
Question 1

a. $y = e^{3x}(x^2 - 1)$

$$\frac{dy}{dx} = 3e^{3x}(x^2 - 1) + e^{3x} \times 2x$$

(1 mark) –using product rule

$$= 3x^2e^{3x} - 3e^{3x} + 2xe^{3x}$$

(1 mark) – correct answer

$$= e^{3x}(3x^2 + 2x - 3) \quad (\text{optional line})$$

b. $f(x) = \log_e(\cos(x))$

Method 1 – short way

$$f'(x) = \log_e(\cos(x))$$

$$f'(x) = \frac{-\sin(x)}{\cos(x)}$$

$$= -\tan(x)$$

(1 mark)

$$f'(\pi) = -\tan(\pi)$$

$$= 0$$

(1 mark)

Method 2 – long way

Let $y = \log_e(\cos(x))$

$$y = \log_e(u) \text{ where } u = \cos(x)$$

$$\frac{dy}{du} = \frac{1}{u} \quad \frac{du}{dx} = -\sin(x)$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \quad (\text{chain rule})$$

$$= \frac{1}{u} \times -\sin(x)$$

$$= \frac{1}{\cos(x)} \times -\sin(x)$$

$$= -\tan(x)$$

(1 mark)

So $f'(\pi) = -\tan(\pi)$

$$= 0$$

(1 mark)

Question 2

a.
$$\int (\sqrt{x} + e^{2x}) dx$$

$$= \int \left(x^{\frac{1}{2}} + e^{2x} \right) dx$$

$$= \frac{2x^{\frac{3}{2}}}{3} + \frac{1}{2} e^{2x} + c$$

(1 mark)

b.
$$f'(x) = \sin(3x)$$

$$f(x) = \int \sin(3x) dx$$

$$= -\frac{1}{3} \cos(3x) + c$$

(1 mark)

Given $f(\pi) = \frac{4}{3}$

$$\frac{4}{3} = -\frac{1}{3} \cos(3\pi) + c$$

$$\frac{4}{3} = -\frac{1}{3} \times -1 + c$$

$$\frac{4}{3} = \frac{1}{3} + c$$

$$c = 1$$

So $f(x) = -\frac{1}{3} \cos(3x) + 1$

(1 mark)**Question 3**

$f(x) = \log_e(x), \quad x > 0$

Show $f(u) - 2f\left(\frac{1}{v}\right) = f(uv^2)$

$$\begin{aligned} LHS &= f(u) - 2f\left(\frac{1}{v}\right) \\ &= \log_e(u) - 2\log_e\left(\frac{1}{v}\right) \\ &= \log_e(u) - \log_e\left(\frac{1}{v}\right)^2 \\ &= \log_e(u) - \log_e\left(\frac{1}{v^2}\right) \\ &= \log_e\left(u \div \frac{1}{v^2}\right) \\ &= \log_e(uv^2) \\ &= f(uv^2) \\ &= RHS \text{ as required.} \end{aligned}$$

(1 mark)**(1 mark)**

Question 4

a. Draw a diagram.

		BLACK					
		1	2	3	4	5	6
R E D	1	x	.	.	.	.	.
	2	.	x	.	.	.	.
	3	.	.	x	.	.	.
	4	.	.	.	x	.	.
	5	.	.	.	.	x	.
	6	.	.	.	.	.	x

$$\Pr(1,1) + \Pr(2,2) + \Pr(3,3) + \Pr(4,4) + \Pr(5,5) + \Pr(6,6)$$

$$= \frac{1}{36} + \frac{1}{36} + \frac{1}{36} + \frac{1}{36} + \frac{1}{36} + \frac{1}{36}$$

$$= \frac{6}{36}$$

$$= \frac{1}{6}$$

(1 mark)

b.

		BLACK					
		1	2	3	4	5	6
R E D	1	.	x	x	x	x	x
	2	.	.	x	x	x	x
	3	.	.	.	x	x	x
	4	.	.	.	.	x	x
	5	.	.	.	.	.	x
	6	.	.	.	.	.	.

$$\Pr(\text{no. on red} < \text{no. on black})$$

$$= \frac{5}{36} + \frac{4}{36} + \frac{3}{36} + \frac{2}{36} + \frac{1}{36}$$

$$= \frac{15}{36}$$

$$= \frac{5}{12}$$

(1 mark)

- c. This represents a binomial distribution with $n = 4$.
 Since odd numbers occur on both die on 9 occasions (from the diagram),

		BLACK					
		1	2	3	4	5	6
R E D	1	x	.	x	.	x	.
	2	.	.	.	.	.	.
	3	x	.	x	.	x	.
	4	.	.	.	.	.	.
	5	x	.	x	.	x	.
	6	.	.	.	.	.	.

$$p = \frac{9}{36} = \frac{1}{4}.$$

(1 mark)

$$\Pr(X \geq 1) = 1 - \Pr(X = 0)$$

$$= 1 - {}^4C_0 \left(\frac{1}{4}\right)^0 \left(\frac{3}{4}\right)^4$$

$$= 1 - \left(\frac{3}{4}\right)^4 \quad \text{Note: } {}^4C_0 = 1 \text{ and } \left(\frac{1}{4}\right)^0 = 1$$

$$= 1 - \frac{81}{256}$$

$$= \frac{175}{256}$$

(1 mark)

Question 5

$$g: R \rightarrow R, g(x) = e^{x+1} - 2$$

$$\text{Let } y = e^{x+1} - 2$$

Swap x and y for inverse

$$x = e^{y+1} - 2$$

(1 mark)

Rearrange

$$x + 2 = e^{y+1}$$

$$\log_e(x+2) = y+1$$

$$y = \log_e(x+2) - 1$$

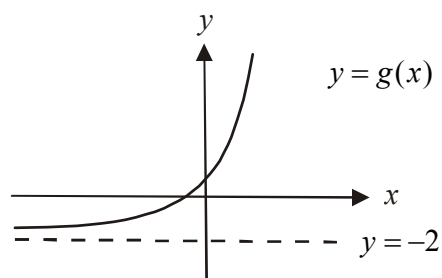
$$\text{So } g^{-1}(x) = \log_e(x+2) - 1$$

Do a quick sketch of $y = g(x)$.

$$d_g = R, \quad r_g = (-2, \infty)$$

$$\text{So, } d_{g^{-1}} = (-2, \infty) \text{ and } r_{g^{-1}} = R$$

$$\text{So } g^{-1}: (-2, \infty) \rightarrow R, g^{-1}(x) = \log_e(x+2) - 1$$

Note that to define a function you must give the rule (equation) **and** the domain.

(1 mark) – correct domain

(1 mark) – correct rule

Question 6

$$mx + y = 2 \quad -(1)$$

$$2x + (m-1)y = m \quad -(2)$$

For no solutions or infinite solutions the determinant of the matrix $\begin{bmatrix} m & 1 \\ 2 & m-1 \end{bmatrix}$ equals zero.

That is, $\begin{vmatrix} m & 1 \\ 2 & m-1 \end{vmatrix} = 0$

(1 mark)

$$m(m-1) - 2 = 0$$

$$m^2 - m - 2 = 0$$

$$(m-2)(m+1) = 0$$

$$m = 2 \text{ or } m = -1$$

(1 mark)

If $m = 2$,

in (1) $2x + y = 2$

in (2) $2x + y = 2$

They are the same equation hence there are an infinite number of solutions.

If $m = -1$,

in (1) $-x + y = 2 \quad -(3)$

in (2) $2x - 2y = -1 \quad -(4)$

$$(4) \div -2 \quad -x + y = \frac{1}{2} \quad -(5)$$

(3) and (5) describe parallel lines with different y -intercepts so there are no points of intersection and hence no solutions.

So for $m = -1$ there is no solution.

(1 mark)**Question 7**

X	2	3	4	5	6
$\Pr(X = x)$	0.2	0.4	0.1	0.2	0.1

a. Median = 3

(1 mark)

b. $\Pr(X \geq 3 | X < 6)$

$$= \frac{\Pr(X \geq 3) \cap \Pr(X < 6)}{\Pr(X < 6)}$$

(1 mark)

$$= \frac{\Pr(X = 3) + \Pr(X = 4) + \Pr(X = 5)}{1 - \Pr(X = 6)}$$

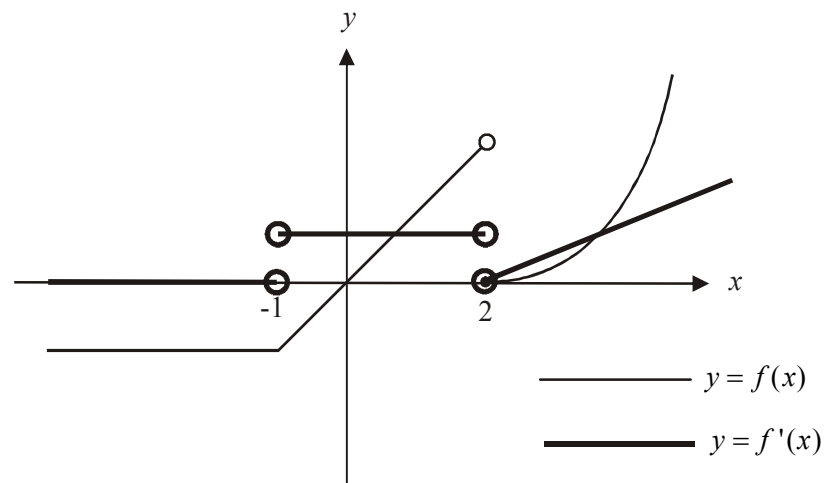
$$= \frac{0.4 + 0.1 + 0.2}{0.9}$$

$$= \frac{7}{9}$$

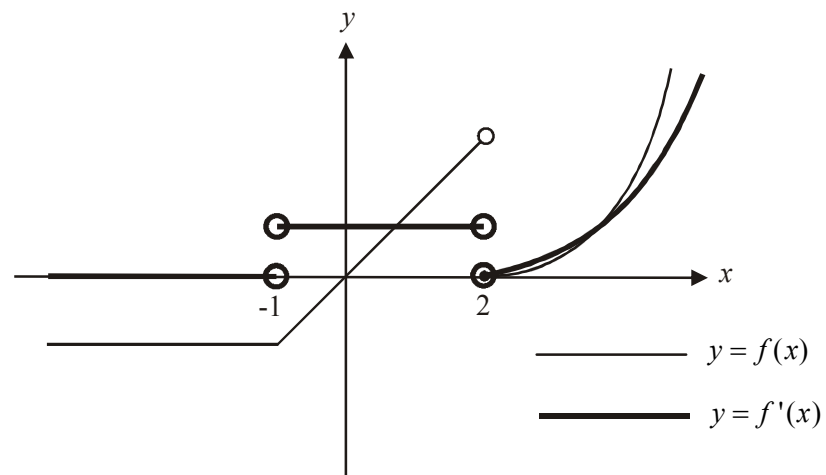
(1 mark)

Question 8

a.



OR



(1 mark) – correct left branch
 (1 mark) – correct middle branch
 (1 mark) – correct right branch
 (curved or straight)

b. $d_{f'} = \mathbb{R} \setminus \{-1, 2\}$

(1 mark)

Question 9Method 1 – intuitively

$$\sin(2x) = \frac{1}{\sqrt{2}}$$



If we were given a restricted domain like $x \in [0, 2\pi]$ then $2x \in [0, 4\pi]$ so

$$2x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{9\pi}{4}, \frac{11\pi}{4}$$

$$x = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{9\pi}{8}, \frac{11\pi}{8}$$

(1 mark)

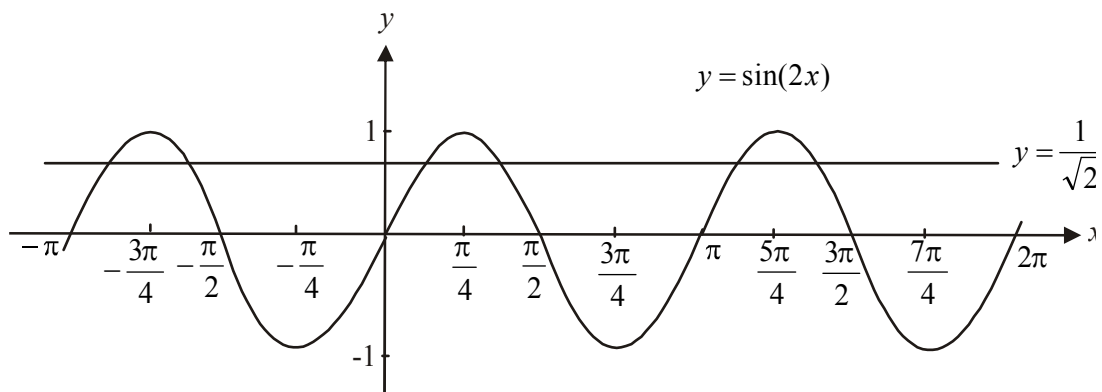
However, the domain is not restricted; we are looking for the general solution where each of the solutions found above is repeated for every clockwise and anticlockwise rotation by 2π .

We can express these values as

$$x = \frac{\pi}{8} + n\pi \quad \text{or} \quad x = \frac{3\pi}{8} + n\pi \quad n \in \mathbb{Z} \text{ (the set of integers)}$$

(1 mark)**(1 mark)**Method 2 – graphically

Sketch the graphs of $y = \sin(2x)$ and $y = \frac{1}{\sqrt{2}}$.



We know that in the first quadrant $\sin(2x) = \frac{1}{\sqrt{2}}$

$$2x = \frac{\pi}{4}$$

$$x = \frac{\pi}{8}$$

By symmetry and using the graph, the points of intersection occur at

$$\begin{aligned} x &= \dots -\frac{3\pi}{4} - \frac{\pi}{8}, -\frac{3\pi}{4} + \frac{\pi}{8}, \frac{\pi}{4} - \frac{\pi}{8}, \frac{\pi}{4} + \frac{\pi}{8}, \frac{5\pi}{4} - \frac{\pi}{8}, \frac{5\pi}{4} + \frac{\pi}{8}, \dots \\ &= \dots -\frac{7\pi}{8}, -\frac{5\pi}{8}, \frac{\pi}{8}, \frac{3\pi}{8}, \frac{9\pi}{8}, \frac{11\pi}{8}, \dots \end{aligned}$$

We can express these values as

$$x = \frac{\pi}{8} + n\pi \quad \text{or} \quad x = \frac{3\pi}{8} + n\pi \quad n \in \mathbb{Z} \text{ (the set of integers)}$$

(1 mark)**(1 mark)****(1 mark)** for graph

Method 3 – using a formulaFor $\sin(x) = a$

$$x = 2n\pi + \sin^{-1}(a) \quad \text{or} \quad x = (2n+1)\pi - \sin^{-1}(a), \quad n \in \mathbb{Z}$$

$$\text{For } \sin(2x) = \frac{1}{\sqrt{2}}$$

$$2x = 2n\pi + \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) \quad \text{or} \quad 2x = (2n+1)\pi - \sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

$$2x = 2n\pi + \frac{\pi}{4}$$

$$2x = (2n+1)\pi - \frac{\pi}{4}$$

$$x = n\pi + \frac{\pi}{8}, n \in \mathbb{Z}$$

$$\begin{aligned} x &= \frac{(2n+1)\pi}{2} - \frac{\pi}{8} \\ &= \frac{4(2n+1)\pi - \pi}{8} \\ &= \frac{8n\pi + 4\pi - \pi}{8} \\ &= \frac{8n\pi + 3\pi}{8} \\ &= n\pi + \frac{3\pi}{8}, \quad n \in \mathbb{Z} \end{aligned}$$

(1 mark)**(1 mark)****(1 mark)** for showing $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$ Method 4 – using a formulaFor $\sin(x) = a$, $x = n\pi + (-1)^n \sin^{-1}(a)$, $n \in \mathbb{Z}$

$$\text{For } \sin(2x) = \frac{1}{\sqrt{2}}$$

$$2x = n\pi + (-1)^n \sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

$$2x = n\pi + (-1)^n \frac{\pi}{4}$$

$$x = \frac{n\pi}{2} + (-1)^n \frac{\pi}{8}, \quad n \in \mathbb{Z}$$

(1 mark) – correct first term**(1 mark)** – correct second term**(1 mark)** for showing $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$

(Note that for this formula

$$\text{when } n = -2, \quad x = -\pi + \frac{\pi}{8} = -\frac{7\pi}{8},$$

$$\text{when } n = 0, \quad x = \frac{\pi}{8}$$

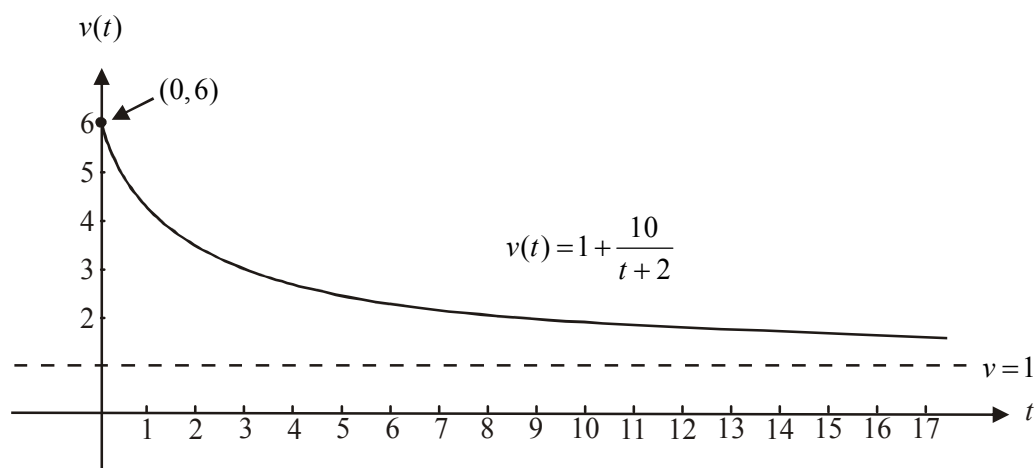
$$\text{when } n = -1, \quad x = -\frac{\pi}{2} - \frac{\pi}{8} = -\frac{5\pi}{8},$$

$$\text{when } n = 1, \quad x = \frac{\pi}{2} - \frac{\pi}{8} = \frac{3\pi}{8}$$

and so on.)

Question 10

a.



(1 mark) – correct endpoint

(1 mark) – correctly labelled asymptote

(1 mark) – correct shape

b.

$$v(t) = 1 + \frac{10}{t+2} < 3$$

$$\frac{10}{t+2} < 2$$

$$10 < 2(t+2) \quad (\text{note that } t \geq 0, \text{ so } t+2 \geq 0 \text{ so we don't have to reverse the inequality sign})$$

$$10 < 2t + 4$$

$$6 < 2t$$

$$3 < t$$

$$\text{So } t > 3 \text{ or } t \in (3, \infty)$$

(1 mark)

c.

$$\text{distance travelled} = \int_0^1 \left(1 + \frac{10}{t+2} \right) dt$$

(1 mark)

$$= [t + 10 \log_e(t+2)]_0^1$$

$$= \{(1 + 10 \log_e(3)) - (0 + 10 \log_e(2))\}$$

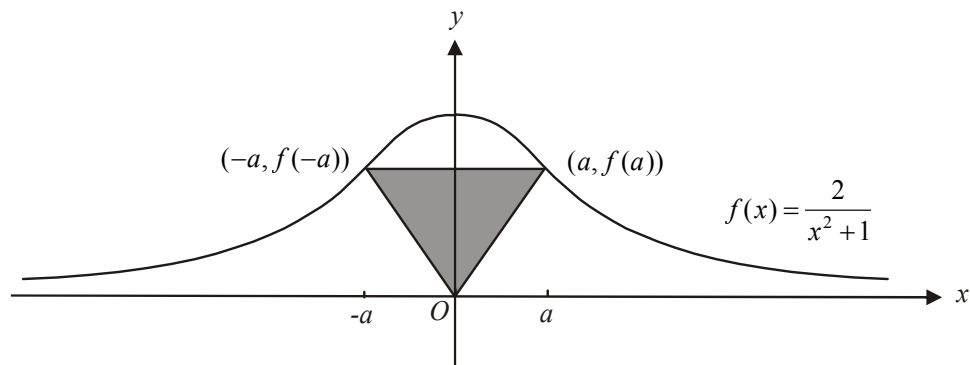
$$= 1 + 10 \log_e(3) - 10 \log_e(2)$$

$$= 1 + 10 \log_e\left(\frac{3}{2}\right) \text{ metres}$$

(1 mark)

Question 11

a.



$$\begin{aligned}
 A &= \frac{1}{2} \times \text{base} \times \text{height} \\
 &= \frac{1}{2} \times 2a \times f(a) \\
 &= a \times f(a) \\
 &= a \times \frac{2}{a^2 + 1} \\
 &= \frac{2a}{a^2 + 1}
 \end{aligned}$$

(1 mark)

b. $\frac{dA}{da} = \frac{(a^2 + 1) \times 2 - 2a \times 2a}{(a^2 + 1)^2}$ (quotient rule)

$$\begin{aligned}
 &= \frac{2a^2 + 2 - 4a^2}{(a^2 + 1)^2} \\
 &= \frac{2 - 2a^2}{(a^2 + 1)^2}
 \end{aligned}$$

(1 mark) correct use of quotient rule

For min/max $\frac{dA}{da} = 0$

$$\frac{2 - 2a^2}{(a^2 + 1)^2} = 0$$

So $2 - 2a^2 = 0$

$$2(1 - a^2) = 0$$

$$2(1 - a)(1 + a) = 0$$

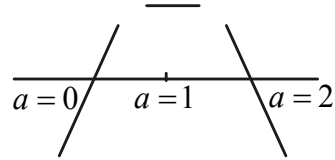
$$a = 1 \text{ or } a = -1$$

but $a > 0$ so $a = 1$

(1 mark)

$$\begin{aligned}\text{At } a=0, \quad \frac{dA}{da} &= \frac{2-2a^2}{(a^2+1)^2} \\ &= 2 \\ &> 0\end{aligned}$$

$$\begin{aligned}\text{At } a=2, \quad \frac{dA}{da} &= \frac{2-2a^2}{(a^2+1)^2} \\ &= \frac{2-8}{25} \\ &= \frac{-6}{25} \\ &< 0\end{aligned}$$



So at $a=1$ there is a local maximum.

(1 mark)

$$\text{From part a., area} = \frac{2a}{a^2+1} = \frac{2}{2} = 1 \text{ square unit.}$$

So the maximum area is 1 square unit.

(1 mark)