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# ***Mathematical Methods (CAS)***

## ***2010***

### ***Trial Examination 2***

## SECTION 1 Multiple-choice questions

### Instructions for Section 1

Answer **all** questions.

Choose the response that is **correct** for the question.

A correct answer scores 1, an incorrect answer scores 0.

Marks will **not** be deducted for incorrect answers.

**No** marks will be given if more than one answer is completed for any question.

### Question 1

The domain and range of  $\log \frac{a}{\sqrt{a-x^2}}$ ,  $a > 0$ , are respectively

- A.  $(-\infty, -\sqrt{a}) \cup (\sqrt{a}, \infty)$  and  $\left(\frac{1}{2}\log a, \infty\right)$
- B.  $[-\sqrt{a}, \sqrt{a}]$  and  $\left(\frac{1}{2}\log a, \infty\right)$
- C.  $(-\infty, -a] \cup [a, \infty)$  and  $(-\infty, \log a]$
- D.  $[-a, a]$  and  $(\log a, \infty)$
- E.  $(-\sqrt{a}, \sqrt{a})$  and  $[\log \sqrt{a}, \infty)$

### Question 2

The graphs of  $y = b - |x - a|$  and  $y = |x + a|$  for  $b > 2a > 0$  intersect at  $(x_1, y_1)$  and  $(x_2, y_2)$ . Which one of the following statements is true?

- A.  $x_1 + x_2 = a$  and  $y_1 + y_2 = b$
- B.  $x_1 + x_2 = -a$  and  $y_1 + y_2 = -b$
- C.  $x_1 + x_2 = 0$  and  $y_1 + y_2 = b$
- D.  $x_1 + x_2 = b$  and  $y_1 + y_2 = a$
- E.  $x_1 + x_2 = 0$  and  $y_1 + y_2 = \frac{b}{2}$

### Question 3

If  $4x - 4y - 3z = 0$  and  $3x + 3y + 2z = 0$ , the  $x$  and  $y$  values that can **never** satisfy both equations simultaneously are

- A.  $x = 0.2$  and  $y = -3.4$
- B.  $x = -1$  and  $y = 17$
- C.  $x = \frac{4}{5}$  and  $y = -\frac{68}{5}$
- D.  $x = -2.1$  and  $y = 35.5$
- E.  $x = 0.05$  and  $y = -0.85$

#### Question 4

A transformation  $T : R^2 \rightarrow R^2$  that maps the curve  $y = f(x)$  onto the curve  $y = 2 - f\left(-\frac{x+1}{3}\right)$  is given by

- A.  $T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix}$
- B.  $T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \end{bmatrix}$
- C.  $T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix}$
- D.  $T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \end{bmatrix}$
- E.  $T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

#### Question 5

All parabolas of the form  $y = ax^2$  can be changed to  $y = x^2$  under

- A. a dilation from the  $y$ -axis by a factor of  $a$
- B. a dilation from the  $y$ -axis by a factor of  $\frac{1}{a}$
- C. a dilation from the  $x$ -axis by a factor of  $\frac{1}{a}$
- D. a dilation from the  $x$ -axis and a dilation from the  $y$ -axis by the same factor  $a$
- E. a dilation from the  $x$ -axis and a dilation from the  $y$ -axis by the same factor  $\frac{1}{a}$

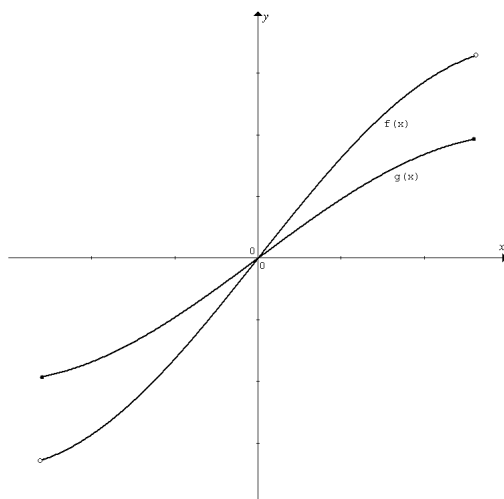
#### Question 6

$\sin(2x)$  is equivalent to

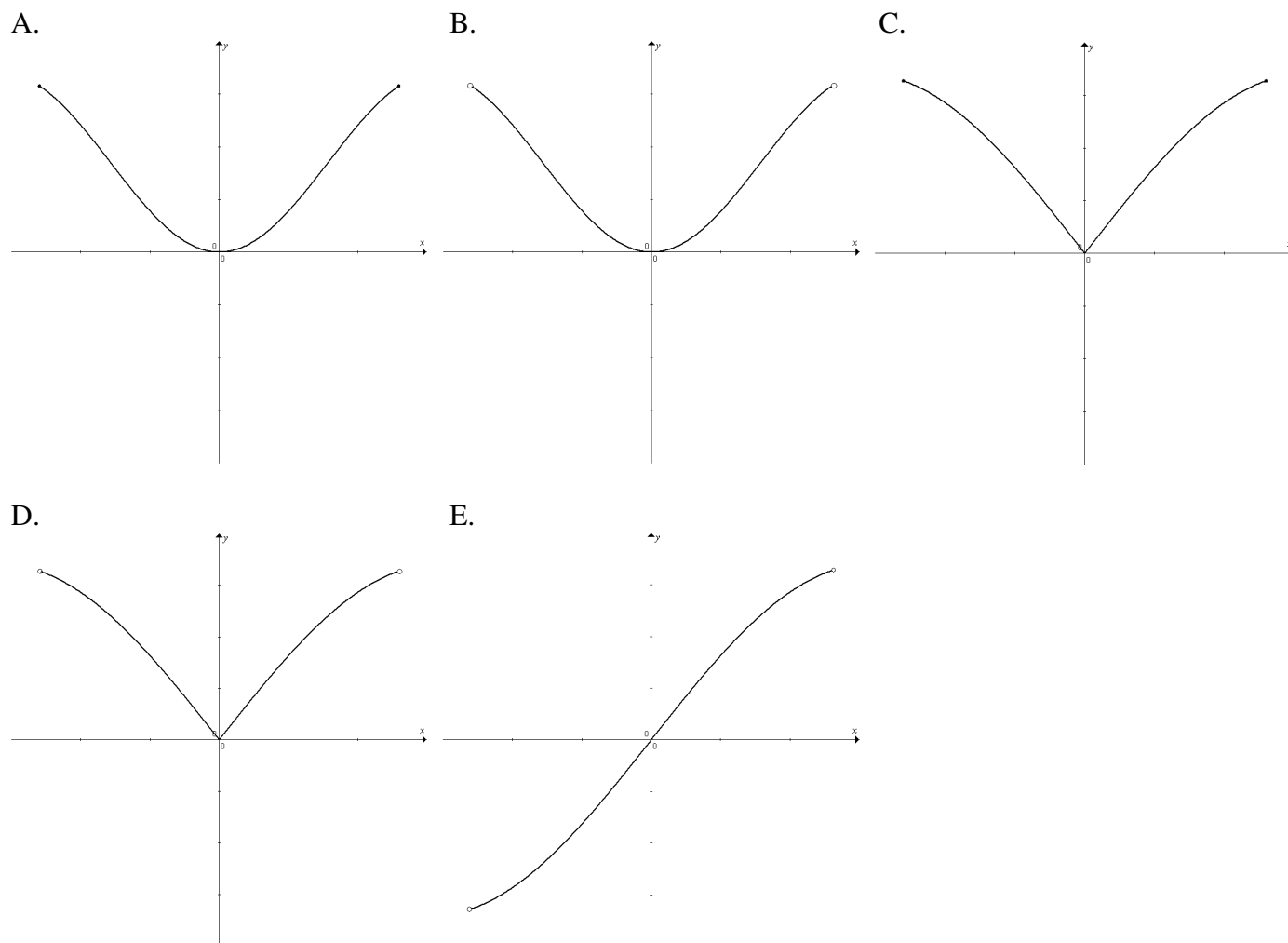
- A.  $(1 - \sin x + \cos x)(1 + \sin x + \cos x)$
- B.  $(1 - \sin x + \cos x)(1 + \sin x - \cos x)$
- C.  $(\sin x + \cos x)(\sin x - \cos x)$
- D.  $(\cos x - \sin x)(\sin x + \cos x)$
- E.  $(\sin x + \cos x)^2$

### Question 7

The graphs of  $f(x)$  and  $g(x)$  are shown below.



The graph of  $f(x) \times g(x)$  is closest to



**Question 8**

Which one of the following functions does **NOT** have the property  $f(f(x)) = x$ ?

- A.  $f(x) = \frac{1}{x}$
- B.  $f(x) = -x$
- C.  $f(x) = x$
- D.  $f(x) = 1$
- E.  $f(x) = -\frac{1}{x}$

**Question 9**

If  $e^{x-1} - e^{2x} = \frac{1}{(2e)^2}$ , then

- A.  $x = -(\log_e 2 + 1)$
- B.  $x = \log_e 2 - 1$
- C.  $x = \log_e 2 + 1$
- D.  $x = \log_e 2 + e$
- E.  $x = -(\log_e 2 + e)$

**Question 10**

The general solution to the equation  $\sqrt{3}\sin(5x) + \cos(5x) = 0$  is

- A.  $x = \frac{2\pi}{15} + \frac{n\pi}{5}, n \in Z$
- B.  $x = \frac{n\pi}{5} - \frac{2\pi}{15}, n \in Z$
- C.  $x = \frac{n\pi}{5} \pm \frac{2\pi}{15}, n \in Z$
- D.  $x = \frac{\pi}{6} + \frac{n\pi}{5}, n \in Z$
- E.  $x = \frac{n\pi}{5} - \frac{\pi}{6}, n \in Z$

**Question 11**

The tangent at the point  $(-2,4)$  on the curve  $y = f(x)$  has equation  $y = \frac{1}{2}x + 5$ .

The tangent at the point  $(-2,1)$  on the curve  $y = -f(x) + 5$  has equation

- A.  $y = -\frac{1}{2}x$
- B.  $y = -2x - 3$
- C.  $y = 2x + 5$
- D.  $y = -\frac{1}{2}x + 3$
- E.  $y = -2x + 1$

**Question 12**

The inverse of the function  $f : [-3,0] \rightarrow R$ ,  $f(x) = -3\sqrt{1 - \frac{x^2}{9}}$  is

- A.  $f^{-1} : (-3,0] \rightarrow R$ ,  $f^{-1}(x) = -\sqrt{9 - x^2}$
- B.  $f^{-1} : [-3,0) \rightarrow R$ ,  $f^{-1}(x) = -3\sqrt{1 - \frac{x^2}{9}}$
- C.  $f^{-1} : [-3,0) \rightarrow R$ ,  $f^{-1}(x) = 3\sqrt{1 - \frac{x^2}{9}}$
- D.  $f^{-1} : (-3,0] \rightarrow R$ ,  $f^{-1}(x) = \sqrt{9 - x^2}$
- E.  $f^{-1} : (0,3] \rightarrow R$ ,  $f^{-1}(x) = \sqrt{9 - x^2}$

**Question 13**

Given  $g(x) = (1 + f(x))\log_e(1 + f(x))$  and  $f(1) = e - 1$ ,  $g'(1)$  is equal to

- A.  $f'(1)$
- B.  $2f'(1)$
- C.  $1 + f'(1)$
- D.  $e + f'(1)$
- E.  $e - 1 + f'(1)$

**Question 14**

The graphs of  $y = \frac{1}{x}$  and  $y = ax^2 - 1$  have exactly one intersection when

- A.  $a > \frac{3}{25}$
- B.  $a > \frac{3}{26}$
- C.  $a > \frac{6}{47}$
- D.  $a > \frac{7}{48}$
- E.  $a > \frac{4}{27}$

**Question 15**

The function  $f : (-\infty, 0] \rightarrow \mathbb{R}$ ,  $f(x) = \frac{1}{2}(x+1)^5 - 1$  is increasing on

- A.  $[-\infty, 0]$
- B.  $[-\infty, 0)$
- C.  $(-\infty, 1]$
- D.  $(-\infty, -1]$  and  $[-1, 0]$
- E.  $\mathbb{R}$

**Question 16**

$g(x)$  is continuous and differentiable over the interval  $(-1, 1)$ .

If  $g(0) = g'(0) = \frac{1}{2}$ , the value of  $g(0.01)$  by Euler's linear approximation method is closest to

- A. 1.01
- B. 0.505
- C. 0.483
- D. 0.482
- E. 0.481

**Question 17**

The speed of a particle moving in a straight line is given by  $v(t) = 10\pi \cos\left(\frac{\pi t}{2a}\right)$ ,  $t \in [0, a]$ .

The average speed of the particle over the interval  $[0, a]$  is

- A. 20
- B.  $20a$
- C.  $20\pi$
- D.  $\frac{10\pi}{a}$
- E.  $\frac{10a}{\pi}$

**Question 18**

At time  $t$  minutes water is drained from a large tank at a rate given by  $20e^{-0.25t}$  litres per minute. The volume of water (in litres) drained from the tank in the first 4 minutes is

- A.  $20(e - 1)$
- B.  $20(1 - e^{-1})$
- C.  $80(1 - e^{-1})$
- D.  $80e^{-1}$
- E.  $80e$

**Question 19**

18 unbiased dice are rolled. The expected number of 1 or 6 appearing uppermost is closest to

- A. 3
- B. 4
- C. 5
- D. 6
- E. 7

**Question 20**

$f(x) = |a(x \cos(2x) + 1)|$ , where  $x \in [0, \pi]$ , is a probability density function. The value of  $|a|$  is closest to

- A. 0.15
- B. 0.16
- C. 0.25
- D. 0.26
- E. 0.35

**Question 21**

If  $\Pr(A) = \frac{2}{3}$  and  $\Pr(A|B) = \Pr(B|A) = \frac{1}{2}$ , then  $\Pr(A' \cap B') =$

- A. 0
- B.  $\frac{1}{6}$
- C.  $\frac{1}{5}$
- D.  $\frac{1}{4}$
- E.  $\frac{2}{5}$

**Question 22**

$X \sim N(3, 3)$  can be transformed to  $Z \sim N(0, 1)$ .

If a probability is given by  $\Pr(Z < \sqrt{3})$  after the transformation and the corresponding probability before the transformation is given by  $\Pr(X < x)$ , then the value of  $x$  is

- A. 9
- B.  $3(\sqrt{3} + 1)$
- C. 6
- D.  $3(\sqrt{3} - 1)$
- E. 2

## Instructions for Section 2

Answer **all** questions.

A decimal approximation will not be accepted if an **exact** answer is required to a question.

In questions where more than one mark is available, appropriate working **must** be shown.

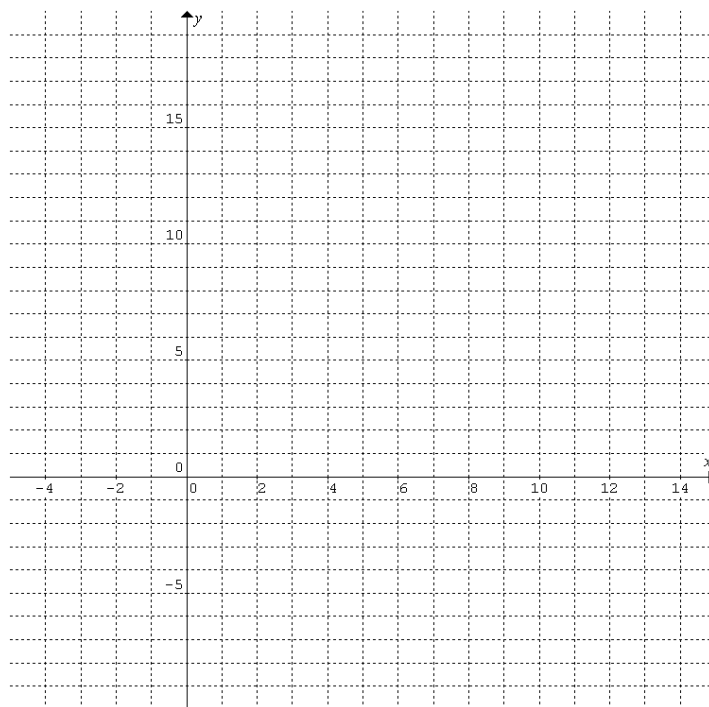
Unless otherwise indicated, the diagrams in this exam are **not** drawn to scale.

### Question 1

Let  $f : [0, 6] \rightarrow \mathbb{R}$ ,  $f(x) = 6x - x^2$ .

a. Sketch the graph of function  $f$ . Label the maximum with its coordinates.

2 marks



b. A trapezium is to be fitted in the region bounded by  $y = f(x)$  and the  $x$ -axis.

Let  $x = a$ , where  $3 < a < 6$ , be the  $x$ -coordinate of one of the vertex of the trapezium on the curve.

i. Find the area of the trapezium in terms of  $a$ .

2 marks

ii. State the area of the largest trapezium and the value of  $a$  when it occurs.

1 mark

c. Let  $P(b, (6b - b^2))$  be a point on the curve  $y = f(x)$ , where  $3 < b < 6$ .

i. Show that the equation of the tangent to the curve  $y = f(x)$  at  $P$  is  $y = 2(3 - b)x + b^2$ . 2 marks

ii. Find the  $x$  and  $y$ -intercepts of the tangent in terms of  $b$ . 1 mark

iii. Hence show that the area enclosed by the tangent and the axes is a minimum when  $b = a$ , the value found in Question 1bii. 3 marks

d.  $A(6,0)$  and  $B(2,8)$  are two points on the curve  $y = f(x)$ .

i. Show that the chord  $AB$  is parallel to the tangent at  $P$ . 2 marks

ii. Hence explain that  $\Delta PAB$  has the greatest area among the triangles in the region bounded by the chord  $AB$  and the curve  $y = f(x)$ .

1 mark

iii. Find the area of the region bounded by the chord  $AB$  and the curve  $y = f(x)$ . 2 marks

## Question 2

Let  $x = a \sin \theta$ , where  $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$  and  $a \in \mathbb{R} \setminus \{0\}$ .

**a.** Find  $\cos \theta$  in terms of parameter  $a$  and variable  $x$ .

2 marks

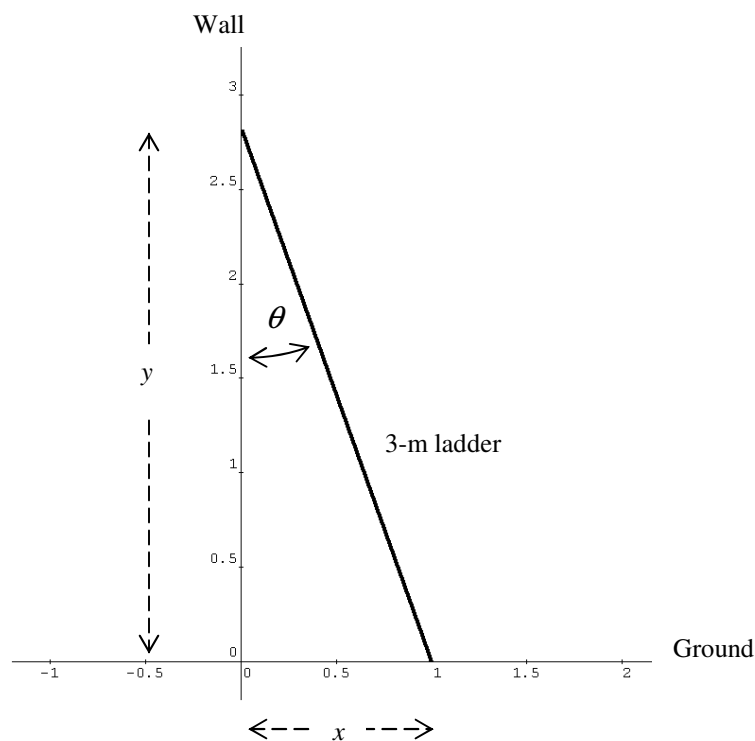
**b.** Given that  $\frac{dx}{d\theta} = k\sqrt{b-x^2}$ , find  $k$  and  $b$  in terms of  $a$ .

3 marks

**c.** Given that  $\frac{d\theta}{dx} = \frac{1}{\frac{dx}{d\theta}}$ , find the derivative of  $\sin^{-1}\left(\frac{x}{a}\right)$  for (i)  $a > 0$  and (ii)  $a < 0$ .

2 marks

A 3-m ladder leans against a vertical wall. The bottom of the ladder slides away from the base of the wall at a speed of 0.6 m/s.



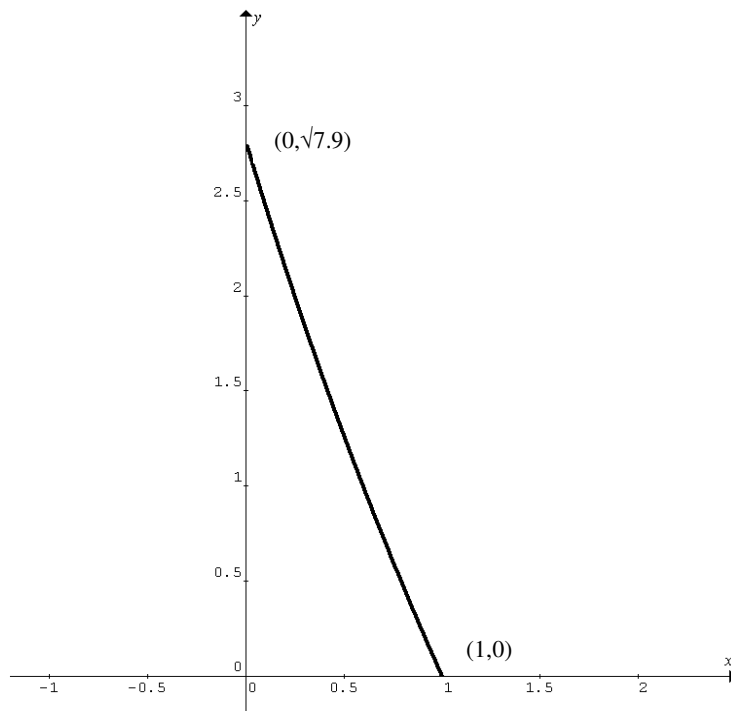
**d.** How fast is  $\theta$  (the angle between the ladder and the wall) changing when the bottom of the ladder is 1 m from the base of the wall? Express your answer in degrees per second, correct to 1 decimal place.

3 marks

**e.** Hence or otherwise, find the speed of the top of the ladder when the bottom of the ladder is 1 m from the base of the wall. Express your answer in m/s, correct to 1 decimal place.

2 marks

Now the ladder is prevented from sliding. A heavy load is placed on the ladder causing it to bend. The bottom of the ladder is 1 m from the base of the wall, and the top of the ladder is  $\sqrt{7.9}$  m from the ground. The shape of the ladder is given by the equation  $y = A[\log_e(x+2)+c]$ , where  $A$  and  $c$  are constants. All length measures are in metres.



f. Show that  $A = -\frac{\sqrt{7.9}}{\log_e 1.5}$  and  $c = -\log_e 3$ .

2 marks

g. Determine the average distance of the ladder from the ground. Express your answer in metres, correct to 2 decimal places.

2 marks

### Question 3

Certain bolts manufactured by Company A have length  $L = 180 + X$  mm, where  $X$  is a random variable with

probability density function 
$$f(x) = \begin{cases} k \cos^4\left(\frac{\pi x}{2}\right) & \text{for } -1 \leq x \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

**a.** Use a double angle formula to show that  $\cos^4\left(\frac{\pi x}{2}\right) = \frac{1}{8}(\cos(2\pi x) + 4\cos(\pi x) + 3)$ . 2 marks

**b.** Hence or otherwise, show that  $k = \frac{4}{3}$ . 2 marks

**c.** Determine  $c$ , correct to 3 decimal places, so that with a probability of 95% a bolt will have any length between  $180 - c$  and  $180 + c$ . 2 marks

**d.** Given that a bolt has a length between  $180 - c$  and  $180 + c$ , find the probability, correct to 3 decimal places, that it is longer than 180.5 mm. 2 marks

**e.** Bolts from a very large batch made by Company A are inspected one by one.

Find the probability, correct to 3 decimal places, which a bolt outside the range from  $180 - c$  to  $180 + c$  first appears in the tenth inspection.

2 marks

**f.** The bolts are sold in boxes of 20. Find the probability, correct to 3 decimal places, that 95% or more of the bolts in a box are in the range from  $180 - c$  to  $180 + c$ .

2 marks

Company B manufactures the same type of bolts. The length of the bolts has a normal distribution. The mean length is 180 mm and 95% of the bolts are in the range from  $180 - c$  to  $180 + c$ , same as those manufactured by Company A.

**g.** Find the standard deviation, correct to 3 decimal places, of the length of the bolts made by Company B.

1 mark

**h.** From which company would you purchase the bolts if you need them as close to 180 mm as possible? Justify your answer with calculations.

2 marks

**Question 4**

Consider the equation  $\cos x + e^{2y} + 1.5z^3 = 0$ .

**a. i.** Express  $y$  in terms of  $x$  and  $z$ .

1 mark

**ii.** Find the exact value(s) of  $y$  when  $x = \pi$  and  $z = e^{\frac{1}{3}y}$ .

2 marks

Let  $\left\{ \begin{array}{l} -\cos x + e^{2y} + 0.5z^3 = -2, \\ 2\cos x + 9e^{2y} + z^3 = -1, \\ 3\cos x + 8e^{2y} + 0.5z^3 = 1 \end{array} \right\}$  be a set of simultaneous equations and  $0 \leq x \leq \pi$ .

**b.** Show that there exists an infinite number of solutions to the set of simultaneous equations.

3 marks

c. If  $x = \pi$ ,  $y = m$  and  $z = n$  are solutions to the equations, find  $m$  and  $n$ .

2 marks

d. Given that  $x = \pi$ ,  $y = 0$  and  $z = -2$  are the **only** solutions to the set of simultaneous equations

$$\begin{cases} -\cos x + e^{2y} + pz^3 = q, \\ 2\cos x + 9e^{2y} + z^3 = -1, \\ 3\cos x + 8e^{2y} + 0.5z^3 = 1 \end{cases}, \text{ find the possible values of coefficients } p \text{ and } q.$$

3 marks

**End of exam 2**