

**Mathematical Association of Victoria
Trial Exam 2009**

**MATHEMATICAL METHODS / MATHEMATICAL METHODS
(CAS)**

STUDENT NAME _____

Written Examination 1

Reading time: 15 minutes

Writing time: 1 hour

QUESTION AND ANSWER BOOK

Structure of book

<i>Number of questions</i>	<i>Number of questions to be answered</i>	<i>Number of marks</i>
9	9	40

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers.
- Students are NOT permitted to bring into the examination room: notes of any kind, blank sheets of paper, white out liquid/tape or a calculator of any type.

Materials supplied

- Question and answer book of 10 pages, with a detachable sheet of miscellaneous formulas at the back
- Working space is provided throughout the book.

Instructions

- Detach the formula sheet from the back of this book during reading time.
- All written responses must be in English.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

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Instructions

Answer **all** questions in the spaces provided.

A decimal approximation will not be accepted if an **exact** answer is required to a question.

In questions where more than one mark is available, appropriate working must be shown.

Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1

Let $f(x) = (e^{2x} - 3)^4$. Evaluate $f'(0)$.

3 marks

Question 2

Solve $2\log_e(x-2) - \log_e(x) = 0$ for x , where $x > 2$.

3 marks

Question 3

Solve $2\sin\left(2x - \frac{\pi}{6}\right) = 1$ for $-\pi < x < \pi$.

3 marks

Question 4

Consider the function $g: [-3, \infty) \rightarrow \mathbb{R}$, with rule $g(x) = 2(x+3)^2 - 1$.

- a. Find the **rule** of the inverse function, g^{-1} .

- b. Find the **domain** of the inverse function, g^{-1} .

2 + 1 = 3 marks

Question 5

Let $h(x) = \frac{1}{\tan(x)} + x$.

- a. Show that $h'(x) = -\frac{1}{\sin^2(x)} + 1$.

- b. Hence find $\int \left(\frac{1}{\sin^2(x)} dx \right)$.

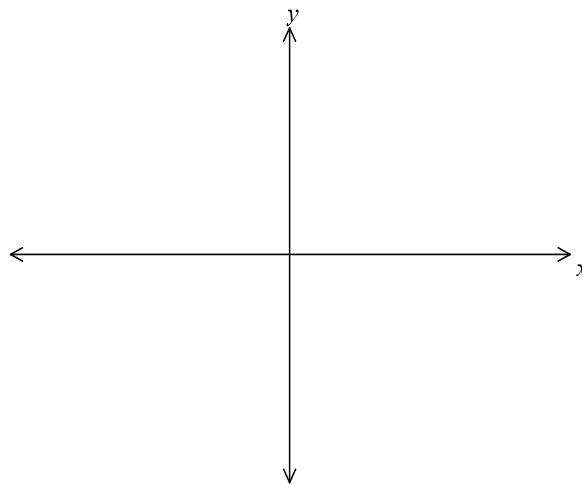
2 + 3 = 5 marks

Question 6

Consider the function $f : \mathbb{R} \setminus \{1\} \rightarrow \mathbb{R}$, $f(x) = 2 - \frac{2}{(x-1)^2}$.

- a. Find the **coordinates** of the x -axis intercepts of the graph of f .

- b. On the axes below, sketch the graph of $y = |f(x)|$. Label all axes intercepts with their coordinates. Label each asymptote with its equation.

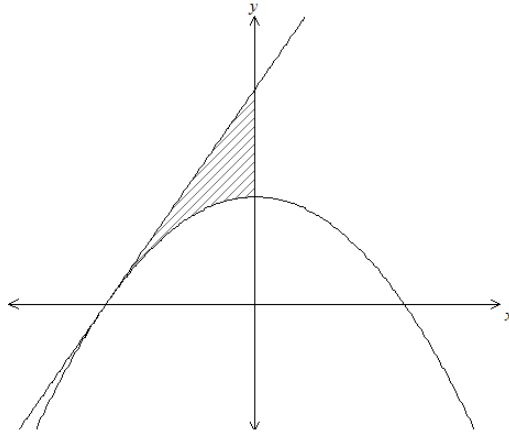


- c. Write down the domain of the derivative function, $\frac{d}{dx}(|f(x)|)$.

1 + 3 + 1 = 5 marks

Question 7

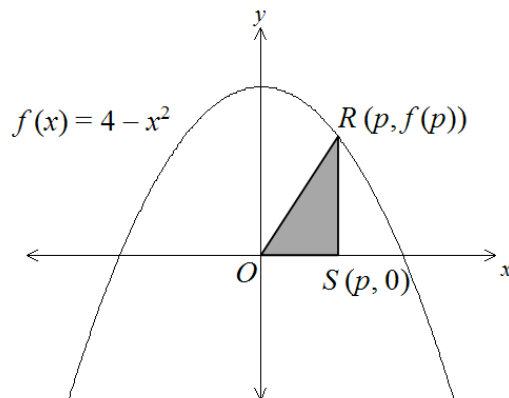
The graph of $f : \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = 4 - x^2$ and the tangent to the graph of f , where it crosses the negative x -axis, are shown.



- a. Find the equation of the tangent to the graph of f where it crosses the negative x -axis.

- b. Find the area of the shaded region.

- c. Consider the triangle ORS , where vertex R is on the graph of f , with coordinates $(p, f(p))$, and vertices O and S are on the x -axis, with coordinates $(0, 0)$ and $(p, 0)$, respectively.



If $p \in [-2, 2]$, find the value(s) of p for which the area of triangle ORS is a maximum.

2 + 3 + 3 = 8 marks

Question 8

A spherical balloon is being inflated at a rate of $10 \text{ cm}^3/\text{s}$. The balloon will burst when the surface area reaches $3600\pi \text{ cm}^2$.

- a. The surface area of a sphere of radius r is given by $4\pi r^2$. Show that the radius of the balloon is 30 cm at the instant when the balloon bursts.

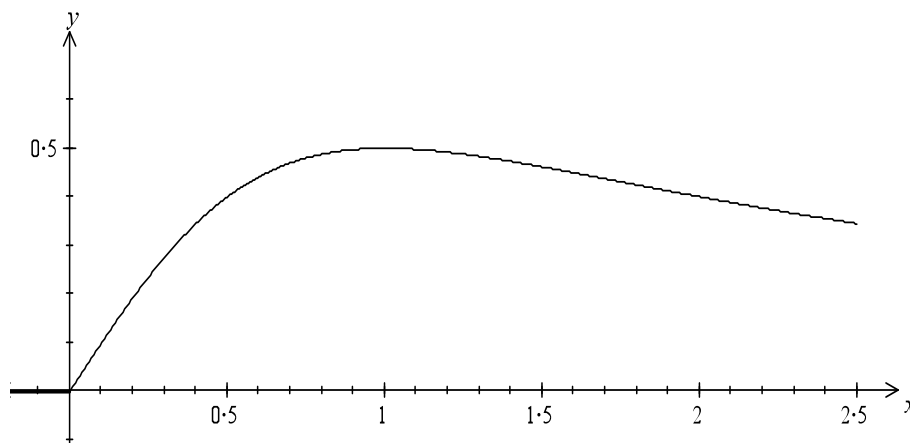
- b. Find the rate at which the radius of the balloon is changing, in cm per second, at the instant when the balloon bursts.

1 + 3 = 4 marks

Question 9

A continuous random variable X has a probability density function $f(x) = \begin{cases} \frac{x}{x^2+1} & \text{for } 0 \leq x \leq a \\ 0 & \text{elsewhere} \end{cases}$, where a

is a real constant. Part of the graph of f , where $0 \leq x \leq a$, is shown below.



- a. Given that $\int \frac{x}{x^2+1} dx = \frac{1}{2} \log_e (x^2+1) + c$, where c is a real constant, find the value of a .

2 marks

- b. Find the mode of X .

- c. Find $\Pr(0.25 \leq X \leq 0.75)$. Put your answer in the form $\log_e\left(\frac{a}{\sqrt{b}}\right)$ where a and b are positive integers.

2 + 2 + 2 = 6 marks

END OF QUESTION AND ANSWER BOOK

Mathematical Methods and Mathematical Methods (CAS)

Formulas

Mensuration

area of a trapezium:	$\frac{1}{2}(a+b)h$	volume of a pyramid:	$\frac{1}{3}Ah$
curved surface area of a cylinder:	$2\pi rh$	volume of a sphere:	$\frac{4}{3}\pi r^3$
volume of a cylinder:	$\pi r^2 h$	area of a triangle:	$\frac{1}{2}bc \sin A$
volume of a cone:	$\frac{1}{3}\pi r^2 h$		

Calculus

$\frac{d}{dx}(x^n) = nx^{n-1}$	$\int x^n dx = \frac{1}{n+1} x^{n+1} + c, n \neq -1$
$\frac{d}{dx}(e^{ax}) = ae^{ax}$	$\int e^{ax} dx = \frac{1}{a} e^{ax} + c$
$\frac{d}{dx}(\log_e(x)) = \frac{1}{x}$	$\int \frac{1}{x} dx = \log_e x + c$
$\frac{d}{dx}(\sin(ax)) = a \cos(ax)$	$\int \sin(ax) dx = -\frac{1}{a} \cos(ax) + c$
$\frac{d}{dx}(\cos(ax)) = -a \sin(ax)$	$\int \cos(ax) dx = \frac{1}{a} \sin(ax) + c$
$\frac{d}{dx}(\tan(ax)) = \frac{a}{\cos^2(ax)} = a \sec^2(ax)$	

product rule: $\frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}$ quotient	rule: $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$
chain rule: $\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$ approximation:	$f(x+h) \approx f(x) + hf'(x)$

Probability

$\Pr(A) = 1 - \Pr(A')$	$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$
$\Pr(A B) = \frac{\Pr(A \cap B)}{\Pr(B)}$	
mean: $\mu = E(X)$	variance: $\text{var}(X) = \sigma^2 = E((X - \mu)^2) = E(X^2) - \mu^2$

probability distribution		mean	variance
discrete	$\Pr(X = x) = p(x)$	$\mu = \sum x p(x)$	$\sigma^2 = \sum (x - \mu)^2 p(x)$
continuous	$\Pr(a < X < b) = \int_a^b f(x) dx$	$\mu = \int_{-\infty}^{\infty} x f(x) dx$	$\sigma^2 = \int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx$

MAV Trial Examination Papers 2009
Mathematical Methods / Mathematical Methods (CAS)
Examination 1 - SOLUTIONS

Question 1

Using the chain rule,

1M

$$f'(x) = 8e^{2x} (e^{2x} - 3)^3$$

1A

$$f'(0) = 8e^0 (e^0 - 3)^3$$

$$= 8 \times (-2)^3$$

$$= -64$$

1A

Question 2

$$2 \log_e (x-2) - \log_e (x) = 0$$

$$\log_e \left(\frac{(x-2)^2}{x} \right) = 0$$

1M

$$\frac{(x-2)^2}{x} = 1$$

$$x^2 - 4x + 4 = x$$

$$x^2 - 5x + 4 = 0$$

1M

$$(x-4)(x-1) = 0$$

$$x = 1 \text{ or } x = 4$$

$$x = 4 \text{ as } x > 2$$

1A

Question 3

$$2 \sin \left(2x - \frac{\pi}{6} \right) = 1$$

$$\sin \left(2x - \frac{\pi}{6} \right) = \frac{1}{2}$$

$$2x - \frac{\pi}{6} = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{\pi}{6} + 2\pi, \frac{5\pi}{6} + 2\pi \dots$$

1M

$$2x = \frac{\pi}{3}, \pi, \frac{\pi}{3} + 2\pi, \pi + 2\pi \dots$$

$$x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{\pi}{6} + \pi, \frac{\pi}{2} + \pi \dots$$

$$x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{7\pi}{6}, \frac{3\pi}{2} \dots$$

1M

However, $-\pi < x < \pi$. By symmetry,

$$x = -\frac{5\pi}{6}, -\frac{\pi}{2}, \frac{\pi}{6}, \frac{\pi}{2}$$

1A

Question 4

- a.** To find the rule of an inverse function, interchange the x and y values and make y the subject of the equation.

$$x = 2(y+3)^2 - 1$$

$$(y+3)^2 = \frac{x+1}{2}$$

$$y + 3 = \sqrt{\frac{x+1}{2}} \quad 1\text{M}$$

(Reject the negative solution because the domain of g is $[-3, \infty)$.)

$$y = \sqrt{\frac{x+1}{2}} - 3$$

$$g^{-1}(x) = \sqrt{\frac{x+1}{2}} - 3 \quad 1\text{A}$$

b. The domain of g^{-1} is the range of g . Domain of g^{-1} is $[-1, \infty)$. 1A

Question 5

a. $h(x) = \frac{1}{\tan(x)} + x = \frac{\cos(x)}{\sin(x)} + x$

$$h'(x) = \frac{-\sec^2(x)}{\tan^2(x)} + 1 \text{ using the quotient rule.} \quad 2\text{M}$$

$$h'(x) = -\frac{1}{\sin^2(x)} + 1, \text{ as required.}$$

Alternatively

$$h(x) = \frac{1}{\tan(x)} + x = (\tan(x))^{-1} + x$$

$$h'(x) = \frac{-\sec^2(x)}{\tan^2(x)} + 1 \text{ using the chain rule.} \quad 2\text{M}$$

$$h'(x) = -\frac{1}{\sin^2(x)} + 1, \text{ as required.}$$

b. From part **a.**,

$$\int \left(-\frac{1}{\sin^2(x)} + 1 \right) dx = \frac{1}{\tan(x)} + x + C_1 \quad 1\text{M}$$

$$\int \left(-\frac{1}{\sin^2(x)} \right) dx + x + C_2 = \frac{1}{\tan(x)} + x + C_1$$

$$\int \left(-\frac{1}{\sin^2(x)} \right) dx = \frac{1}{\tan(x)} + x - x + C_3 \quad 1\text{M}$$

Where $C_3 = C_1 - C_2$

$$\int \left(\frac{1}{\sin^2(x)} \right) dx = -\frac{1}{\tan(x)} + C_3 \quad 1\text{A}$$

Question 6

- a. To find x -axes intercepts, $0 = 2 - \frac{2}{(x-1)^2}$

$$\frac{2}{(x-1)^2} = 2$$

$$(x-1)^2 = 1$$

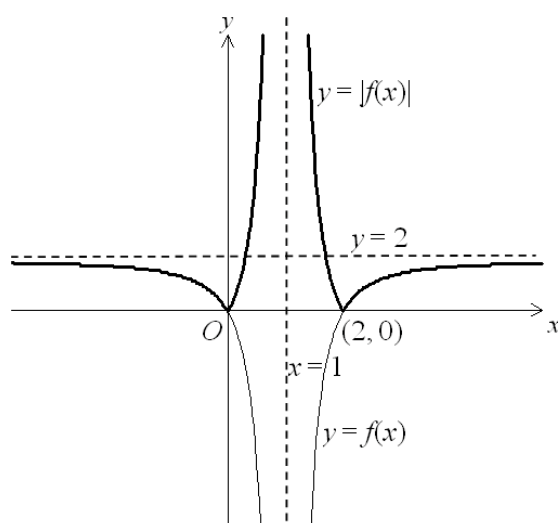
$$x-1 = \pm\sqrt{1}$$

$$x = 2 \text{ or } x = 0$$

The coordinates are $(0, 0)$ or $(2, 0)$

1A

- b. Note that the required graph is $y = |f(x)|$.



Correct shape (including cusps)

1A

Correct asymptotes labelled

1A

Axes intercepts labelled

1H

- c. The domain of the derivative function is $R \setminus \{0, 1, 2\}$.

Alternatively, $(-\infty, 0) \cup (0, 1) \cup (1, 2) \cup (2, \infty)$.

1A

Question 7

- a. The graph of f crosses the negative x -axis at $(-2, 0)$. The gradient of the tangent at this point is $f'(-2) = 4$. 1M

Using $y - y_1 = m(x - x_1)$ for the equation of a straight line, the equation of the tangent is

$$y - 0 = 4(x - (-2))$$

$$y = 4x + 8$$

1A

- b. $A = \int_{-2}^0 ((4x + 8) - (4 - x^2)) dx$ 1M

$$A = \int_{-2}^0 (x^2 + 4x + 4) dx$$

$$A = \left[\frac{x^3}{3} + 2x^2 + 4x \right]_{-2}^0$$

1M

$$A = \left[0 - \left(\frac{(-2)^3}{3} + 2 \times (-2)^2 + 4 \times (-2) \right) \right]$$

$$A = \frac{8}{3} \text{ square units}$$

1A

c. $A = \frac{1}{2} \text{ base} \times \text{height}$

$$A = \frac{1}{2} p (4 - p^2)$$

1M

$$A = 2p - \frac{p^3}{2}$$

For maximum area, $\frac{dA}{dp} = 0$

$$2 - \frac{3p^2}{2} = 0$$

1M

$$\frac{3p^2}{2} = 2$$

$$p^2 = \frac{4}{3}$$

$$p = \pm \frac{2}{\sqrt{3}} = \pm \frac{2\sqrt{3}}{3}$$

1A

Maximum area when $p = \frac{2\sqrt{3}}{3}$ or $p = -\frac{2\sqrt{3}}{3}$

Question 8

a. $4\pi r^2 = 3600\pi$

$$r^2 = 900$$

$$r = \sqrt{900} = 30 \quad \text{Radius is 30 cm, as required.}$$

1M

b. Require $\frac{dr}{dt}$ when $r = 30$, given that $\frac{dV}{dt} = 10 \text{ cm}^3/\text{s}$.

$$V = \frac{4}{3}\pi r^3$$

$$\frac{dV}{dr} = 4\pi r^2$$

1M

For these related rates

$$\frac{dr}{dt} = \frac{dr}{dV} \times \frac{dV}{dt}$$

Substitute $\frac{dV}{dr} = 4\pi r^2$ and $\frac{dV}{dt} = 10 \text{ cm}^3/\text{s}$.

$$\frac{dr}{dt} = \frac{1}{4\pi r^2} \times 10$$

1M

At $r = 30$,

$$\frac{dr}{dt} = \frac{1}{4\pi(30)^2} \times 10$$

$$\frac{dr}{dt} = \frac{10}{3600\pi} = \frac{1}{360\pi}$$

1A

The rate of change in radius is $\frac{1}{360\pi}$ cm/s.

Question 9

a. $\int_{-\infty}^{\infty} \left(\frac{x}{1+x^2} \right) dx = \int_0^a \left(\frac{x}{1+x^2} \right) dx = 1$

$$\left[\frac{1}{2} \log_e(1+x^2) \right]_0^a = 1$$

1M

$$\frac{1}{2} (\log_e(1+a^2) - \log_e(1)) = 1$$

$$\log_e(1+a^2) = 2$$

$$1+a^2 = e^2$$

$$a = \sqrt{e^2 - 1} \text{ as } a > 0$$

1A

b. The mode is the x -value at the turning point.

$$\frac{d}{dx} \left(\frac{x}{1+x^2} \right) = 0$$

$$\frac{1+x^2-2x^2}{(1+x^2)^2} = 0$$

1M

$$\frac{1-x^2}{(1+x^2)^2} = 0$$

$$1-x^2 = 0 \text{ as } 1+x^2 \neq 0$$

$$x = 1 \text{ as } x > 0$$

1A

c. $\Pr(0.25 \leq X \leq 0.75) = \int_{0.25}^{0.75} \left(\frac{x}{1+x^2} \right) dx$

1M

$$= \left[\frac{1}{2} \log_e(1+x^2) \right]_{0.25}^{0.75}$$

$$= \frac{1}{2} \left(\log_e \left(\frac{25}{16} \right) - \log_e \left(\frac{17}{16} \right) \right)$$

$$= \frac{1}{2} \left(\log_e \left(\frac{25}{17} \right) \right)$$

$$= \log_e \left(\frac{5}{\sqrt{17}} \right)$$

1A