Year 2010
VCE
Mathematical Methods
CAS
Solutions
Trial Examination 1

KILBAHA MULTIMEDIA PUBLISHING **PO BOX 2227** **KEW VIC 3101** **AUSTRALIA**	**TEL: (03) 9817 5374** **FAX: (03) 9817 4334** **kilbaha@gmail.com** **http://kilbaha.com.au**

Question 1

$\Delta = \begin{vmatrix} -3 & p \\ 4 & -5 \end{vmatrix} = 15 - 4p$

$(1) \times 4 \Rightarrow -12x + 4py = 4q$

$(2) \times 3 \Rightarrow \quad 12x - 15y = 60$ M1

i. for a unique solution $\Delta \neq 0 \Rightarrow \ p \neq \dfrac{15}{4}$ and $q \in R$ A1

ii. for no solution $\Delta = 0 \Rightarrow \ p = \dfrac{15}{4}$ and $q \neq -15$ A1

iii. for infinitely many solutions $\Delta = 0 \Rightarrow \ p = \dfrac{15}{4}$ and $q = -15$ A1

Question 2

$f(x) = e^{\cos(2x)}$ chain rule

$y = e^{\cos(2x)} = e^{u} \quad u = \cos(2x)$

$\dfrac{dy}{du} = e^{u} \quad \dfrac{du}{dx} = -2\sin(2x)$ M1

$f'(x) = -2\sin(2x)e^{\cos(2x)}$

$f'\left(\dfrac{\pi}{6}\right) = -2\sin\left(\dfrac{\pi}{3}\right)e^{\cos\left(\frac{\pi}{3}\right)} = -2\left(\dfrac{\sqrt{3}}{2}\right)e^{\frac{1}{2}}$

$f'\left(\dfrac{\pi}{6}\right) = -\sqrt{3e}$ A1

Question 3

i. Let $y = e^{-2x}\left(2\cos(3x) - 3\sin(3x)\right)$ using the product rule

$\dfrac{dy}{dx} = -2e^{-2x}\left(2\cos(3x) - 3\sin(3x)\right) + e^{-2x}\left(-6\sin(3x) - 9\cos(3x)\right)$ M1

$\dfrac{dy}{dx} = e^{-2x}\left(-4\cos(3x) + 6\sin(3x) - 6\sin(3x) - 9\cos(3x)\right) = -13e^{-2x}\cos(3x)$

so that $\dfrac{d}{dx}\left(e^{-2x}\left(2\cos(3x) - 3\sin(3x)\right)\right) = -13e^{-2x}\cos(3x)$ A1

ii. $\int e^{-2x}\cos(3x)\,dx = \dfrac{e^{-2x}}{13}\left(3\sin(3x) - 2\cos(3x)\right) + c$ A1

Question 4

$\sqrt{3}\sin(2x)+\cos(2x)=0$

$\sqrt{3}\sin(2x)=-\cos(2x)$

$\tan(2x)=-\dfrac{1}{\sqrt{3}}$ M1

$2x=n\pi+\tan^{-1}\left(-\dfrac{1}{\sqrt{3}}\right)$

A1

$2x=n\pi-\dfrac{\pi}{6}$

$x=\dfrac{n\pi}{2}-\dfrac{\pi}{12}=\dfrac{\pi}{12}(6n-1)$ where $n\in Z$ A1

Question 5

i. $f:\quad y=e^{2x}-e^{-2x}$

$f^{-1}:\quad x=e^{2y}-e^{-2y}$ let $u=e^{2y}$ M1

$x=u-\dfrac{1}{u}$

$u^2-xu-1=0$ solve using the quadratic formulae

$a=1\ \ b=-x\ \ c=-1$

$u=\dfrac{-b\pm\sqrt{\Delta}}{2a}$

$u=e^{2y}=\dfrac{x\pm\sqrt{x^2+4}}{2}$ must take the positive, since $e^{2y}>0$ for $y\in R$ M1

and the domain and range of f and f^{-1} are both R

$y=f^{-1}(x)=\dfrac{1}{2}\log_e\left(\dfrac{x+\sqrt{x^2+4}}{2}\right)$

ii. $e^{2x}-e^{-2x}=4$

hence $x=f^{-1}(4)=\dfrac{1}{2}\log_e\left(\dfrac{4+\sqrt{16+4}}{2}\right)$ M1

$x=\dfrac{1}{2}\log_e\left(\dfrac{4+2\sqrt{5}}{2}\right)$

$x=\dfrac{1}{2}\log_e\left(2+\sqrt{5}\right)$ A1 alternatively

$e^{2x} - e^{-2x} = 4$ let $u = e^{-2x}$

$u - \dfrac{1}{u} = 4$

$u^2 - 4u - 1 = 0$ M1

$u^2 - 4u + 4 = 1 + 4$

$(u-2)^2 = 5$

$u - 2 = \pm\sqrt{5}$

$u = e^{2x} = 2 \pm \sqrt{5}$ must take the positive

$x = \dfrac{1}{2}\log_e\left(2+\sqrt{5}\right)$ A1

iii. $\overline{y} = \dfrac{1}{2-0}\displaystyle\int_0^2 \left(e^{2x} - e^{-2x}\right)dx$ A1

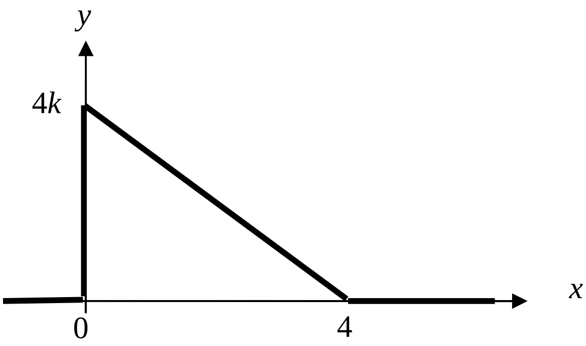
$\overline{y} = \dfrac{1}{2}\left[\dfrac{1}{2}e^{2x} + \dfrac{1}{2}e^{-2x}\right]_0^2$

$\overline{y} = \dfrac{1}{4}\left[\left(e^4 + e^{-4}\right) - (1+1)\right]$

$\overline{y} = \dfrac{1}{4}\left(e^4 + e^{-4}\right) - \dfrac{1}{2}$ A1

Question 6

i.



area of a triangle $\frac{1}{2}(4)(4k)=8k=1$ M1

$k=\frac{1}{8}$

alternatively

$k\int_0^4 (4-x)\,dx=1$ M1

$k\left[4x-\frac{x^2}{2}\right]_0^4=k(16-8-0)=1$

$8k=1$

$k=\frac{1}{8}$

ii. median is m

$\frac{1}{8}\int_0^m (4-x)\,dx=\frac{1}{2}$ A1

$\left[4x-\frac{x^2}{2}\right]_0^m=4$

$4m-\frac{1}{2}m^2=4$

$m^2-8m+8=0$ M1

$m^2-8m+16=16-8=8$

$(m-4)^2=\pm\sqrt{8}$

$m=4\pm 2\sqrt{2}$ must take negative since $m\in[0,4]$

$m=4-2\sqrt{2}$ A1

Question 7

i. $y = \dfrac{x-2}{x+2} = \dfrac{x+2-4}{x+2} = 1 - \dfrac{4}{x+2}$

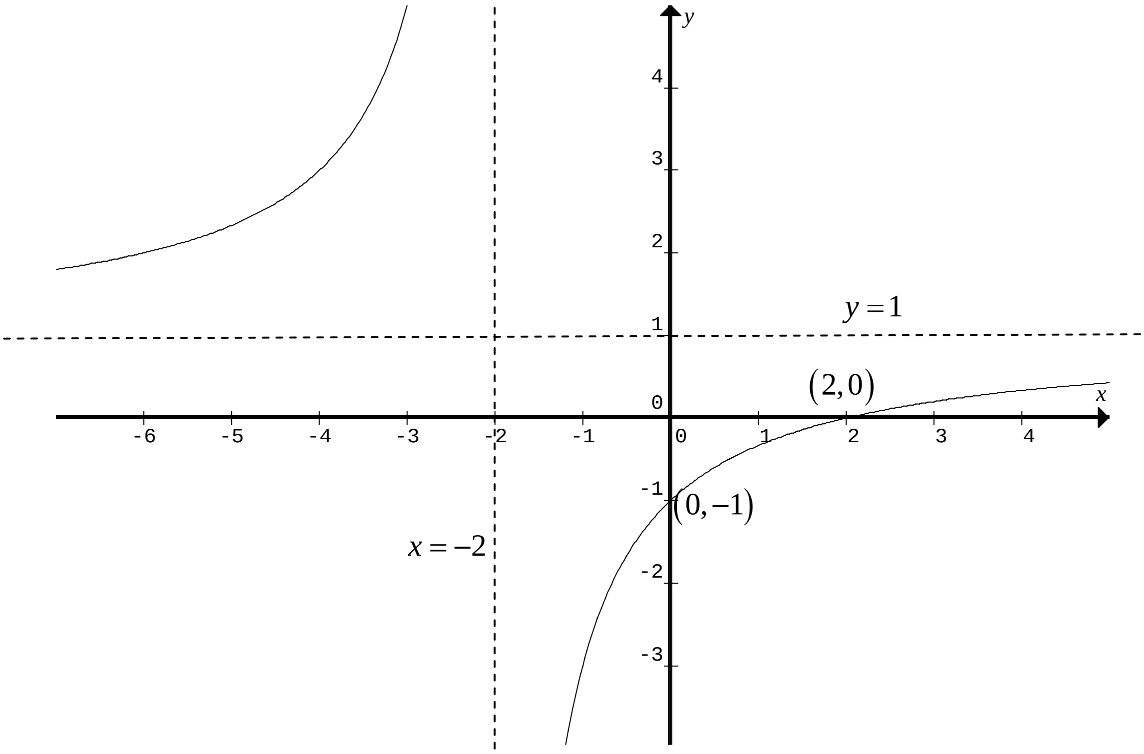
crosses the x-axis at $y = 0 \Rightarrow x = 2$ $(2,0)$

crosses the y-axis at $x = 0 \Rightarrow y = -1$ $(0,-1)$, graph shape G1

$x = -2$ is a vertical asymptote

$y = 1$ is a horizontal asymptote,

both intercepts and both asymptotes A1



ii. The required area is $A = -\displaystyle\int_0^2 \frac{x-2}{x+2}\,dx$ since the area is below the x-axis

$A = \displaystyle\int_0^2 \left(\frac{4}{x+2} - 1\right) dx$ A1

$A = \left[4\log_e|x+2| - x\right]_0^2$ M1

$A = \left(4\log_e(4) - 2\right) - \left(4\log_e(2) - 0\right)$

$A = 4\log_e(2) - 2$

$A = \log_e(16) - 2$ and $A > 0$

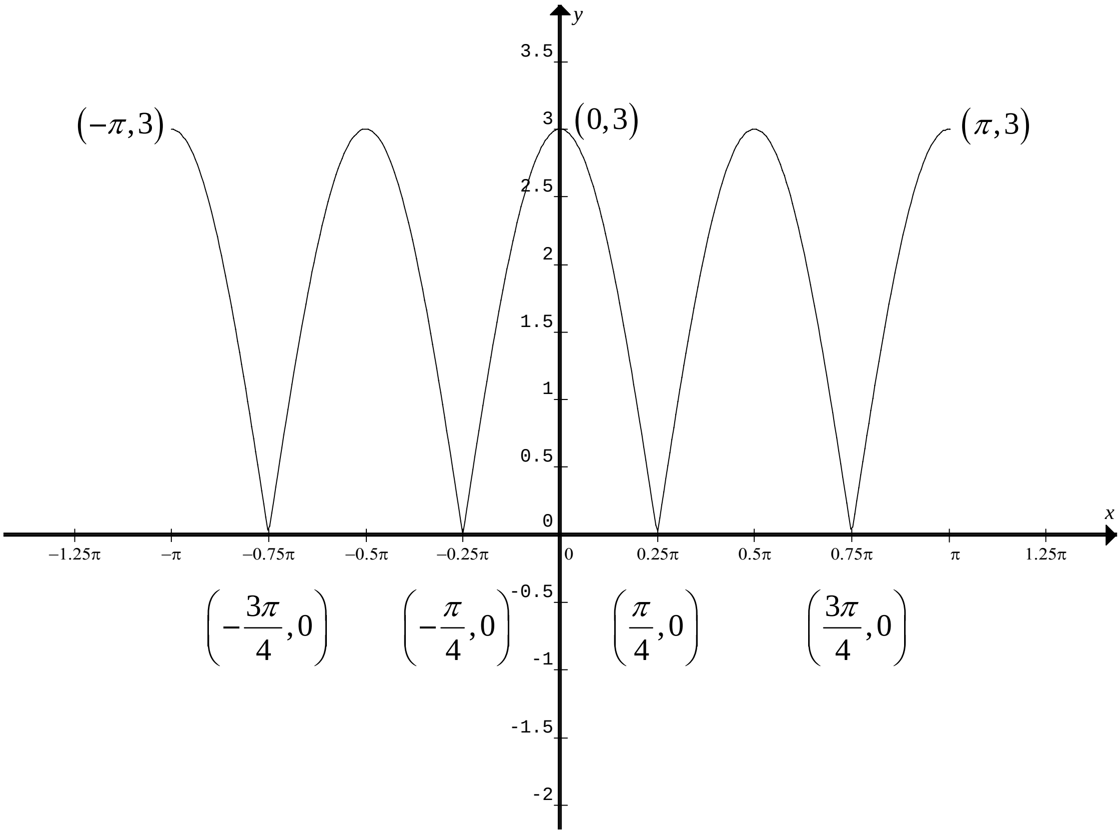
$a = 16$ $b = -2$ A1

Question 8

$f(g(x)) = f(-3\cos(2x)) = |-3\cos(2x)| = 3|\cos(2x)|$, graph shape G1

and domain of $f(g(x)) =$ domain $g = [-\pi, \pi]$ correct period

and end-points $(\pm\pi, 3)$, axial intercepts and end-points. A1



Question 9

Area of a triangle $A = \frac{1}{2}bc\sin(\theta) = \frac{1}{2}(4)(9)\sin(\theta) = 18\sin(\theta)$

$A = 18\sin(\theta) \quad \Rightarrow \quad \frac{dA}{d\theta} = 18\cos(\theta)$

Now given $\frac{d\theta}{dt} = 1^0\text{ /min} = \frac{\pi}{180}{}^c\text{/ min}$ A1

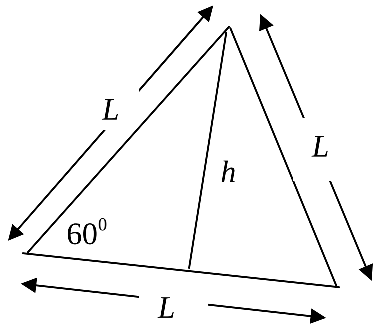
By the chain rule $\frac{dA}{dt} = \frac{dA}{d\theta}\frac{d\theta}{dt} = \frac{18\pi}{180}\cos(\theta)$ M1

$\left.\frac{dA}{dt}\right|_{\theta=45^0} = \frac{\pi}{10}\cos(45^0)$

$= \frac{\pi\sqrt{2}}{20}\text{ cm}^2\text{/min}$ A1

Question 10

a.



Each triangle is equilateral, the area of one triangle is

$$A = \frac{1}{2}Lh = \frac{1}{2}L^2\sin\left(60^0\right) = \frac{\sqrt{3}L^2}{4} \qquad \text{A1}$$

Area of the top is six triangles, so the volume V

$$V = 6\left(\frac{\sqrt{3}L^2}{4}\right)H = 10$$

$$H = \frac{20}{3\sqrt{3}L^2} = \frac{20\sqrt{3}}{9L^2} \qquad \text{A1}$$

b. The total surface area is 12 triangles (top and bottom) and 6 rectangles

$$A = 12\left(\frac{\sqrt{3}L^2}{4}\right) + 6HL \qquad \text{M1}$$

$$A = 3\sqrt{3}L^2 + 6L\left(\frac{20\sqrt{3}}{9L^2}\right) \quad \text{since} \quad L > 0 \ \text{and} \ H > 0$$

$$A = 3\sqrt{3}L^2 + \frac{40\sqrt{3}}{3L}$$

c. for a minimum surface area

$$\frac{dA}{dL} = 6\sqrt{3}L - \frac{40\sqrt{3}}{3L^2} = 0 \qquad \text{A1}$$

$$6\sqrt{3}L = \frac{40\sqrt{3}}{3L^2}$$

$$L^3 = \frac{40}{18}$$

$$L = \sqrt[3]{\frac{20}{9}} \text{ metres} \qquad \text{A1}$$

Question 11

i. total number of balls $r+b+w$

$\Pr(\text{same colour}) = \Pr(RRR) + \Pr(BBB) + \Pr(WWW)$

$$= \frac{r(r-1)(r-2)}{(r+b+w)(r+b+w-1)(r+b+w-2)} + \frac{b(b-1)(b-2)}{(r+b+w)(r+b+w-1)(r+b+w-2)} + \frac{w(w-1)(w-2)}{(r+b+w)(r+b+w-1)(r+b+w-2)}$$ M1

$$= \frac{r(r-1)(r-2)+b(b-1)(b-2)+w(w-1)(w-2)}{(r+b+w)(r+b+w-1)(r+b+w-2)}$$ A1

ii. $\Pr(\text{all different colour})$

$= \Pr(RBW) + \Pr(RWB) + \Pr(WRB) + \Pr(WBR) + \Pr(BWR) + \Pr(BRW)$

$$= \frac{6rbw}{(r+b+w)(r+b+w-1)(r+b+w-2)}$$ A1

END OF SUGGESTED SOLUTIONS