

SECTION 1

1	2	3	4	5	6	7	8	9	10	11
D	E	A	D	B	C	D	E	A	C	C

12	13	14	15	16	17	18	19	20	21	22
B	B	A	C	D	B	D	D	D	C	E

Q1 $Period = \frac{\pi}{\frac{1}{3}} = 3\pi$ D

Q2 $f(x) = x^3 + 2x$, $[1, 5]$
When $x=1$, $y=3$; when $x=5$, $y=135$
 $Av.rate = \frac{135-3}{5-1} = \frac{132}{4} = 33$ E

Q3 For $f(x) = |x^2 - 9|$, the range is $[0, \infty)$.
For $f(x) = |x^2 - 9| + 3$, the range is $[3, \infty)$. A

Q4 $f(x) = \frac{1}{2}e^{3x}$, $g(x) = \log_e(2x) + 3$
 $g(f(x)) = \log_e(2f(x)) + 3 = \log_e(e^{3x}) + 3 = 3x + 3 = 3(x+1)$ D

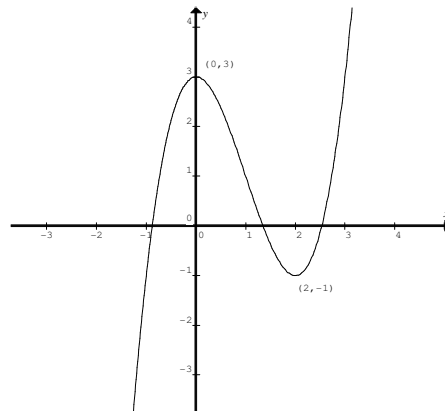
Q5
 $1x + 0y + 0z = 5$
 $0x + 1y + 1z = 10$
 $0x - 1y + 1z = 6$
 $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \\ 6 \end{bmatrix}$ B

Q6 $g'(x) = x^2 - 2x$, $(1, 0)$
 $g(x) = \frac{x^3}{3} - x^2 + c$
 $g(1) = \frac{1^3}{3} - 1^2 + c = 0$, $\therefore c = \frac{2}{3}$
 $\therefore g(x) = \frac{x^3}{3} - x^2 + \frac{2}{3}$ C

Q7 $(m-1)x + 5y = 7$ and $3x + (m-3)y = 0.7m$
Change to standard form:
 $y = -\frac{m-1}{5}x + \frac{7}{5}$ and $y = -\frac{3}{m-3}x + \frac{0.7m}{m-3}$
Equate coefficients: $-\frac{m-1}{5} = -\frac{3}{m-3}$ and $\frac{7}{5} = \frac{0.7m}{m-3}$
 $\therefore m = 6$ D

Q8 $f(x) = 3\log_e(2x)$
 $f(5x) = 3\log_e(2(5x)) = \log_e((10x)^3) = \log_e(y)$
 $\therefore y = (10x)^3 = 1000x^3$ E

Q9 The graph of $f(x) = x^3 - 3x^2 + 3$:



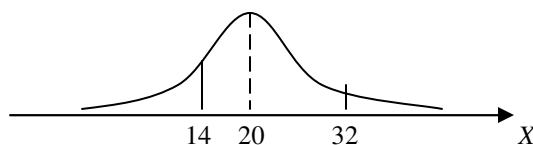
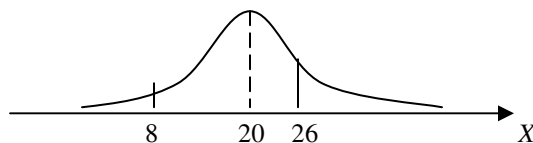
The graph is 1 to 1 in the interval $(-\infty, 0]$, $\therefore a \leq 0$ A

Q10 $Average = \frac{\int_0^\pi e^{2x} \cos(3x) dx}{\pi} \approx -26.3$ C

Q11 $\Pr(x < a) = \int_{\frac{3\pi}{4}}^a \cos(2x) dx = 0.25$
By CAS or by checking each alternative, $a \approx 2.88$ C

Q12 Binomial: $n = 15$, $p = \frac{3}{5}$
 $\Pr(X < 7) = \Pr(X \leq 6) = 0.0950$ B

Q13 $Z = \frac{X - \mu}{\sigma}$, $\therefore X = \mu + \sigma Z$
 $\Pr(-2 < Z < 1) = \Pr(8 < X < 26) = \Pr(14 < X < 32)$ B



Q14 $\Pr(all.three.are.black) = \frac{6}{10} \times \frac{5}{9} \times \frac{4}{8} = \frac{1}{6}$ A

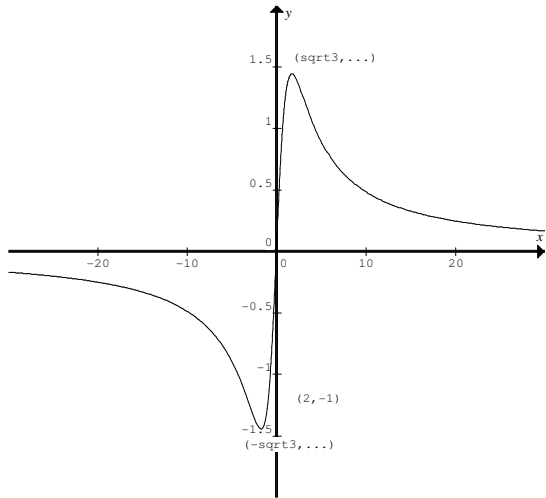
Q15 $\sum \Pr(X=x) = a + b + 0.4 = 1$ and

$$\mu = 0 \times a + 1 \times b + 2 \times 0.4 = 1$$

Solve the simultaneous equations: $a = 0.4$ and $b = 0.2$

C

Q16 The graph of $f(x) = \frac{5x}{x^2 + 3}$:

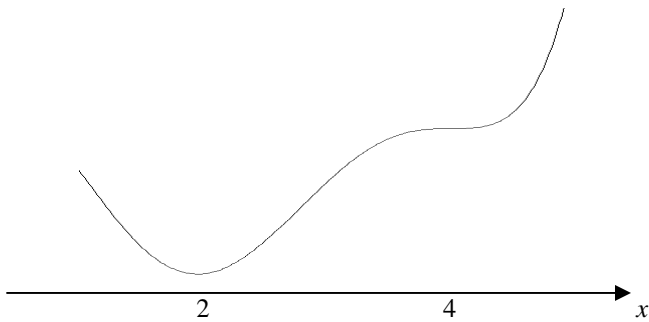


$f'(x)$ is negative for $x < -\sqrt{3}$ OR $x > \sqrt{3}$.

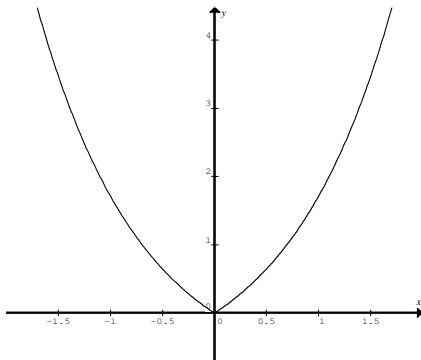
Q17 Sketch according to the given conditions:

D

B



Q18 The graph of $f(x) = e^{|x|} - 1$:



$f(x)$ is not differentiable at $x = 0$.

D

Q19 Gradient function $f'(x)$ has three x -intercepts. They are at $x < 0$, $x = 0$ and $x > 0$. $\therefore$ function $f(x)$ has stationary points at those locations. On the right of the third x -intercept, $f'(x) > 0$ $\therefore f(x)$ has a positive slope. D

$$Q20 \quad 2 \int_0^{5a} \left(f\left(\frac{x}{5}\right) + 3 \right) dx = 2 \int_0^{5a} f\left(\frac{x}{5}\right) dx + 2 \int_0^{5a} 3 dx$$

$$= 2 \int_0^a 5 f(u) \frac{du}{dx} dx + 2 \int_0^a 3 dx \quad (\text{SM})$$

$$= 10 \int_0^a f(u) du + 2[3x]_0^a$$

$$= 10a + 30a = 40a$$

D

Alternatively:

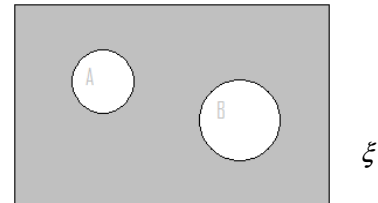
$f\left(\frac{x}{5}\right)$ is the transformation (horizontal dilation by a factor of 5)

of $f(x)$, $\therefore$ the area under the graph of $f\left(\frac{x}{5}\right)$ is 5 times that of

$$f(x), \text{ i.e. } \int_0^{5a} f\left(\frac{x}{5}\right) dx = 5 \int_0^a f(x) dx = 5a \quad (\text{FM})$$

$$2 \int_0^{5a} \left(f\left(\frac{x}{5}\right) + 3 \right) dx = 2 \int_0^{5a} f\left(\frac{x}{5}\right) dx + 2 \int_0^{5a} 3 dx = 40a$$

Q21 The Venn diagram shows mutually exclusive A and B.



The shaded region represents $A' \cap B'$.

$$\Pr(A' \cap B') = \Pr(\xi) - (\Pr(A) + \Pr(B)) = 1 - (p + q)$$

C

Q22 For $a > 1$ and $b > 1$, the range of the interval $[3, ab + 2]$ is greater than the sum of the range of the interval $[3, a + 2]$ and the range of the interval $[3, b + 2]$.

Proof: $a > 1$ and $b > 1$, $(a-1)b > (a-1)$, $ab - b > a - 1$, $ab > a + b - 1$, $ab - 1 > a + b - 2$, $ab - 1 > a - 1 + b - 1$, $\therefore (ab + 2) - 3 > (a + 2) - 3 + (b + 2) - 3$.

$\therefore f(x)$ must be a decreasing function for

$$\int_3^{ab+2} f(x) dx = \int_3^{a+2} f(x) dx + \int_3^{b+2} f(x) dx \text{ to hold.}$$

Only $f(x) = \frac{1}{x-2}$ in the given choices is a decreasing function.

E

SECTION 2

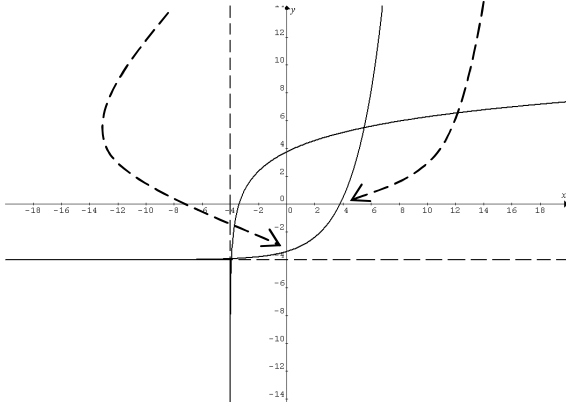
Q1ai The domain of g^{-1} is the range of g , i.e. R .

The equation of the inverse of g is $x = 2\log_e(y+4)+1$.

Express y in terms of x : $y = e^{\frac{x-1}{2}} - 4$. $\therefore g^{-1}(x) = e^{\frac{x-1}{2}} - 4$

Q1aii x -intercept, $y = 0$, $x = 2\log_e 4 + 1 = 4\log_e 2 + 1$

y -intercept: $x = 0$, $y = e^{-\frac{1}{2}} - 4$



Q1aiii Let $x = 2\log_e(x+4)+1$. By CAS calculator, $x \approx -3.914$ or $x \approx 5.503$.

Q1aiv $Area = \int_{-3.914}^{5.503} \left((2\log_e(x+4)+1) - \left(e^{\frac{x-1}{2}} - 4 \right) \right) dx \approx 52.63$
unit squares, by CAS calculator.

Q1bi $f(x) = k \log_e(x+a)+c$, $a=1$

Q1bii $y=1$ when $x=0$, $\therefore 1 = k \log_e(1)+c$, $\therefore c=1$

Q1biii From the results of Q1bi and Q1bii, and given $P(p,10)$,
 $10 = k \log_e(p+1)+1$, $\therefore k = \frac{9}{\log_e(p+1)}$.

Q1biv $f(x) = k \log_e(x+1)+1$, $f'(x) = \frac{k}{x+1}$.

At $x=p$, $f'(p) = \frac{k}{p+1} = \frac{9}{(p+1)\log_e(p+1)}$

Q1bv Equation of the tangent at $P(p,10)$:

$$y = \frac{9}{(p+1)\log_e(p+1)}(x-p)+10$$

$$\text{At } (-1,0), 0 = \frac{9}{(p+1)\log_e(p+1)}(-1-p)+10,$$

$$0 = \frac{-9}{\log_e(p+1)}+10, \log_e(p+1) = \frac{9}{10}, p = e^{0.9} - 1.$$

$$\begin{aligned} \text{Q2a } \Pr(\text{third is } R) &= \Pr(SSR) + \Pr(SRR) \\ &= 1(1-p)p + 1(1-p)(1-(p-0.2)) = 0.12 \text{ when } p = 0.9 \end{aligned}$$

$$\text{Q2b } \Pr(SSSS) = 1 \times p^3 = 0.9^3 = 0.729$$

Q2c

$$\begin{bmatrix} 0.9 & 0.7 \\ 0.1 & 0.3 \end{bmatrix}^n \rightarrow \begin{bmatrix} 0.875 & 0.875 \\ 0.125 & 0.125 \end{bmatrix} \text{ as } n \rightarrow \infty$$

$\therefore$ steady state $\Pr(S) = 0.875$

Q2di

The transition matrix is $\begin{bmatrix} p & p-0.2 \\ 1-p & 1-(p-0.2) \end{bmatrix} = \begin{bmatrix} p & p-0.2 \\ 1-p & 1.2-p \end{bmatrix}$.

State matrix for the first statue is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

State matrix for the second statue is

$$\begin{bmatrix} p & p-0.2 \\ 1-p & 1.2-p \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} p \\ 1-p \end{bmatrix}.$$

State matrix for the third statue is

$$\begin{bmatrix} p & p-0.2 \\ 1-p & 1.2-p \end{bmatrix} \begin{bmatrix} p \\ 1-p \end{bmatrix} = \begin{bmatrix} 1.2p-0.2 \\ \dots \end{bmatrix} = \begin{bmatrix} 0.7 \\ \dots \end{bmatrix}.$$

$\therefore 1.2p-0.2 = 0.7$, $p = 0.75$

$$\begin{aligned} \text{Q2dii } \Pr(S|S) &= p = 0.75, \Pr(R|S) = 0.25, \\ \Pr(S|R) &= p-0.2 = 0.55, \Pr(R|R) = 0.45 \end{aligned}$$

$$\Pr(\text{no } S's) = \Pr(RRR) = 1 \times 0.45 \times 0.45 = 0.2025$$

$$\Pr(1S) = \Pr(RSR) + \Pr(RRS)$$

$$= 1 \times 0.55 \times 0.25 + 1 \times 0.45 \times 0.55 = 0.385$$

$$\Pr(2S's) = \Pr(RSS) = 1 \times 0.55 \times 0.75 = 0.4125$$

x	0	1	2
$\Pr(X=x)$	0.2025	0.385	0.4125

$$E(x) = 0 \times 0.2025 + 1 \times 0.385 + 2 \times 0.4125 = 1.21$$

Q2e Binomial: n statues, $p_s = 0.2$

$$\Pr(X \geq 2) \geq 0.9, 1 - \Pr(X \leq 1) \geq 0.9$$

$$\therefore \Pr(X \leq 1) \leq 0.1, \text{binomcdf}(n, 0.2, 1) \leq 0.1$$

Use CAS calculator to find n , $n \geq 18$ $\therefore$ minimum n is 18.

$$\text{Q3ai } \frac{\overline{AZ}}{10} = \cos(x), \overline{AZ} = 10\cos(x), \therefore \overline{AB} = 20\cos(x)$$

$$\text{Q3aii } \frac{\overline{WZ}}{10} = \sin(x), \overline{WZ} = 10\sin(x)$$

Q3b

$$\begin{aligned} \text{Total surface area } S &= (20\cos(x))^2 + 4\left(\frac{1}{2}(20\cos(x))(10\sin(x))\right) \\ &= 400(\cos^2(x) + \cos(x)\sin(x)) \end{aligned}$$

$$\begin{aligned} \text{Q3c } \overline{WY} &= \sqrt{WZ^2 + ZY^2} = \sqrt{100\sin^2(x) - 100\cos^2(x)} \\ &= 10\sqrt{\sin^2(x) - \cos^2(x)} = 10\sqrt{1 - 2\cos^2(x)} \end{aligned}$$

$$\begin{aligned} \text{Q3d Volume } T &= \frac{1}{3} \times (20\cos(x))^2 \times 10\sqrt{1 - 2\cos^2(x)} \\ &= \frac{4000}{3} \cos^2(x) \sqrt{1 - 2\cos^2(x)} = \frac{4000}{3} \sqrt{\cos^4(x)(1 - 2\cos^2(x))} \\ &= \frac{4000}{3} \sqrt{\cos^4(x) - 2\cos^6(x)} \end{aligned}$$

$$\begin{aligned} \text{Q3e } \frac{dT}{dx} &= \frac{4000}{3} \times \frac{-4\cos^3(x)\sin(x) + 12\cos^5(x)\sin(x)}{2\sqrt{\cos^4(x) - 2\cos^6(x)}} \\ &= \frac{8000}{3} \times \frac{-\cos^3(x)\sin(x) + 3\cos^5(x)\sin(x)}{\sqrt{\cos^4(x) - 2\cos^6(x)}} \end{aligned}$$

$$\text{Let } \frac{dT}{dx} = 0, \therefore -\cos^3(x)\sin(x) + 3\cos^5(x)\sin(x) = 0$$

$$\therefore \cos^3(x)\sin(x)(3\cos^2(x) - 1) = 0$$

$$\text{Since } \frac{\pi}{4} < x < \frac{\pi}{2}, \therefore 3\cos^2(x) - 1 = 0, \cos(x) = \frac{1}{\sqrt{3}},$$

$$x = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right).$$

$$\therefore \text{maximum } T = \frac{4000}{3} \sqrt{\frac{1}{9} - \frac{2}{27}} = \frac{4000}{3\sqrt{27}} = \frac{4000}{9\sqrt{3}} = \frac{4000\sqrt{3}}{27} \text{ m}^3$$

$$\text{Q3f Let } \frac{4000}{3} \sqrt{\cos^4(x) - 2\cos^6(x)} = \frac{1}{2} \times \frac{4000}{3\sqrt{27}}$$

$$\therefore \sqrt{\cos^4(x) - 2\cos^6(x)} = \frac{1}{2\sqrt{27}}$$

$$\cos^4(x) - 2\cos^6(x) = \frac{1}{4 \times 27} = \frac{1}{108}$$

$$\therefore x \approx 0.81 \text{ or } 1.23 \text{ radians by CAS calculator.}$$

$$\text{Q4a } f(x) = \frac{1}{27}(2x-1)^3(6-3x) + 1 = -\frac{1}{9}(2x-1)^3(x-2) + 1$$

$$f'(x) = -\frac{1}{9}((2x-1)^3 + 6(2x-1)^2(x-2)) = -\frac{1}{9}(2x-1)^2(8x-13)$$

$$\therefore \text{stationary points are at } x = \frac{1}{2} \text{ (inflection) and } x = \frac{13}{8}$$

(maximum). The nature of each point is determined by sketching $f(x)$.

$$\text{Q4b } f(x) = \frac{1}{27}(ax-1)^3(b-3x) + 1$$

$$f'(x) = \frac{1}{9}(a(ax-1)^2(b-3x) - (ax-1)^3)$$

$$= \frac{1}{9}(ax-1)^2(a(b-3x) - (ax-1))$$

$$= \frac{1}{9}(ax-1)^2(ab+1-4ax)$$

$$\text{Stationary points are at } x = \frac{1}{a} \text{ and } x = \frac{ab+1}{4a}.$$

$$\text{Q4c } x = \frac{1}{a} \text{ and } x = \frac{ab+1}{4a} \text{ are undefined when } a = 0, \text{ i.e. no stationary points when } a = 0.$$

$$\text{Q4d One stationary point when } \frac{1}{a} = \frac{ab+1}{4a}, \text{ i.e. } a = \frac{3}{b}.$$

Q4e The maximum number of stationary points is 3 for quartic polynomial functions. They are either local max. or min. In this

case, quartic $f(x)$ is in the form of $\frac{1}{27}(ax-1)^3(b-3x) + 1$, it

has a stationary inflection point and a maximum (or minimum).
 $\therefore$ the maximum number of stationary points is 2.

$$\text{Q4f } f'(x) = \frac{1}{9}(ax-1)^2(ab+1-4ax)$$

$$\text{Stationary points are at } x = \frac{1}{a} \text{ and } x = \frac{ab+1}{4a}.$$

Given two stationary points $(1,1)$ and (p,p) .

$$\text{Consider the possibility: } p = \frac{1}{a} \text{ and } 1 = \frac{ab+1}{4a}$$

$$\therefore f(p) = \frac{1}{27}(ap-1)^3(b-3p) + 1 = 1, \therefore f(p) \neq p \text{ and } p \neq \frac{1}{a}.$$

$$\text{Consider the possibility: } 1 = \frac{1}{a} \text{ and } p = \frac{ab+1}{4a}$$

$$\therefore a = 1 \text{ and } b = 4p - 1$$

$$\therefore f(1) = 1$$

$$\therefore f(p) = \frac{1}{27}(p-1)^3(4p-1-3p) + 1 = \frac{1}{27}(p-1)^4 + 1 = p$$

$$\therefore \frac{1}{27}(p-1)^4 - (p-1) = 0, (p-1)\left(\frac{1}{27}(p-1)^3 - 1\right) = 0$$

$$\text{Since } p \neq 1, \text{ i.e. } p-1 \neq 0, \therefore \frac{1}{27}(p-1)^3 - 1 = 0$$

$$\text{Hence } (p-1)^3 = 27, p-1 = 3, p = 4.$$

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