

Year 2010
VCE
Mathematical Methods
CAS
Solutions
Trial Examination 1



KILBAHA MULTIMEDIA PUBLISHING
PO BOX 2227
KEW VIC 3101
AUSTRALIA

TEL: (03) 9817 5374
FAX: (03) 9817 4334
kilbaha@gmail.com
<http://kilbaha.com.au>

IMPORTANT COPYRIGHT NOTICE

- This material is copyright. Subject to statutory exception and to the provisions of the relevant collective licensing agreements, no reproduction of any part may take place without the written permission of Kilbaha Pty Ltd.
 - The contents of this work are copyrighted. Unauthorised copying of any part of this work is illegal and detrimental to the interests of the author.
 - For authorised copying within Australia please check that your institution has a licence from Copyright Agency Limited. This permits the copying of small parts of the material, in limited quantities, within the conditions set out in the licence.
 - Teachers and students are reminded that for the purposes of school requirements and external assessments, students must submit work that is clearly their own.
 - Schools which purchase a licence to use this material may distribute this electronic file to the students at the school for their exclusive use. This distribution can be done either on an Intranet Server or on media for the use on stand-alone computers.
 - Schools which purchase a licence to use this material may distribute this printed file to the students at the school for their exclusive use.
-
- **The Word file (if supplied) is for use ONLY within the school.**
 - **It may be modified to suit the school syllabus and for teaching purposes.**
 - **All modified versions of the file must carry this copyright notice.**
 - **Commercial use of this material is expressly prohibited.**

Question 1

$$\Delta = \begin{vmatrix} -3 & p \\ 4 & -5 \end{vmatrix} = 15 - 4p$$

$$(1) \times 4 \Rightarrow -12x + 4py = 4q$$

$$(2) \times 3 \Rightarrow 12x - 15y = 60$$

M1

i. for a unique solution $\Delta \neq 0 \Rightarrow p \neq \frac{15}{4}$ and $q \in R$

A1

ii. for no solution $\Delta = 0 \Rightarrow p = \frac{15}{4}$ and $q \neq -15$

A1

iii. for infinitely many solutions $\Delta = 0 \Rightarrow p = \frac{15}{4}$ and $q = -15$

A1

Question 2

$$f(x) = e^{\cos(2x)} \quad \text{chain rule}$$

$$y = e^{\cos(2x)} = e^u \quad u = \cos(2x)$$

$$\frac{dy}{du} = e^u \quad \frac{du}{dx} = -2\sin(2x)$$

M1

$$f'(x) = -2\sin(2x)e^{\cos(2x)}$$

$$f'\left(\frac{\pi}{6}\right) = -2\sin\left(\frac{\pi}{3}\right)e^{\cos\left(\frac{\pi}{3}\right)} = -2\left(\frac{\sqrt{3}}{2}\right)e^{\frac{1}{2}}$$

$$f'\left(\frac{\pi}{6}\right) = -\sqrt{3}e$$

A1

Question 3

i. Let $y = e^{-2x}(2\cos(3x) - 3\sin(3x))$ using the product rule

$$\frac{dy}{dx} = -2e^{-2x}(2\cos(3x) - 3\sin(3x)) + e^{-2x}(-6\sin(3x) - 9\cos(3x))$$

M1

$$\frac{dy}{dx} = e^{-2x}(-4\cos(3x) + 6\sin(3x) - 6\sin(3x) - 9\cos(3x)) = -13e^{-2x}\cos(3x)$$

so that $\frac{d}{dx}(e^{-2x}(2\cos(3x) - 3\sin(3x))) = -13e^{-2x}\cos(3x)$

A1

ii. $\int e^{-2x}\cos(3x)dx = \frac{e^{-2x}}{13}(3\sin(3x) - 2\cos(3x)) + c$

A1

Question 4

$$\sqrt{3} \sin(2x) + \cos(2x) = 0$$

$$\sqrt{3} \sin(2x) = -\cos(2x)$$

$$\tan(2x) = -\frac{1}{\sqrt{3}} \quad \text{M1}$$

$$2x = n\pi + \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) \quad \text{A1}$$

$$2x = n\pi - \frac{\pi}{6}$$

$$x = \frac{n\pi}{2} - \frac{\pi}{12} = \frac{\pi}{12}(6n-1) \quad \text{where } n \in \mathbb{Z} \quad \text{A1}$$

Question 5

i. $f: y = e^{2x} - e^{-2x}$
 $f^{-1}: x = e^{2y} - e^{-2y} \quad \text{let } u = e^{2y}$ M1

$$x = u - \frac{1}{u}$$

$$u^2 - xu - 1 = 0 \quad \text{solve using the quadratic formulae}$$

$$a = 1 \quad b = -x \quad c = -1$$

$$u = \frac{-b \pm \sqrt{\Delta}}{2a}$$

$$u = e^{2y} = \frac{x \pm \sqrt{x^2 + 4}}{2} \quad \text{must take the positive, since } e^{2y} > 0 \quad \text{for } y \in \mathbb{R} \quad \text{M1}$$

and the domain and range of f and f^{-1} are both $\mathbb{R}$

$$y = f^{-1}(x) = \frac{1}{2} \log_e \left(\frac{x + \sqrt{x^2 + 4}}{2} \right)$$

ii. $e^{2x} - e^{-2x} = 4$

$$\text{hence } x = f^{-1}(4) = \frac{1}{2} \log_e \left(\frac{4 + \sqrt{16 + 4}}{2} \right) \quad \text{M1}$$

$$x = \frac{1}{2} \log_e \left(\frac{4 + 2\sqrt{5}}{2} \right)$$

$$x = \frac{1}{2} \log_e (2 + \sqrt{5}) \quad \text{A1}$$

alternatively

$$e^{2x} - e^{-2x} = 4 \quad \text{let} \quad u = e^{-2x}$$

$$u - \frac{1}{u} = 4$$

$$u^2 - 4u - 1 = 0$$

M1

$$u^2 - 4u + 4 = 1 + 4$$

$$(u - 2)^2 = 5$$

$$u - 2 = \pm\sqrt{5}$$

$$u = e^{2x} = 2 \pm \sqrt{5} \quad \text{must take the positive}$$

$$x = \frac{1}{2} \log_e (2 + \sqrt{5})$$

A1

iii. $\bar{y} = \frac{1}{2-0} \int_0^2 (e^{2x} - e^{-2x}) dx$

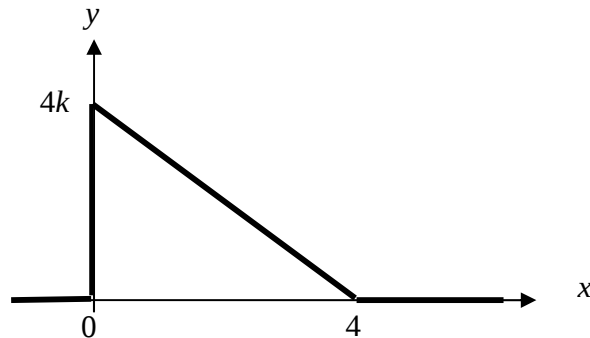
A1

$$\bar{y} = \frac{1}{2} \left[\frac{1}{2} e^{2x} + \frac{1}{2} e^{-2x} \right]_0^2$$

$$\bar{y} = \frac{1}{4} [(e^4 + e^{-4}) - (1 + 1)]$$

$$\bar{y} = \frac{1}{4} (e^4 + e^{-4}) - \frac{1}{2}$$

A1

Question 6**i.**

$$\text{area of a triangle } \frac{1}{2}(4)(4k) = 8k = 1$$

M1

$$k = \frac{1}{8}$$

alternatively

$$k \int_0^4 (4-x) dx = 1$$

M1

$$k \left[4x - \frac{x^2}{2} \right]_0^4 = k(16 - 8 - 0) = 1$$

$$8k = 1$$

$$k = \frac{1}{8}$$

ii. median is m

$$\frac{1}{8} \int_0^m (4-x) dx = \frac{1}{2}$$

A1

$$\left[4x - \frac{x^2}{2} \right]_0^m = 4$$

$$4m - \frac{1}{2}m^2 = 4$$

$$m^2 - 8m + 8 = 0$$

M1

$$m^2 - 8m + 16 = 16 - 8 = 8$$

$$(m-4)^2 = \pm\sqrt{8}$$

$$m = 4 \pm 2\sqrt{2} \quad \text{must take negative since } m \in [0, 4]$$

$$m = 4 - 2\sqrt{2}$$

A1

Question 7

i. $y = \frac{x-2}{x+2} = \frac{x+2-4}{x+2} = 1 - \frac{4}{x+2}$

crosses the x-axis at $y = 0 \Rightarrow x = 2$ (2,0)

crosses the y-axis at $x = 0 \Rightarrow y = -1$ (0,-1), graph shape

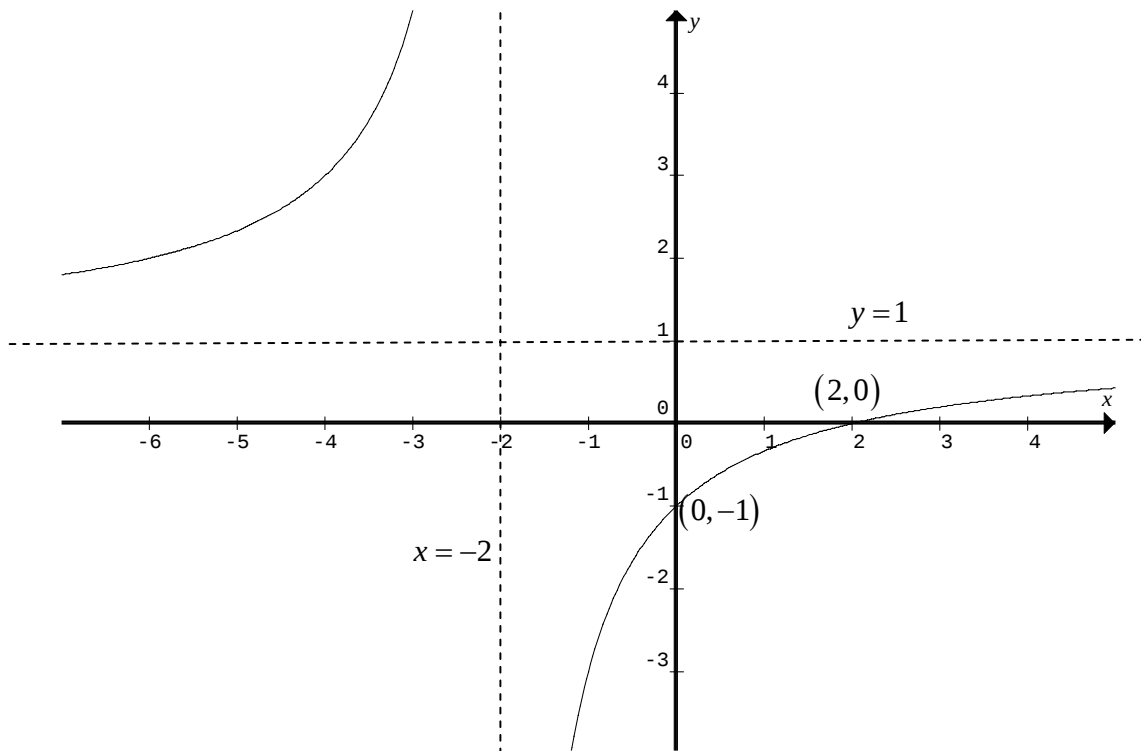
G1

$x = -2$ is a vertical asymptote

$y = 1$ is a horizontal asymptote,

both intercepts and both asymptotes

A1



ii. The required area is $A = -\int_0^2 \frac{x-2}{x+2} dx$ since the area is below the x-axis

$$A = \int_0^2 \left(\frac{4}{x+2} - 1 \right) dx$$

A1

$$A = \left[4 \log_e |x+2| - x \right]_0^2$$

M1

$$A = (4 \log_e(4) - 2) - (4 \log_e(2) - 0)$$

$$A = 4 \log_e(2) - 2$$

$$A = \log_e(16) - 2 \text{ and } A > 0$$

$$a = 16 \quad b = -2$$

A1

Question 8

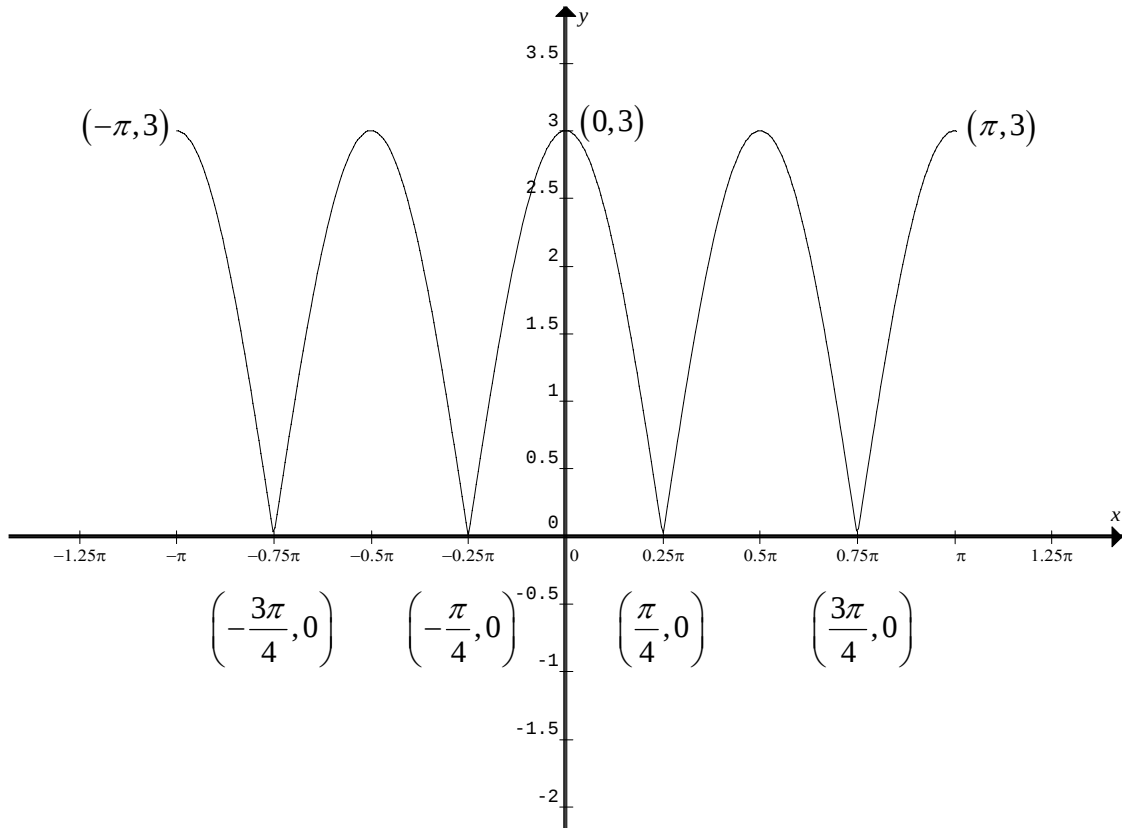
$$f(g(x)) = f(-3\cos(2x)) = |-3\cos(2x)| = 3|\cos(2x)|, \text{ graph shape}$$

G1

and domain of $f(g(x)) = \text{domain } g = [-\pi, \pi]$ correct period

and end-points $(\pm\pi, 3)$, axial intercepts and end-points.

A1

**Question 9**

$$\text{Area of a triangle } A = \frac{1}{2}bc \sin(\theta) = \frac{1}{2}(4)(9)\sin(\theta) = 18\sin(\theta)$$

$$A = 18\sin(\theta) \Rightarrow \frac{dA}{d\theta} = 18\cos(\theta)$$

$$\text{Now given } \frac{d\theta}{dt} = 1^\circ/\text{min} = \frac{\pi}{180} \text{ } ^\circ/\text{min}$$

A1

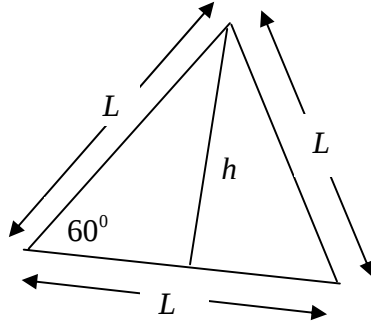
$$\text{By the chain rule } \frac{dA}{dt} = \frac{dA}{d\theta} \frac{d\theta}{dt} = \frac{18\pi}{180} \cos(\theta)$$

M1

$$\left. \frac{dA}{dt} \right|_{\theta=45^\circ} = \frac{\pi}{10} \cos(45^\circ)$$

$$= \frac{\pi\sqrt{2}}{20} \text{ cm}^2/\text{min}$$

A1

Question 10**a.**

Each triangle is equilateral, the area of one triangle is

$$A = \frac{1}{2} Lh = \frac{1}{2} L^2 \sin(60^\circ) = \frac{\sqrt{3}L^2}{4} \quad \text{A1}$$

Area of the top is six triangles, so the volume V

$$V = 6 \left(\frac{\sqrt{3}L^2}{4} \right) H = 10$$

$$H = \frac{20}{3\sqrt{3}L^2} = \frac{20\sqrt{3}}{9L^2} \quad \text{A1}$$

b. The total surface area is 12 triangles (top and bottom) and 6 rectangles

$$A = 12 \left(\frac{\sqrt{3}L^2}{4} \right) + 6HL \quad \text{M1}$$

$$A = 3\sqrt{3}L^2 + 6L \left(\frac{20\sqrt{3}}{9L^2} \right) \text{ since } L > 0 \text{ and } H > 0$$

$$A = 3\sqrt{3}L^2 + \frac{40\sqrt{3}}{3L}$$

c. for a minimum surface area

$$\frac{dA}{dL} = 6\sqrt{3}L - \frac{40\sqrt{3}}{3L^2} = 0 \quad \text{A1}$$

$$6\sqrt{3}L = \frac{40\sqrt{3}}{3L^2}$$

$$L^3 = \frac{40}{18}$$

$$L = \sqrt[3]{\frac{20}{9}} \text{ metres} \quad \text{A1}$$

Question 11

i. total number of balls $r + b + w$

$$\Pr(\text{same colour}) = \Pr(RRR) + \Pr(BBB) + \Pr(WWW)$$

$$= \frac{r(r-1)(r-2)}{(r+b+w)(r+b+w-1)(r+b+w-2)}$$

$$+ \frac{b(b-1)(b-2)}{(r+b+w)(r+b+w-1)(r+b+w-2)}$$

M1

$$+ \frac{w(w-1)(w-2)}{(r+b+w)(r+b+w-1)(r+b+w-2)}$$

$$= \frac{r(r-1)(r-2) + b(b-1)(b-2) + w(w-1)(w-2)}{(r+b+w)(r+b+w-1)(r+b+w-2)}$$

A1

ii. $\Pr(\text{all different colour})$

$$= \Pr(RBW) + \Pr(RWB) + \Pr(WRB) + \Pr(WBR) + \Pr(BWR) + \Pr(BRW)$$

$$= \frac{6rbw}{(r+b+w)(r+b+w-1)(r+b+w-2)}$$

A1

END OF SUGGESTED SOLUTIONS