

MAV Trial Examination Papers 2009
Mathematical Methods Examination 2
SOLUTIONS

SECTION 1 - ANSWERS

- | | | | | | |
|-------|-------|-------|-------|-------|-------|
| 1. E | 2. D | 3. A | 4. B | 5. C | 6. B |
| 7. E | 8. A | 9. E | 10. A | 11. C | 12. E |
| 13. C | 14. A | 15. A | 16. B | 17. A | 18. B |
| 19. B | 20. D | 21. D | 22. A | | |

SECTION 1 - SOLUTIONS

Question 1

Answer E

$$f: [-4, 2] \rightarrow R, f(x) = (x-1)^2$$

The local minimum occurs at the turning point

$(1, 0)$ and the endpoint maximum occurs at $(-4, f(-4)) = (-4, 25)$.

Hence the range is $[0, 25]$.

Question 2

Answer D

The graph of g is a transformation of the graph of $y = |x|$, as follows.

Reflection in the x -axis: $y = -|x|$

Translation 2 units right: $y = -|x-2|$

Translation 3 units up: $y = -|x-2| + 3$

The rule is $g(x) = -|x-2| + 3$

Question 3

Answer A

$$\begin{aligned}\frac{2x-1}{x+3} &= \frac{2(x+3)-7}{x+3} \\ &= \frac{2(x+3)}{x+3} - \frac{7}{x+3} \\ &= 2 - \frac{7}{x+3} \\ &= 2 - 7f(x+3)\end{aligned}$$

Alternatively, use the division algorithm.

Question 4

Answer B

From the graph of the quartic function the coefficient of x^4 is negative and the single roots are $x = a$ and $x = c$. The corresponding factors are $(x-a)$ and $(x-c)$. The double root is $(x-b)$ (notwithstanding the fact that a and b are negative numbers).

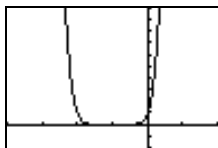
A possible rule is

$$f(x) = -(x-a)(x-c)(x-b)^2$$

Question 5
Answer C

From the graph of $f(x) = (2x+1)^8 = \left(2\left(x + \frac{1}{2}\right)\right)^8$, note that f will be a one-to-one function for

$$x \in \left[-\frac{1}{2}, \infty\right).$$



Therefore, f will have an inverse function when $m \geq -\frac{1}{2}$.

Question 6
Answer B

The domain of f is $\mathbb{R} \setminus \{0\}$ and the domain of g is $\mathbb{R}$. Hence, the domain of $f - g$ is where the $\text{dom } f \cap \text{dom } g$ which is $\mathbb{R} \setminus \{0\}$.

Question 7
Answer E

$$\cos(3x) = -\frac{1}{2}$$

$$3x = \frac{2\pi}{3}, \frac{4\pi}{3}, \dots$$

$$x = \frac{2\pi}{9}, \frac{4\pi}{9}, \frac{8\pi}{9} \text{ as the period is } \frac{2\pi}{3}.$$

$$\text{The sum is } \frac{14\pi}{9}.$$

Question 8
Answer A

$$y = 2 \sin(3x - 1) = 2 \sin\left(3\left(x - \frac{1}{3}\right)\right)$$

There is a dilation of a factor of 2 from the x -axis, a dilation of a factor of $\frac{1}{3}$ from the y -axis and a translation of $\frac{1}{3}$ of a unit to the right.

Question 9
Answer E

$$\log_e(2x) - \sqrt{x^3} = \log_e(2x) - x^{\frac{3}{2}}$$

$$\begin{aligned} \frac{d}{dx} \left(\log_e(2x) - x^{\frac{3}{2}} \right) &= \frac{2}{2x} - \frac{3x^{\frac{1}{2}}}{2} \\ &= \frac{1}{x} - \frac{3\sqrt{x}}{2} \end{aligned}$$

Question 10
Answer A

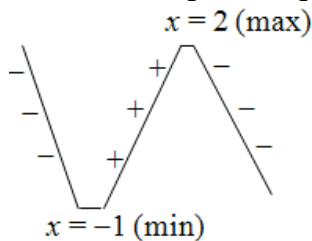
Using the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}, \text{ where } y = g(u) \text{ and } u = f(x)$$

$$\begin{aligned} \frac{d(g(x))}{dx} &= \frac{d(-\cos(f(x)))}{d(f(x))} \times \frac{d(f(x))}{dx} \\ &= \sin(f(x)) \times f'(x) \end{aligned}$$

Question 11
Answer C

Consider the sign of the gradient.



Local min. at $x = -1$ and max. at $x = 2$.

Question 12
Answer E

The derivative of f is not defined at $x = 1$ as there is a cusp at $x = 1$.

f' exists for $x \in \mathbb{R} \setminus \{1\}$.

Question 13
Answer C

The area is given by

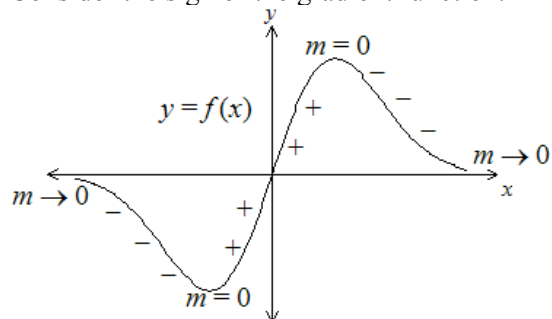
$$\int_{-2}^0 f(x) dx - \int_0^1 f(x) dx + \int_1^2 f(x) dx$$

Note that $\int_1^0 f(x) dx = -\int_0^1 f(x) dx$

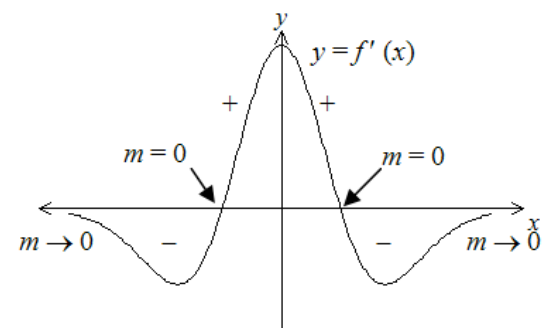
The equivalent option is $\int_{-2}^0 f(x) dx + \int_0^1 f(x) dx + \int_1^2 f(x) dx$

Question 14
Answer A

Consider the sign of the gradient function.



Option A shows the graph of the gradient function with this sign profile.



Question 15
Answer A

$$\begin{aligned}\int_{-2}^1 \left(2x - \frac{u(x)}{2} \right) dx &= \int_{-2}^1 (2x) dx - \frac{1}{2} \int_{-2}^1 u(x) dx \\ &= \left[x^2 \right]_{-2}^1 - \frac{1}{2} \times 8 \\ &= [1 - 4] - 4 \\ &= -7\end{aligned}$$

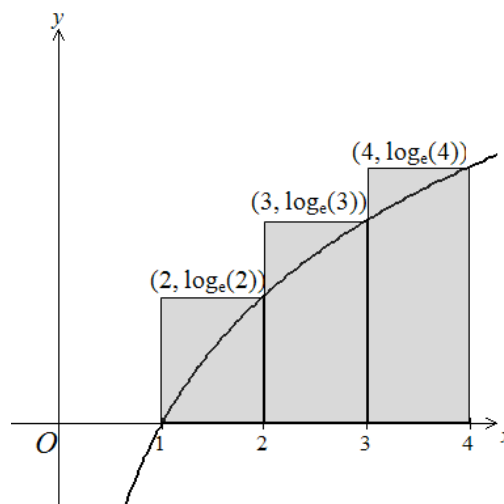
Question 16
Answer B

The area of each rectangle = length $\times$ width

$$\text{Area} = \log_e(2) \times 1 + \log_e(3) \times 1 + \log_e(4) \times 1$$

$$= \log_e(2 \times 3 \times 4)$$

$$= \log_e(24)$$


Question 17
Answer A

The average rate of change is given by the average gradient of the graph of h .

$$\begin{aligned}\text{av. rate} &= \frac{h(\pi) - h\left(\frac{\pi}{2}\right)}{\pi - \frac{\pi}{2}} \\ &= \frac{(\sin(\pi) + \cos(\pi)) - (\sin(\pi/2) + \cos(\pi/2))}{\pi/2} = \frac{-4}{\pi} \\ &= \frac{-2}{\pi/2}\end{aligned}$$

Question 18
Answer B

Let R represent the probability it will rain.

The probability it will rain on Wednesday = $RRR + RR'R$

$$= 0.35 \times 0.35 + 0.65 \times 0.22$$

$$= 0.2655$$

Question 19
Answer B

$$\text{The median} = \frac{3+5}{2} = 4$$

Question 20
Answer D

$$\begin{aligned}\Pr(X = 6 \mid X \geq 3) &= \frac{\Pr(X = 6)}{\Pr(X \geq 3)} \\ &= \frac{0.15}{0.6} \\ &= 0.25\end{aligned}$$

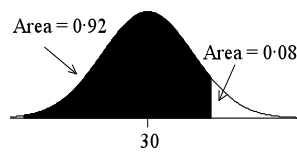
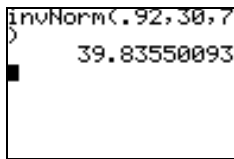
Question 21

Answer D

Let X represent the Exam Result

$$X \sim N(30, 49)$$

Using the inverse normal distribution, 40 is the cut off score.



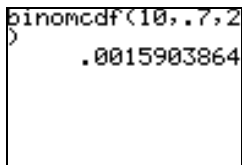
Question 22

Answer A

Let X represent being well

$$X \sim \text{Bi}(10, 0.7)$$

$$\Pr(X \leq 2) = 0.0016 \text{ correct to 4 decimal places}$$



END OF SECTION 1 SOLUTIONS

SECTION 2 - SOLUTIONS

Question 1

a. i A is the amplitude

$$A = 10 \text{ cm}$$

[1A]

B is the vertical translation

$$B = 10 \text{ cm}$$

[1A]

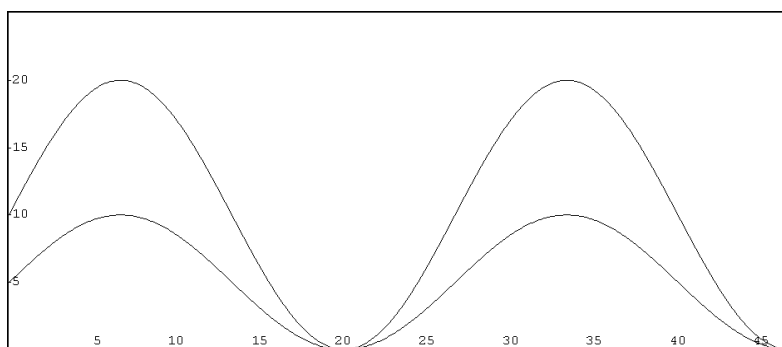
$$\text{ii. } P = \frac{2\pi}{n} = \frac{4}{3} \times 20 = \frac{80}{3}$$

[1M]

$$n = \frac{6\pi}{80}$$

$$= \frac{3\pi}{40} \text{ as required}$$

b. i.



Correct shape

[1A]

Coordinates labelled

[1A]

$$(0, 5), \left(\frac{20}{3}, 10\right), (20, 0), \left(\frac{100}{3}, 10\right), \left(\frac{140}{3}, 0\right)$$

$$\text{ii. } d_b = 5 \sin\left(\frac{3\pi}{40}x\right) + 5$$

[1A]

iii. The areas are the same.

[1A]

$$\text{iv. } \frac{3}{2} \int_0^{\frac{140}{3}} d \, dx$$

Must have dx

[1A]

Could use $\frac{3}{2}d$ or $3(d - d_b)$ or $3d_b$.

$$= \frac{3}{2} \int_0^{\frac{140}{3}} \left(10 \sin\left(\frac{3\pi}{40}x\right) + 10\right) dx$$

$$= 763.66 \text{ cm}^3 \text{ correct to 2 decimal places}$$

[1A]

$$\text{c. } d_b = 5 \sin\left(\frac{3\pi}{40}\left(\frac{20}{3} - 2\right)\right) + 5$$

[1H]

The height is 9.46 cm correct to two decimal places.

[1A]

Question 2

Consider the function $f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = 2xe^{-\frac{x}{2}}$.

- a. Using the product rule [1M]

$$f'(x) = 2e^{-\frac{x}{2}} - xe^{-\frac{x}{2}}$$

$$= (2-x)e^{-\frac{x}{2}} \quad [1A]$$

- b. For a stationary point, $f'(x) = 0$.

$$(2-x)e^{-\frac{x}{2}} = 0 \quad [1M]$$

Using the null factor law, noting that $e^{-\frac{x}{2}} > 0$ for all values of x ,

$$(2-x) = 0$$

$$x = 2 \quad [1A]$$

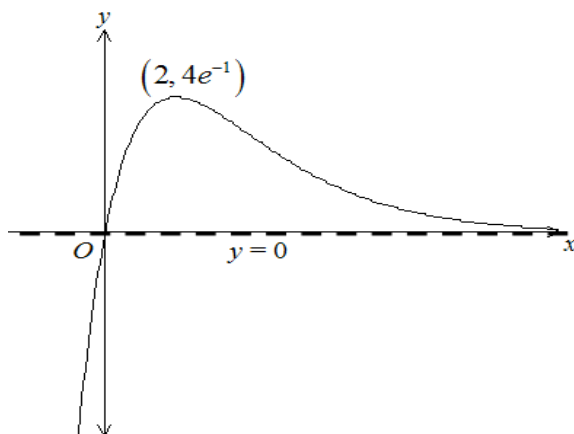
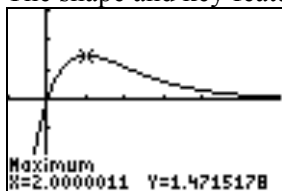
$$f(2) = 2 \times 2e^{-\frac{2}{2}}$$

$$= 4e^{-1} = \frac{4}{e}$$

Turning point at $(2, 4e^{-1}) = \left(2, \frac{4}{e}\right)$ [1A]

The height is 9.46 cm correct to two decimal places. [1A]

- c. The shape and key features of the graph may be obtained using a graphical calculator.



Correct shape, crossing at the origin:
Asymptote and turning point labelled:

[1A]
[1A]

- d. i. $f'(1) = (2-1)e^{-\frac{1}{2}}$
 $= e^{-\frac{1}{2}} = \frac{1}{\sqrt{e}}$

The gradient at $x = 1$ is $e^{-\frac{1}{2}}$ or $\frac{1}{\sqrt{e}}$.

[1A]

$$\text{ii. } m = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$\frac{f(a) - 0}{a - 0} = e^{-\frac{1}{2}}$$

$$\frac{2ae^{-\frac{a}{2}} - 0}{a - 0} = e^{-\frac{1}{2}}$$

[1M]

$$2e^{-\frac{a}{2}} = e^{-\frac{1}{2}}$$

$$e^{-\frac{a}{2}} = \frac{e^{-\frac{1}{2}}}{2}$$

$$-\frac{a}{2} = \log_e \left(\frac{e^{-\frac{1}{2}}}{2} \right)$$

[1M]

$$-\frac{a}{2} = \log_e \left(e^{-\frac{1}{2}} \right) - \log_e (2)$$

$$-\frac{a}{2} = -\frac{1}{2} - \log_e (2)$$

$$a = 1 + 2 \log_e (2), \text{ as required}$$

[1A]

$$\text{iii. The equation of the line } PQ \text{ is } y = \left(e^{-\frac{1}{2}} \right) x$$

$$\text{Area} = \int_0^{1+2\log_e(2)} \left(\left(2xe^{-\frac{x}{2}} \right) - \left(\left(e^{-\frac{1}{2}} \right) x \right) \right) dx$$

[1M]

$$= 0.9522$$

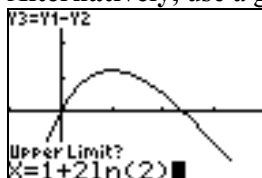
[1A]

The integral may be evaluated using a graphical calculator.

```
Plot1 Plot2 Plot3
Y1=2Xe^(-X/2)
Y2=(e^(-.5))X
Y3=Y1-Y2
Y4=
Y5=
Y6=
Y7=
```

```
fInt(Y3,X,0,1+2
ln(2))
.9522416447
```

Alternatively, use a graphical approach.



$$\text{e. The gradient of the tangent : } m_T = e^{-\frac{1}{2}}.$$

$$\text{Gradient of normal: } m_N = -\frac{1}{m_T} = -e^{\frac{1}{2}}$$

$$\text{Normal at } (1, f(1)) = \left(1, 2e^{-\frac{1}{2}} \right)$$

[1M]

The equation of the normal is

$$y - 2e^{-\frac{1}{2}} = -e^{\frac{1}{2}}(x - 1)$$

$$y = -e^{\frac{1}{2}}x + e^{\frac{1}{2}} + 2e^{-\frac{1}{2}}$$

[1A]

Alternatively

$$y = -\sqrt{e}x + \sqrt{e} + \frac{2}{\sqrt{e}}$$

Question 3

a. Using Pythagoras' theorem,

$$BD = \sqrt{x^2 + 6^2} = \sqrt{x^2 + 36}$$

[1A]

b.

i. $\text{time} = \frac{\text{distance}}{\text{speed}}$

[1M]

$$\text{Time from } A \text{ to } B = \frac{14 - x}{20}$$

$$\text{Time from } B \text{ to } D = \frac{\sqrt{x^2 + 36}}{20}$$

[1M]

$$T(x) = \frac{14 - x}{20} + \frac{\sqrt{x^2 + 36}}{12}, 0 \leq x \leq 14$$

[1A]

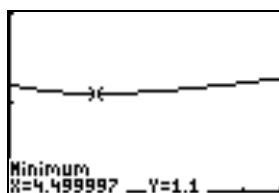
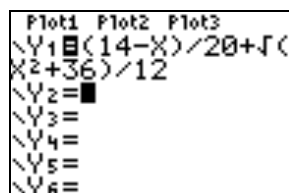
Using technology to find the coordinates of the local minimum of the graph of T ,

ii. Minimum time occurs when $x = 4.5$ km.

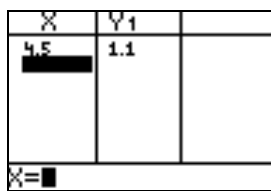
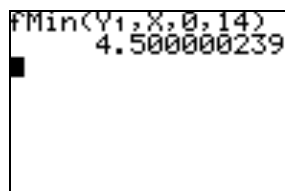
[1A]

iii. The minimum time is 1.1 hours

[1A]



Alternatively

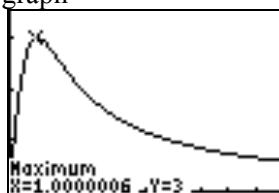
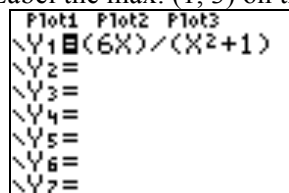


c. i. The maximum concentration is 3 units/cm³ and it occurs 1 hour after the dose is administered.

[2A]

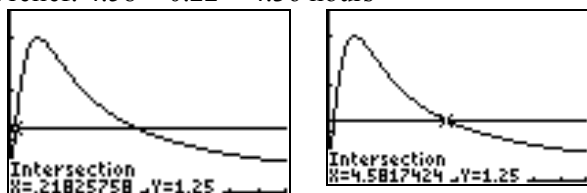
ii. Label the max. (1, 3) on the graph

[1A]

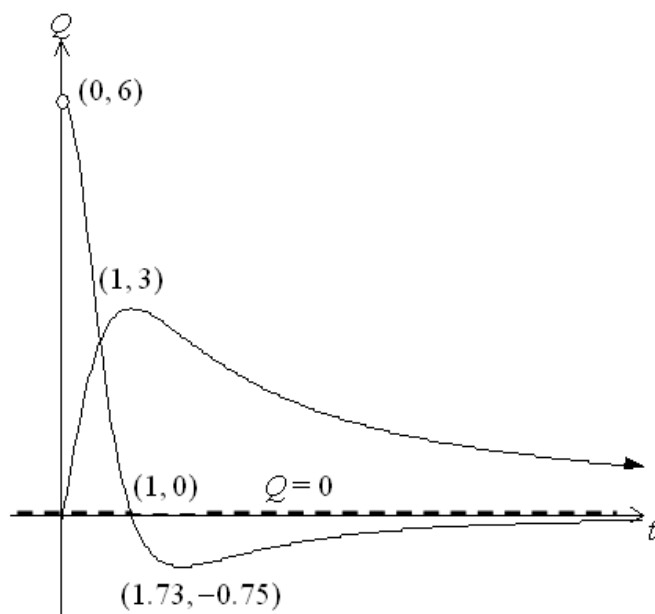
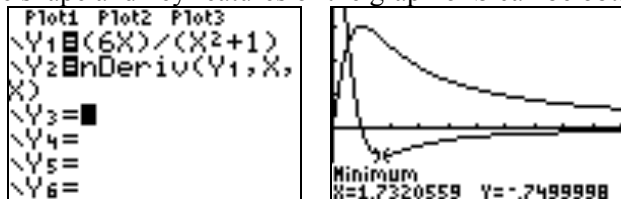


- d. $Q(t) > 1.25$ when $0.22 < t < 4.58$. [1M]

Pain relief: $4.58 - 0.22 = 4.36$ hours [1A]



- e. The shape and key features of the graph of S can be obtained using a graphical calculator. [1A]



Correct shape [1A]

Local min. and x-intercept labelled [1A]

Asymptote labelled and (0, 6) shown as an “open circle” [1A]

Question 4

a. i. $\Pr(3 \text{ red}) = \frac{10}{13} \times \frac{9}{12} \times \frac{8}{11} = \frac{60}{143}$ [1A]

ii. $\Pr(2 \text{ green}) = \frac{3}{12} \times \frac{2}{11} = \frac{1}{22}$ [1A]

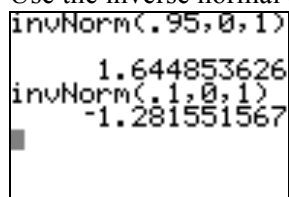
b. i. $\Pr(\text{tea}) = 0.1$ [1A]

ii. Let coffee be c, tea t and nothing n.

$\Pr(ct) + \Pr(tt) + \Pr(nt) = 0.7 \times 0.1 + 0.1 \times 0.3 + 0.2 \times 0.4 = 0.18$ [1M]

[1A]

- c. Use the inverse normal distribution, with $\mu = 0$ and $\sigma = 1$.



$$\frac{30 - \mu}{\sigma} = 1.64485 \dots (1)$$

[1M]

$$\frac{15 - \mu}{\sigma} = -1.28155 \dots (2)$$

$$30 - \mu = 1.64485\sigma \dots (1)$$

$$15 - \mu = -1.28155\sigma \dots (2)$$

$$(1) - (2)$$

$$15 = 2.9264\sigma$$

$$\sigma = 5.1 \text{ g}$$

[1A]

$$30 - \mu = 1.64485 \times 5.12575$$

$$\mu = 21.6 \text{ g}$$

[1A]

d. 0.85^4
 ≈ 0.5220

[1M]

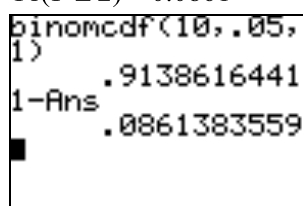
[1A]

e. Let $Y \sim Bi(10, 0.05)$

[1M]

$$\Pr(Y \geq 2) \approx 0.0861$$

[1A]



f. Let $W \sim Bi(n, 0.05)$

$$\Pr(W \geq 1) > 0.95$$

$$1 - \Pr(W = 0) > 0.95$$

[1M]

$$\Pr(W = 0) < 0.05$$

$$0.95^n < 0.05$$

[1M]

$$n > \frac{\log_{10}(0.05)}{\log_{10}(0.95)} \approx 58.404$$

$$n = 59$$

[1A]

END OF SECTION 2 SOLUTIONS