	1	2
$\Pr(X = x)$	$4c$	$3c$	$2c$

$$\sum \Pr(X = x) = 4c + 3c + 2c = 9c = 1 \Rightarrow c = \frac{1}{9}$$

X	0	1	2
$\Pr(X = x)$	$\frac{4}{9}$	$\frac{1}{3}$	$\frac{2}{9}$

Since $\Pr(X = 0) > \Pr(X = 1)$ and $\Pr(X = 0) > \Pr(X = 2)$ the mode is zero

Since $\Pr(X = 0) < \frac{1}{2}$ but $\Pr(X = 0) + \Pr(X = 1) > \frac{1}{2}$ the median is 1

END OF SECTION 1 SUGGESTED ANSWERS

SECTION 2

Question 1

a.i $f(x) = ax^3 - 6ax + 12$ for $a > 0$

$$f'(x) = 3ax^2 - 6a$$

for stationary points $f'(x) = 0$

$$f'(x) = 3ax^2 - 6a = 0$$

$$3a(x^2 - 2) = 0$$

$$x^2 = 2$$

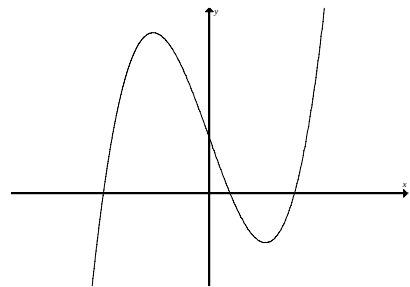
$$x = \pm\sqrt{2}$$

A1

ii. $f(\sqrt{2}) = a(\sqrt{2})^3 - 6a\sqrt{2} + 12 = 12 - 4\sqrt{2}$

$$f(-\sqrt{2}) = a(-\sqrt{2})^3 + 6a\sqrt{2} + 12 = 12 + 4\sqrt{2}$$

since $a > 0$



$(-\sqrt{2}, 12 + 4a\sqrt{2})$ is a local maximum

A1

$(\sqrt{2}, 12 - 4a\sqrt{2})$ is a local minimum.

A1

if there are three x -intercepts

$$12 - 4a\sqrt{2} < 0 \Rightarrow a\sqrt{2} > 3 \Rightarrow a > \frac{3}{\sqrt{2}}$$

$$a > \frac{3\sqrt{2}}{2}$$

A1

iii. $(\sqrt{2}, 12 - 4a\sqrt{2})$ is a local minimum, if there is only one x -intercept

$$12 - 4a\sqrt{2} > 0 \Rightarrow a\sqrt{2} < 3 \text{ since } a > 0$$

$$0 < a < \frac{3\sqrt{2}}{2} \text{ shown}$$

M1

iv. $f(-1) = -a + 6a + 12 = 12 + 5a > 0$ since $0 < a < \frac{3\sqrt{2}}{2}$

M1

since $f(-1) > 0$ and $f(0) = 12$ and $f(1) = 12 - 5a > 0$ and

$$f'(0) = -6a < 0 \text{ and } f'(x) < 0 \text{ for } -1 \leq x \leq 1 \text{ since } [-1, 1] \subset [-\sqrt{2}, \sqrt{2}],$$

there is no zero in $[-1, 1]$

A1

b.i. $g(x) = 12 \tan(x) - \frac{5}{\cos(x)} + \cos(x)$

$$g'(x) = \frac{12}{\cos^2(x)} - \frac{5 \sin(x)}{\cos^2(x)} - \sin(x)$$

M1

$$g'(x) = \frac{12 - 5 \sin(x) - \sin(x) \cos^2(x)}{\cos^2(x)}$$

$$g'(x) = \frac{12 - 5 \sin(x) - \sin(x)(1 - \sin^2(x))}{\cos^2(x)}$$

$$g'(x) = \frac{\sin^3(x) - 6 \sin(x) + 12}{\cos^2(x)}$$

A1

$$g'(x) = \frac{f(\sin(x))}{\cos^2(x)} \quad \text{with } a = 1$$

ii. since $a = 1$ is inside $0 < a < \frac{3\sqrt{2}}{2}$ and $f(x) \neq 0$ for $x \in [-1, 1]$

let $u = \sin(x)$ since $u \in [-1, 1] \Rightarrow x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ from **a.iv**

so that $g'(x) \neq 0$

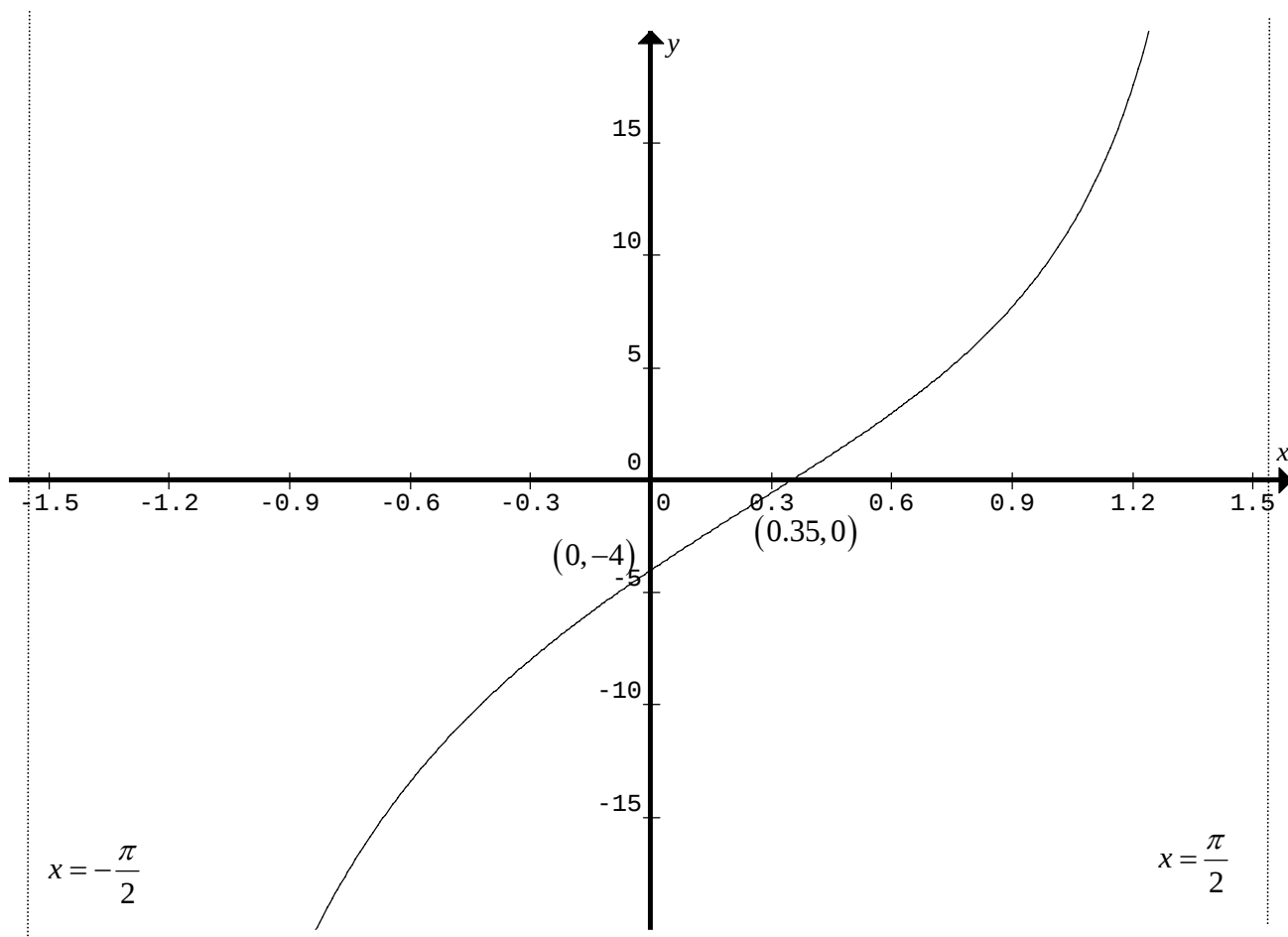
A1

so $g(x)$ has no stationary points it is a one-one function and hence has an inverse which is a function.

A1

iii. y-intercept $(0, -4)$, x-intercept $(0.35, 0)$, no turning points, G1

$x = -\frac{\pi}{2}$, $\frac{\pi}{2}$ vertical asymptotes, graph only for $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ A1



Question 2

a.i. $A(0, 18.5)$ $B(150, 15.5)$ $D(600, 18.5)$

the sine curve $ABCD$, has an amplitude of 3, and passes through A and B .

$$x=0 \quad y=18.5 \Rightarrow a=18.5 \text{ and } b=-3$$

A1

$$\text{the period } T = \frac{2\pi}{n} = 600 \Rightarrow n = \frac{\pi}{300}$$

A1

$$\text{at the smooth join } \frac{dy}{dx} = nb \cos(nx) = -\frac{\pi}{100} \cos\left(\frac{\pi x}{300}\right) \text{ at } x=600$$

M1

$$\left. \frac{dy}{dx} \right|_{x=600} = \frac{-\pi}{100} \cos(2\pi) = \frac{-\pi}{100} = m \text{ the gradient of the line}$$

A1

$$\text{at } D \quad x=600 \quad y=18.5 \quad y=mx+c$$

$$18.5 = \frac{-\pi}{100}(600) + c \Rightarrow c = 18.5 + 6\pi$$

A1

ii. when $x=1000$ $y = \left(-\frac{\pi}{100}\right)1000 + 6\pi + 18.5 = 18.5 - 4\pi$

$$D(600, 18.5) \quad E(1000, 18.5 - 4\pi)$$

$$d(DE) = \sqrt{(600-1000)^2 + (18.5 - (18.5 - 4\pi))^2}$$

$$d(DE) = \sqrt{(-400)^2 + (4\pi)^2}$$

$$d(DE) = 4\sqrt{\pi^2 + 10000}$$

A1

b.i. $y = 18.5 - 3\sin\left(\frac{\pi x}{300}\right) \Rightarrow \frac{dy}{dx} = -\frac{\pi}{100} \cos\left(\frac{\pi x}{300}\right)$

$$s = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

$$s = \int_0^{600} \sqrt{1 + \left(\frac{\pi}{100} \cos\left(\frac{\pi x}{300}\right)\right)^2} dx + \int_{600}^{1000} \sqrt{1 + \frac{\pi^2}{10000}} dx$$

A2

ii. $s = 600.148 + d(DE) = 600.148 + 400.197$

$s = 1000.35$ bike track length

A1

c.i. translate $f(x)$ 1.5 metres up parallel to the y-axis

$$f_n(x) = \begin{cases} 20 - 3\sin\left(\frac{\pi x}{300}\right) & \text{for } 0 \leq x \leq 600 \\ -\frac{\pi x}{100} + 20 + 6\pi & \text{for } 600 \leq x \leq 1000 \end{cases} \quad \text{A1}$$

translate $f(x)$ 1.5 metres down parallel to the y-axis

$$f_s(x) = \begin{cases} 17 - 3\sin\left(\frac{\pi x}{300}\right) & \text{for } 0 \leq x \leq 600 \\ -\frac{\pi x}{100} + 17 + 6\pi & \text{for } 600 \leq x \leq 1000 \end{cases} \quad \text{A1}$$

$$\text{ii. } A = \int_0^{1000} (f_n(x) - f_s(x)) dx = \int_0^{1000} 3 dx$$

area of asphalt is 3,000 m² A1

$$\text{iii. } A = \int_0^{600} \left(17 - 3\sin\left(\frac{\pi x}{300}\right) \right) dx + \int_{600}^{1000} \left(-\frac{\pi x}{100} + 17 + 6\pi \right) dx \quad \text{A1}$$

$$A = \left[17x + \frac{900}{\pi} \cos\left(\frac{\pi x}{300}\right) \right]_0^{600} + \left[-\frac{\pi x^2}{200} + (17 + 6\pi)x \right]_{600}^{1000}$$

$$A = \left(17 \times 600 + \frac{900}{\pi} \cos(2\pi) \right) - \left(17 \times 0 + \frac{900}{\pi} \cos(0) \right) \\ + \left(-\frac{\pi(1000)^2}{200} + 1000(17 + 6\pi) \right) - \left(-\frac{\pi(600)^2}{200} + 600(17 + 6\pi) \right)$$

$$A = 10200 + 6800 - 800\pi$$

$$A = 17000 - 800\pi \text{ m}^2 \quad \text{A1}$$

Alternatively area of a rectangle + area of a trapezium

$$= 17 \times 600 + \frac{1}{2}(400)(17 + 17 - 4\pi) = 17000 - 800\pi$$

iv. $T(450, 23)$ A1

Question 3

a. X is the time of the quarter in minutes

$$X \stackrel{d}{=} N(\mu = 27, \sigma^2 = 2^2)$$

i.
$$\Pr(X > 30 | X > 25) = \frac{\Pr(X > 30)}{\Pr(X > 25)} = \frac{0.0668}{0.8413}$$

$$= 0.0794 \quad \text{A1}$$

ii. $\Pr(X > T) = 0.75$

$$\frac{T - 27}{2} = -0.6745 \Rightarrow T = 25.65 \quad \text{M1}$$

$$\text{longer than 25 minutes 39 seconds} \quad \text{A1}$$

iii. $Y \stackrel{d}{=} Bi(n = 4, p = 0.0668)$

$$\Pr(Y \geq 1) = 1 - \Pr(Y = 0) \quad \text{M1}$$

$$\Pr(Y \geq 1) = 1 - (1 - 0.0668)^4$$

$$\Pr(Y \geq 1) = 0.2416 \quad \text{A1}$$

b.i. plays for the next three weeks

$$(0.75)^3 = 0.4219 \quad \text{A1}$$

ii. $P \rightarrow P = 0.75$, $P \rightarrow N = 0.25$, $N \rightarrow P = 0.4$, $N \rightarrow N = 0.6$

$$\Pr(2 \text{ matches}) = NPP + PNP + PPN \quad \text{M1}$$

$$= 0.25 \times 0.4 \times 0.75 + 0.75 \times 0.25 \times 0.4 + 0.75 \times 0.75 \times 0.25$$

$$= 0.2906 \quad \text{A1}$$

iii.
$$\frac{0.4}{0.4 + 0.25} = 0.6154$$

or alternatively

$$\begin{matrix} & N & P \\ N & \begin{bmatrix} 0.6 & 0.25 \end{bmatrix} \\ P & \begin{bmatrix} 0.4 & 0.75 \end{bmatrix} \end{matrix}$$

$$\begin{bmatrix} 0.6 & 0.25 \\ 0.4 & 0.75 \end{bmatrix}^n \begin{bmatrix} 0 \\ 1 \end{bmatrix} \approx \frac{1}{13} \begin{bmatrix} 5 \\ 8 \end{bmatrix} \text{ for } n \geq 9 \quad \frac{8}{13} = 0.6154 \quad \text{M1}$$

$$\text{in the long run, the percentage of games played } 61.54\% \quad \text{A1}$$

c.i. the function is continuous at $s = 4 \Rightarrow f(4) = k\sqrt{4} = 2k = a \cos(0)$

$$\Rightarrow a = 2k$$

A1

the total area under the curve is one.

$$k \int_0^4 \sqrt{s} \, ds + a \int_4^{10} \cos\left(\frac{\pi(s-4)}{12}\right) ds = 1$$

M1

$$k \left[\frac{2}{3} s^{\frac{3}{2}} \right]_0^4 + \frac{12a}{\pi} \left[\sin\left(\frac{\pi(s-4)}{12}\right) \right]_4^{10} = 1$$

$$\Rightarrow \frac{16k}{3} + \frac{12a}{\pi} = 1$$

A1

ii. $k = \frac{a}{2}$, $\frac{16k}{3} + \frac{12a}{\pi} = 1$ becomes $\frac{8a}{3} + \frac{12a}{\pi} = 1$

$$\frac{8\pi a + 36a}{3\pi} = 1 \Rightarrow a = \frac{3\pi}{4(2\pi + 9)}$$

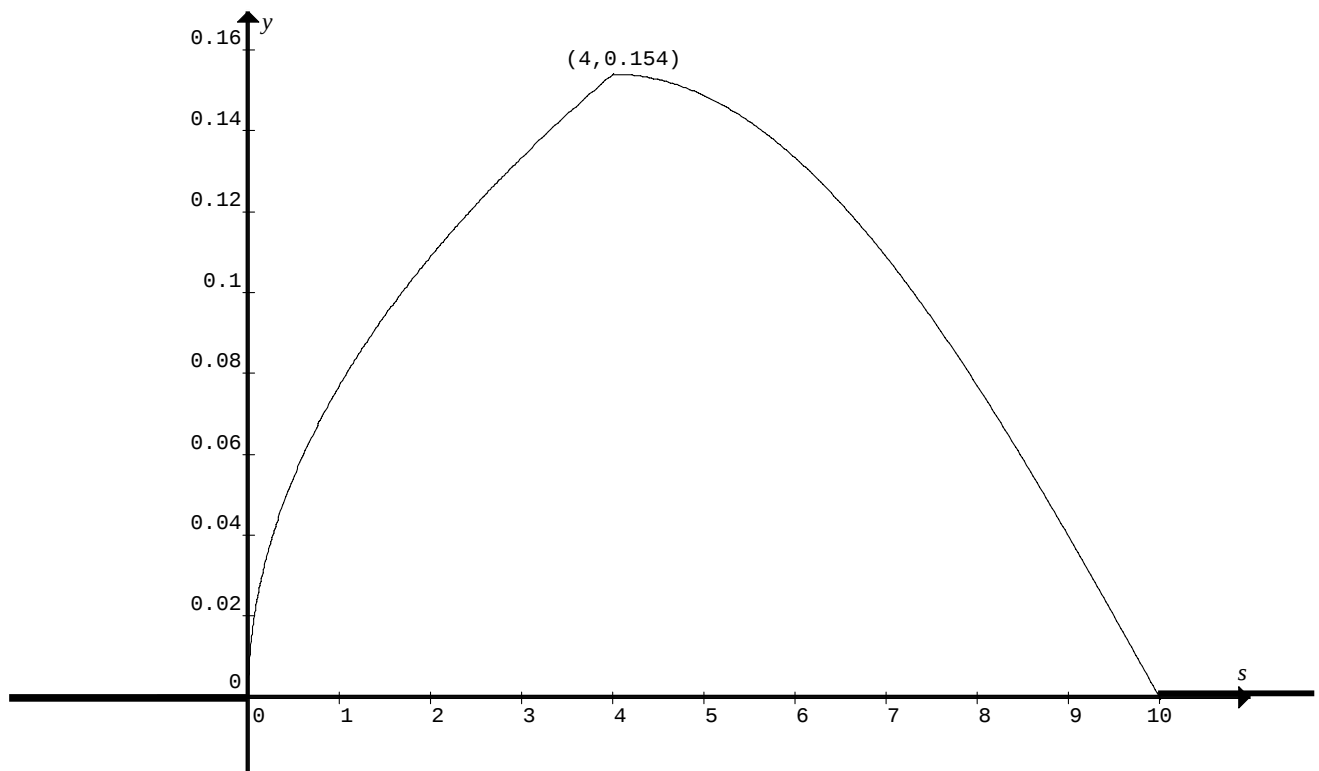
A1

iii. must be continuous at $(4, 0.154)$ the maximum

A1

and must show zero for $s \geq 10$ and $s \leq 0$

G1



iv. $\Pr(S < 6) = 1 - \Pr(S > 6)$

$$\Pr(S < 6) = 1 - \frac{3\pi}{4(2\pi + 9)} \int_6^{10} \cos\left(\frac{\pi(s-4)}{12}\right) ds \quad \text{M1}$$

$$\Pr(S < 6) = 1 - \frac{3\pi}{4(2\pi + 9)} \left[\frac{12}{\pi} \sin\left(\frac{\pi(s-4)}{12}\right) \right]_6^{10}$$

$$\Pr(S < 6) = 1 - \frac{9}{2\pi + 9} \left[\sin\left(\frac{6\pi}{12}\right) - \sin\left(\frac{2\pi}{12}\right) \right]$$

$$\Pr(S < 6) = 1 - \frac{9}{2\pi + 9} \left[1 - \frac{1}{2} \right]$$

$$\Pr(S < 6) = 1 - \frac{9}{2(2\pi + 9)} = \frac{4\pi + 9}{2(2\pi + 9)} \quad \text{A1}$$

d.i. $P \stackrel{d}{=} \text{Bi}\left(n = 8, p = \frac{2}{3}\right)$

$$\Pr(P > 4) = 0.7414 \quad \text{A1}$$

ii. $T \stackrel{d}{=} \text{Bi}(n = 22, p = 0.74135)$

$$\Pr(T \geq 16) = 0.6651 \quad \text{A1}$$

Question 4

a.i. $y = f(x) = \frac{1}{x}$ $P\left(a, \frac{1}{a}\right)$
 $\frac{dy}{dx} = f'(x) = -\frac{1}{x^2}$ $f'(a) = -\frac{1}{a^2}$ A1

equation of the tangent is

$$y - \frac{1}{a} = -\frac{1}{a^2}(x - a) = -\frac{x}{a^2} + \frac{1}{a}$$

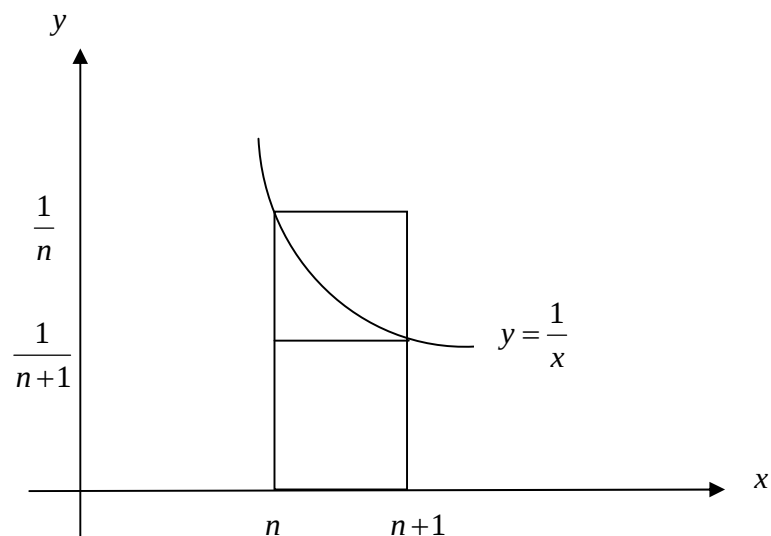
$$y = -\frac{x}{a^2} + \frac{2}{a}$$
 A1

ii. crosses the y-axis when $x = 0 \Rightarrow y_s = \frac{2}{a}$
 $S\left(0, \frac{2}{a}\right)$ A1

crosses the x-axis when $y = 0 \Rightarrow -\frac{x}{a^2} + \frac{2}{a} \Rightarrow x_Q = 2a$
 $Q(2a, 0)$ A1

iii. area of the triangle OQS $\frac{1}{2}(2a)\left(\frac{2}{a}\right) = 2$ independent of a A1

b.i.



consider a rectangle, with width one unit, from x values $x = n$ to $n + 1$ when

$$x = n, y = \frac{1}{n} \quad \text{when} \quad x = n + 1, y = \frac{1}{n + 1} \quad \text{M1}$$

the area of the lower rectangle $<$ true area $<$ the area of the upper rectangle,

$$\text{so that} \quad \frac{1}{n + 1} < \int_n^{n+1} \frac{1}{x} dx < \frac{1}{n} \quad \text{A1}$$

$$\text{ii.} \quad \int_n^{n+1} \frac{1}{x} dx = [\log_e(x)]_n^{n+1}$$

$$= \log_e(n + 1) - \log_e(n) = \log_e\left(\frac{n + 1}{n}\right)$$

$$= \log_e\left(1 + \frac{1}{n}\right) \quad \text{A1}$$

from **b.i**

$$\int_n^{n+1} \frac{1}{x} dx = \log_e\left(1 + \frac{1}{n}\right) < \frac{1}{n}$$

$$\Rightarrow n \log_e\left(1 + \frac{1}{n}\right) < 1$$

$$\Rightarrow \log_e\left(1 + \frac{1}{n}\right)^n < 1$$

$$\left(1 + \frac{1}{n}\right)^n < e \quad \text{A1}$$

also from **b.i**

$$\frac{1}{n + 1} < \log_e\left(1 + \frac{1}{n}\right) = \int_n^{n+1} \frac{1}{x} dx$$

$$\Rightarrow 1 < (n + 1) \log_e\left(1 + \frac{1}{n}\right)$$

$$\Rightarrow 1 < \log_e\left(1 + \frac{1}{n}\right)^{n+1}$$

$$e < \left(1 + \frac{1}{n}\right)^{n+1} \quad \text{A1}$$

$$\text{so that} \quad \left(1 + \frac{1}{n}\right)^n < e < \left(1 + \frac{1}{n}\right)^{n+1} \quad \text{shown}$$

END OF SECTION 2 SUGGESTED ANSWERS