2009 Mathematical Methods 3/4 Trial Exam 2 Solutions
Free download and print from www.itute.com

SECTION 1

1	2	3	4	5	6	7	8	9	10	11
C	A	C	E	E	A	C	B	E	E	E

12	13	14	15	16	17	18	19	20	21	22
D	B	D	A	C	C	B	A	C	E	E

Q1 $\frac{dy}{dx}=60x^2+82x+28=0$ has 2 solutions.
$\therefore$ $y=20x^3+41x^2+28x+25$ has 2 stationary points, a local maximum and a local minimum. Between these 2 points is an inflection point. C

Q2 $3(a-x)^2e^x-3(a-x)e^{\frac{x}{2}}+1=0$. Let $u=(a-x)e^{\frac{x}{2}}$, $3u^2-3u+1=0$. Since $b^2-4ac=(-3)^2-4(3)(1)=-3$, a negative value, no real *u* and hence no real *x* will satisfy the equation. A

Q3 For the function $\tan\left(\frac{x}{k}\right)$ with no domain restrictions, the asymptotes closest to the origin O are $x=\pm\frac{k\pi}{2}$. For function *f* with domain $D=\left(-\frac{k\pi}{4},\frac{k\pi}{4}\right)$, it has no asymptotes. C

Q4 The intersection of $y=x^2-b$ and $y=\sqrt{x+b}$ is on the line $y=x$. $\therefore x^2-b=\sqrt{x+b}$ has the same solution as $x^2-b=x$.
$x^2-x-b=0$, $x=\frac{1+\sqrt{1+4b}}{2}$. E

Q5 $10^{(\log_5 x)(\log_2 y)}=(2\times5)^{(\log_5 x)(\log_2 y)}=2^{(\log_5 x)(\log_2 y)}\times5^{(\log_5 x)(\log_2 y)}$
$=\left(2^{\log_2 y}\right)^{\log_5 x}\times\left(5^{\log_5 x}\right)^{\log_2 y}=y^{\log_5 x}x^{\log_2 y}$. E

Q6 $1-3f(2-2x)=4x^2$, $f(2-2x)=\frac{1-4x^2}{3}$.
Let $X=2-2x$, $\therefore 2x=2-X$,
$\therefore f(X)=\frac{1-(2-X)^2}{3}=\frac{[1-(2-X)][1+(2-X)]}{3}$
$=\frac{(X-1)(3-X)}{3}$. $\therefore f(x)=\frac{(x-1)(3-x)}{3}$. A

Q7 C

Q8 EITHER $ax+b\ge0$ and $cx-d>0$ OR $ax+b\le0$ and $cx-d<0$.
$\therefore$ EITHER $x\ge-\frac{b}{a}$ and $x>\frac{d}{c}$ OR $x\le-\frac{b}{a}$ and $x<\frac{d}{c}$.
$\therefore$ EITHER $x>\frac{d}{c}$ OR $x\le-\frac{b}{a}$,
which is $R\backslash\left\{x:-\frac{b}{a}<x\le\frac{d}{c}\right\}$. B

Q9 $(x+5)P(x)=x^4+c$, $\therefore P(x)=\frac{x^4+c}{x+5}$, $x\ne-5$.

$$\begin{array}{r}
x^3-5x^2+25x-125 \\
x+5\,\big)\,\overline{x^4+0x^3+0x^2+0x+c} \\
\underline{x^4+5x^3} \\
-5x^3+0x^2 \\
\underline{-5x^3-25x^2} \\
25x^2+0x \\
\underline{25x^2+125x} \\
-125x+c \\
\underline{-125x-625} \\
0
\end{array}$$

E

Q10 $e^x+e^y=2$ ……(1), $e^x-e^y=1$ ……(2)
(1) + (2), $2e^x=3$, $x=\log_e 1.5$.
(1) − (2), $2e^y=1$, $y=\log_e 0.5$.
$x+y=\log_e 1.5+\log_e 0.5=\log_e(1.5\times0.5)=\log_e 0.75$. E

Q11 Given $f(x)=1+\log_e x$, then $f(y)=1+\log_e y$
To check which one is false, let $y=1$. $f(1)=1+\log_e 1=1$.
E is false because $f(x+y)=f(x+1)=1+\log_e(x+1)$, but $f(x)+f(y)-f(x)f(y)=f(x)+f(1)-f(x)f(1)$
$=f(x)+1-f(x)=1$.
$\therefore f(x+y)\ne f(x)+f(y)-f(x)f(y)$. E

Q12 Draw a tangent to the curve at $x=-5$, and determine its gradient to be ≈-0.7. D

Q13 $P'(x)=\frac{\sqrt{x}\,g'\left(\sqrt{x}\right)\frac{1}{2\sqrt{x}}-g\left(\sqrt{x}\right)\frac{1}{2\sqrt{x}}}{x}$
$=\frac{\sqrt{x}\,g'\left(\sqrt{x}\right)-g\left(\sqrt{x}\right)}{2x\sqrt{x}}$
$=\frac{xg'\left(\sqrt{x}\right)-\sqrt{x}\,g\left(\sqrt{x}\right)}{2x^2}$. B

Q14 $\int_{\frac{1}{3}}^{3}\left(\log_e(2x)-\frac{1}{2x}\right)dx=\int_{\frac{1}{3}}^{3}\log_e(2x)dx-\left[\frac{1}{2}\log_e x\right]_{\frac{1}{3}}^{3}$

$=\int_{\frac{1}{3}}^{3}\log_e(2x)dx-\left[\frac{1}{2}\log_e 3-\frac{1}{2}\log_e\left(\frac{1}{3}\right)\right]$

$=\int_{\frac{1}{3}}^{3}\log_e(2x)dx-\left[\frac{1}{2}\log_e 3+\frac{1}{2}\log_e 3\right]$

$=\int_{\frac{1}{3}}^{3}\log_e(2x)dx-\log_e 3$. D

Q15

	$x<0$	$x=0$	$0<x<2$	$x=2$	$x>2$
$f'(x)$	negative	0	positive	0	negative

A

Q16 Average rate of change $=\dfrac{f\left(\frac{4\pi}{3}\right)-f\left(\frac{\pi}{3}\right)}{\frac{4\pi}{3}-\frac{\pi}{3}}$

$=\dfrac{\left(2\sin\left(\frac{2\pi}{3}\right)-\cos\left(\frac{4\pi}{3}\right)\right)-\left(2\sin\left(\frac{\pi}{6}\right)-\cos\left(\frac{\pi}{3}\right)\right)}{\pi}$

$=\dfrac{\left(\sqrt{3}+\frac{1}{2}\right)-\left(1-\frac{1}{2}\right)}{\pi}=\dfrac{\sqrt{3}}{\pi}$. C

Q17 $\Pr(X=8.2)=1-(0.1+0.15+0.2+0.25+0.2+0.05)=0.05$.

$\overline{X}=2(0.1)+3.3(0.15)+5(0.2)+7(0.25)+8.2(0.05)+9(0.2)+9.5(0.05)$

$=6.13$. C

Q18 Binomial distribution.

At each corner the drunkard is equally likely to move $\longrightarrow$ or $\searrow$, $\therefore p=\frac{1}{2}$ and $q=\frac{1}{2}$. The drunkard has to move 3 times $\longrightarrow$ and 2 times $\searrow$ in any order before reaching Q, $\therefore n=5$ and $X=3$.

$\Pr(X=3)={}^{5}C_3\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^2=\frac{5}{16}$. B

Q19 A

Q20 Two-state Markov chain.

The two states are *A* and *B*. Let $\Pr(B\mid A)=x$, then

$\Pr(A\mid A)=1-x$.

$\Pr(BAA)=\Pr(B\mid A)\Pr(A\mid B)\Pr(A\mid A)$.

$\therefore \frac{1}{16}=x\times\frac{1}{3}\times(1-x)$, $16x^2-16x+3=0$, $(4x-1)(4x-3)=0$,

$x=\frac{1}{4}$ or $\frac{3}{4}$. C

Q21 $\Pr(X<85\mid X>p)=\dfrac{\Pr(X<85\cap X>p)}{\Pr(X>p)}$

$=\dfrac{\Pr(p<X<85)}{\Pr(X>p)}=\dfrac{\Pr(X<85)-\Pr(X<p)}{1-\Pr(X<p)}$.

$\therefore \dfrac{\Pr(X<85)-\Pr(X<p)}{1-\Pr(X<p)}=0.85$.

By calculator, $\Pr(X<85)=0.9612$.

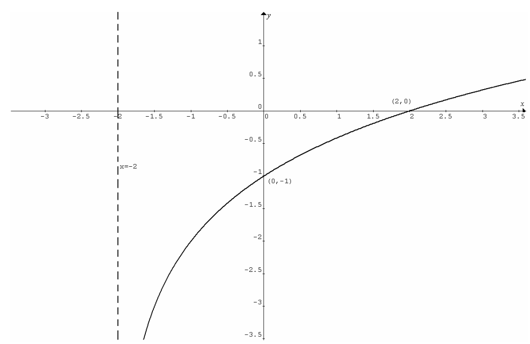
$\therefore \dfrac{0.9612-\Pr(X<p)}{1-\Pr(X<p)}=0.85$.

Hence $\Pr(X<p)=0.7413$ and $p=75.5$. E

Q22 E

SECTION 2

Q1a



Q1b Equation of the inverse is $x=\log_2(y+2)-2$. Express *y* in terms of *x*: $y=2^{x+2}-2$.

The range of the inverse is the domain of $f(x)$, $(-2,\infty)$.

Q1c Let $P(x,y)$ be the point closest to $O(0,0)$, and let *D* be the distance *OP*.

$D=\sqrt{x^2+y^2}$,

$D=\sqrt{x^2+(\log_2(x+2)-2)^2}=\sqrt{x^2+\left(\dfrac{\log_e(x+2)}{\log_e 2}-2\right)^2}$.

Use calculator to find the minimum point $(0.4280,-0.7202)$.

Q1d Area $=-\int_0^2\left(\dfrac{\log_e(x+2)}{\log_e 2}-2\right)dx$, which is the same as the area under the inverse of $f(x)$ between $x=-1$ and $x=0$,

i.e. $\int_{-1}^{0}\left(2^{x+2}-2\right)dx=\int_{-1}^{0}\left(e^{(\log_e 2)(x+2)}-2\right)dx$

$=\left[\dfrac{e^{(\log_e 2)(x+2)}}{\log_e 2}-2x\right]_{-1}^{0}=\dfrac{4}{\log_e 2}-\left(\dfrac{2}{\log_e 2}+2\right)$

$=\dfrac{2}{\log_e 2}-2$.

Q1ei Let (x, y) be the coordinates of the vertex of the rectangle opposite to the vertex at *O*.
For area *A* to be the greatest, point (x, y) must be on the curve $y=\frac{\log_e(x+2)}{\log_e 2}-2$.

$A=-xy=-x\left(\frac{\log_e(x+2)}{\log_e 2}-2\right)$.

Use calculator to find the *x*-coordinate of the maximum point to be 0.9194 (0.91938). Substitute $x=0.91938$ into $y=\frac{\log_e(x+2)}{\log_e 2}-2$ to obtain $y=-0.4543$.
Length $=0.9194$, width $=0.4543$.

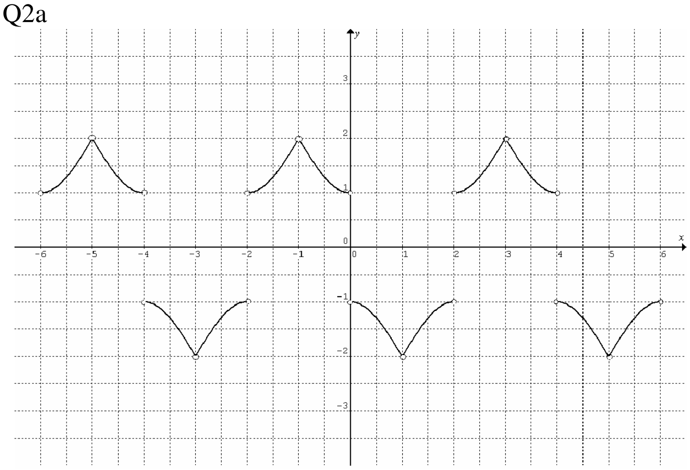
Q1eii $A=-x\left(\frac{\log_e(x+2)}{\log_e 2}-2\right)$,

$\frac{dA}{dx}=-\left(\frac{\log_e(x+2)}{\log_e 2}-2\right)-\frac{x}{(x+2)\log_e 2}$.

Given $\frac{dx}{dt}=\log_e 2$.

$\frac{dA}{dt}=\frac{dA}{dx}\frac{dx}{dt}=-\log_e(x+2)+2\log_e 2-\frac{x}{x+2}=\log_e\left(\frac{4}{x+2}\right)-\frac{x}{x+2}$.

Q2a



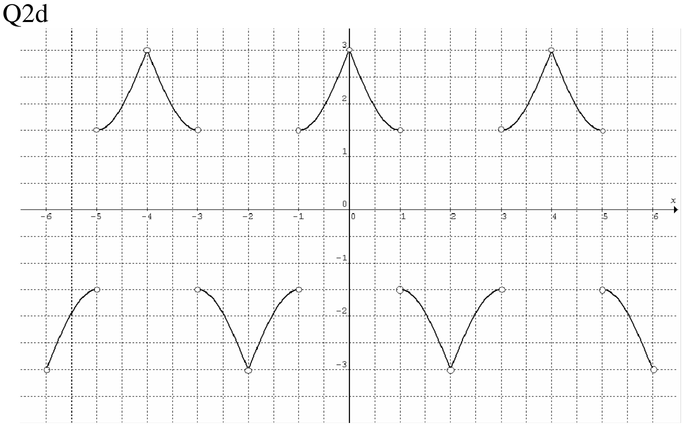
Q2b Domain is $R\setminus\{n: n=0,\pm1,\pm2,\pm3,......\}$.
Range is $(-2,-1)\cup(1,2)$.

Q2c Use property 1: $f(-x)=-f(x)$ and property 2: $f(x-2)=-f(x)$.
$f(-3.5)=-f(3.5)=f(1.5)=-f(-0.5)$.

Use property 3: $f(x)=2-\cos\left(\frac{\pi}{2}x\right)$ when $x\in(-1,0)$.

$-f(-0.5)=-\left(2-\cos\left(-\frac{\pi}{4}\right)\right)=-\left(2-\frac{1}{\sqrt{2}}\right)=\frac{1}{\sqrt{2}}-2$.

Q2d



Q2e The transformed function in part d is an even function, property 1 becomes $f(-x)=f(x)$.

Property 3 becomes $f(x)=1.5\left(2-\cos\left(\frac{\pi}{2}(x-1)\right)\right)$ when $x\in(0,1)$, i.e. $f(x)=3-1.5\cos\left(\frac{\pi}{2}(x-1)\right)$ when $x\in(0,1)$.

Q3a Ratio $r:h=20:25$, $\therefore$ radius $r=\frac{4h}{5}$.

Volume $V=\frac{1}{3}\pi r^2 h=\frac{1}{3}\pi\left(\frac{4h}{5}\right)^2 h=\frac{16\pi h^3}{75}$.

Q3b Full volume $=\frac{16\pi 15^3}{75}=720\pi$ cm^3.
Time $=1$ hour $=3600$ s.
Rate of flow $=\frac{720\pi}{3600}=\frac{\pi}{5}$ cm^3 s^{-1}.

Q3c $\frac{dV}{dh}=\frac{16\pi h^2}{25}$.

$\frac{dV}{dt}=\frac{dV}{dh}\frac{dh}{dt}$, $\frac{dh}{dt}=\frac{\frac{dV}{dt}}{\frac{dV}{dh}}=\frac{-\frac{\pi}{5}}{\frac{16\pi h^2}{25}}=-\frac{5}{16h^2}$.

When $h=5$, $\frac{dh}{dt}=-\frac{1}{80}$.

Rate of decrease $=\frac{1}{80}$ cm s^{-1}.

Q3d Consider the air (cone-shape) above the liquid.
When the depth of liquid is 5 cm, the height of air in the cone $h=25-5=20$ cm.

For the air, $\frac{dh}{dt}=\frac{\frac{dV}{dt}}{\frac{dV}{dh}}=\frac{-\frac{\pi}{5}}{\frac{16\pi h^2}{25}}=-\frac{1}{1280}$ cm s^{-1}.

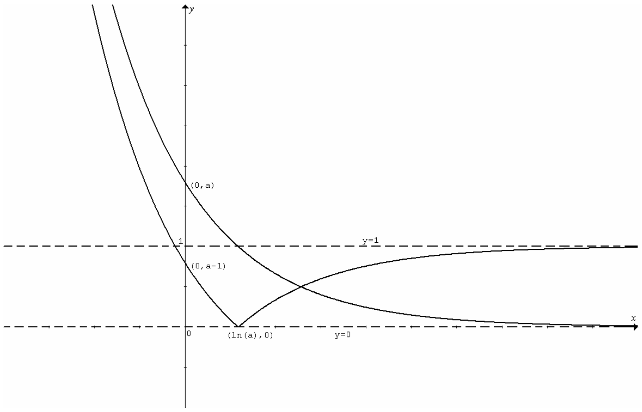
For the liquid, the rate of increase $=\frac{1}{1280}$ cm s^{-1}.

Q3e $\frac{dt}{dh}=\frac{1}{\frac{dh}{dt}}=-\frac{16h^2}{5}$, $t=\int_{15}^{0}-\frac{16h^2}{5}dh=\left[-\frac{16h^3}{15}\right]_{15}^{0}=3600$ s.
Required time $=1$ hour.

Q4a When $x=0$, $y=f(0)=\left|ae^{0}-1\right|=a-1$,

$y=g(0)=ae^{0}=a$.

When $f(x)=0$, $ae^{-x}-1=0$, $ae^{-x}=1$, $e^{x}=a$, $x=\log_e a$.



Q4b $\left|ae^{-x}-1\right|=ae^{-x}$, $-\left(ae^{-x}-1\right)=ae^{-x}$, $2ae^{-x}=1$, $e^{-x}=\dfrac{1}{2a}$,

$e^{x}=2a$, $x=\log_e(2a)$, and $y=ae^{-x}=\dfrac{1}{2}$.

Intersection $\left(\log_e(2a),\dfrac{1}{2}\right)$.

Q4c Area of the region $=\displaystyle\int_0^{\log_e(2a)}\left(g(x)-f(x)\right)dx$

$$=\int_0^{\log_e(a)}\left(g(x)-f(x)\right)dx+\int_{\log_e(a)}^{\log_e(2a)}\left(g(x)-f(x)\right)dx$$

$$=\int_0^{\log_e(a)}\left(ae^{-x}-\left(ae^{-x}-1\right)\right)dx+\int_{\log_e(a)}^{\log_e(2a)}\left(ae^{-x}-^{-}\left(ae^{-x}-1\right)\right)dx$$

$$=\int_0^{\log_e(a)}1dx+\int_{\log_e(a)}^{\log_e(2a)}\left(2ae^{-x}-1\right)dx$$

$$=\left[x\right]_0^{\log_e(a)}+\left[-2ae^{-x}-x\right]_{\log_e(a)}^{\log_e(2a)}$$

$$=\log_e(a)+\left(-2ae^{-\log_e(2a)}-\log_e(2a)\right)-\left(-2ae^{-\log_e(a)}-\log_e(a)\right)$$

$$=\log_e(a)+\left(-1-\log_e(2a)\right)-\left(-2-\log_e(a)\right)$$

$$=1+\log_e\left(\frac{a}{2}\right).$$

Q5a $\displaystyle\int_{-\infty}^{\infty}ke^{-\frac{1}{2}\left(\frac{x-25}{5}\right)^2}dx=1$, $k\displaystyle\int_{-\infty}^{\infty}e^{-\frac{1}{2}\left(\frac{x-25}{5}\right)^2}dx=1$.

Evaluate the definite integral by calculator: $k(12.5331)=1$, $k\approx 0.08$.

Q5b $\mu=25$, $\sigma=5$,

$\Pr(L>20)=\Pr(L>\mu-\sigma)\approx 0.68+\dfrac{1}{2}(1-0.68)=0.84$,

i.e. 84%.

Q5c Binomial distribution: $n=5$,

$p=\Pr(L>30)=\Pr(L>\mu+\sigma)\approx 0.16$.

$\Pr(X=2)=binompdf(5,0.16,2)\approx 0.15$.

Q5d Binomial distribution: $n=5$,

$p=\Pr(L>30\mid L>20)=\dfrac{\Pr(L>30)}{\Pr(L>20)}\approx\dfrac{0.16}{0.84}\approx 0.19$.

$\Pr(X=2)=binompdf(5,0.19,2)\approx 0.19$.

Q5ei Now the fish in the first pond are all longer than 20 cm.

$\displaystyle\int_{20}^{\infty}ke^{-\frac{1}{2}\left(\frac{x-25}{5}\right)^2}dx=1$, $k\displaystyle\int_{20}^{\infty}e^{-\frac{1}{2}\left(\frac{x-25}{5}\right)^2}dx=1$.

By calculator, $k(10.544689)=1$, $k\approx 0.0948$.

Hence $f(x)=\begin{cases}0.0948e^{-\frac{1}{2}\left(\frac{x-25}{5}\right)^2}, & x>20\\ 0, & elsewhere.\end{cases}$

Q5eii $p=\displaystyle\int_{30}^{\infty}0.0948e^{-\frac{1}{2}\left(\frac{x-25}{5}\right)^2}dx\approx 0.19$.

$\Pr(X=2)=binompdf(5,0.19,2)\approx 0.19$.

Q5f $\mu=\displaystyle\int_{20}^{\infty}xf(x)dx=\int_{20}^{\infty}x0.0948e^{-\frac{1}{2}\left(\frac{x-25}{5}\right)^2}dx\approx 26.44$ cm.

The mean (26.44) is different from the mode (25), $\therefore$ no longer a normal distribution.

Q5g Mean price in dollars

$=\displaystyle\int_{30}^{\infty}0.01x^2f(x)dx=\int_{30}^{\infty}0.01x^2 0.0948e^{-\frac{1}{2}\left(\frac{x-25}{5}\right)^2}dx\approx 2.017$.

Total price $=\$2.017\times 1000\approx\$2000$.

Please inform mathline@itute.com re conceptual, mathematical and/or typing errors