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**MATHS METHODS (CAS) 3 & 4
TRIAL EXAMINATION 2
SOLUTIONS
2009**

SECTION 1 – Multiple-choice answers

1. C	9. B	17. D
2. A	10. D	18. A
3. D	11. B	19. D
4. E	12. D	20. B
5. A	13. E	21. E
6. C	14. B	22. E
7. A	15. C	
8. A	16. A	

SECTION 1 – Multiple-choice solutions

Question 1

$E(X)$ = mean of X

$$= -2 \times 0.3 + -1 \times 0.2 + 0 \times 0.4 + 1 \times 0.1$$

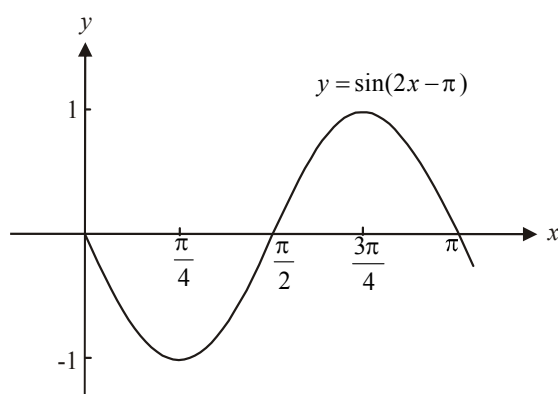
$$= -0.7$$

The answer is C.

Question 2

f^{-1} exists if f is a 1:1 function.

Sketch the graph of $y = \sin(2x - \pi)$ for $x \geq 0$.

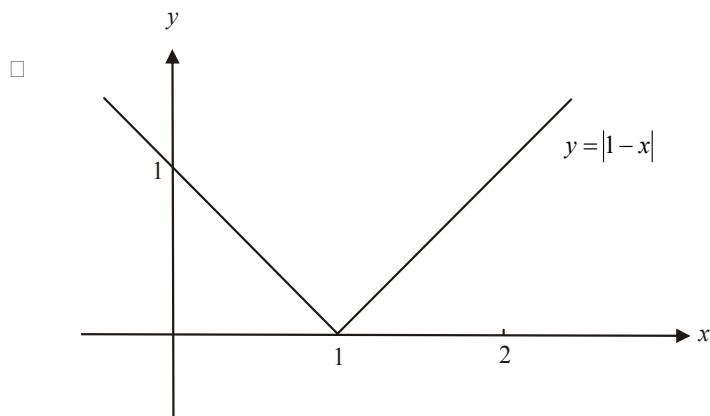


If $a = \frac{\pi}{4}$, then f is a 1:1 function and f^{-1} exists.

The answer is A.

Question 3

Sketch the function $y = |1 - x|$.



The rate of change is the gradient of the function. At $x = 2$, the gradient = 1.
The answer is D.

Question 4

$$5x + (a - 3)y = 1$$

$$ax + 2y = a$$

$$\begin{bmatrix} 5 & a-3 \\ a & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ a \end{bmatrix}$$

For no solutions or an infinite number of solutions the determinant equals zero.

$$5 \times 2 - a(a - 3) = 0$$

$$10 - a^2 + 3a = 0$$

$$-(a^2 - 3a - 10) = 0$$

$$-(a - 5)(a + 2) = 0$$

$$a = 5 \text{ or } a = -2$$

If $a = 5$, $5x + 2y = 1$

$$5x + 2y = 5$$

We have parallel lines, hence no solution.

If $a = -2$, $5x - 5y = 1$

$$\text{So } x - y = \frac{1}{5}$$

$$-2x + 2y = -2$$

$$\text{So } x - y = 1$$

We have parallel lines, hence no solution.

So $a \in \{-2, 5\}$ for no solutions.

The answer is E.

Question 5

This is a binominal distribution with $n = 8$ and $p = 0.1$.

$$\begin{aligned}
 \Pr(X \geq 2) &= 1 - \Pr(X < 2) \\
 &= 1 - \{\Pr(X = 0) + \Pr(X = 1)\} \\
 &= 1 - \{ {}^8C_0 (0.1)^0 (0.9)^8 + {}^8C_1 (0.1)^1 (0.9)^7 \} \\
 &= 1 - (0.430467... + 0.382637...) \\
 &= 0.186896... \\
 &= 0.1869 \text{ (to four decimal places)}
 \end{aligned}$$

The answer is A.

Question 6

$$\begin{aligned}
 \text{Average value} &= \frac{1}{b-a} \int_a^b f(x) dx \\
 &= \frac{1}{2} \int_1^3 \left(\frac{1}{x+2} - 3 \right) dx \\
 &= \frac{1}{2} \left(\log_e \left(\frac{5}{3} \right) - 6 \right) \\
 &= \frac{1}{2} \log_e \left(\frac{5}{3} \right) - 3
 \end{aligned}$$

The answer is C.

Question 7

$$\begin{aligned}
 \int_0^4 (2 - 5f(x)) dx &= \int_0^4 2 dx - 5 \int_0^4 f(x) dx \\
 &= [2x]_0^4 - 5 \times 3 \\
 &= 8 - 0 - 15 \\
 &= -7
 \end{aligned}$$

The answer is A.

Question 8

Sketch the function.

$$\text{At } x = \frac{\pi}{12},$$

$$4 \cos(2x)$$

$$= 4 \cos\left(\frac{\pi}{6}\right)$$

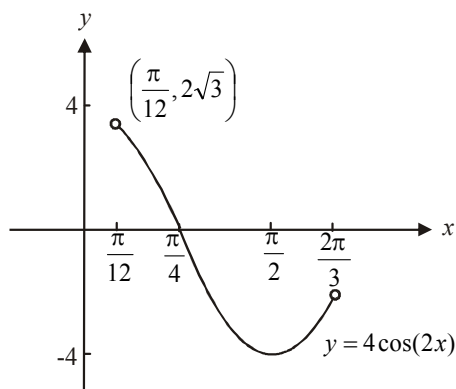
$$= 4 \times \frac{\sqrt{3}}{2}$$

$$= 2\sqrt{3}$$

From the diagram,

$$r_f = [-4, 2\sqrt{3})$$

The answer is A.



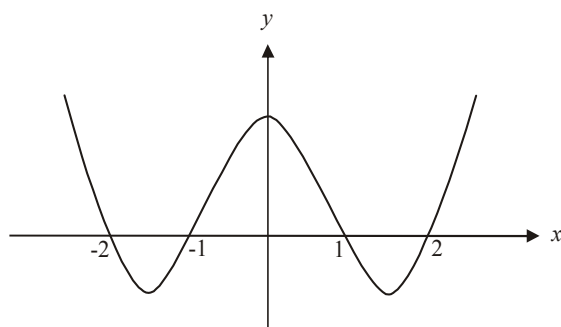
Question 9

$$\begin{aligned}
 \Pr(X < 5 | X < 10) &= \frac{\Pr(X < 5 \cap X < 10)}{\Pr(X < 10)} \\
 &= \frac{\Pr(X < 5)}{0.5} \\
 &= \frac{0.00621}{0.5} \\
 &= 0.01242
 \end{aligned}$$

The answer is B.

Question 10

$$\begin{aligned}
 y &= x^4 - 5x^2 + 4 \\
 &= (x^2 - 4)(x^2 - 1) \\
 &= (x - 2)(x + 2)(x - 1)(x + 1)
 \end{aligned}$$



The graph is symmetrical about the y -axis.

Use your calculator to find the minimum turning points.

They occur at $(-1.58114, -2.25)$ and, by symmetry at $(1.58114, -2.25)$.

The gradient is positive for $x \in (-1.5811\dots, 0) \cup (1.5811\dots, \infty)$

The closest answer is D.

Question 11

$$\begin{aligned}
 \text{Average rate of change} &= \frac{f(2) - f(1)}{2 - 1} \\
 &= \log_e(5) - \log_e(3) \\
 &= \log_e\left(\frac{5}{3}\right)
 \end{aligned}$$

The answer is B.

Question 12

Option A is incorrect because h is discontinuous at $x = 1$ and $x = 3$.

Option B is incorrect because h does not exist at $x = 3$.

Option C is incorrect because $h(x) \leq 0$ for $x \in [-3, 0]$.

Option D is correct because $h(x)$ exists at $x = 1$.

Option E is incorrect because $h'(1)$ does not exist. This is because the limits for $h'(x)$ from the left and right hand side of $x = 1$ are not equal.

The answer is D.

Question 13

Let $\begin{bmatrix} x' \\ y' \end{bmatrix}$ represent an image point.

$$\begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' - 2 \\ y' + 3 \end{bmatrix}$$

$$\begin{aligned} \begin{bmatrix} x \\ y \end{bmatrix} &= \frac{-1}{4} \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x' - 2 \\ y' + 3 \end{bmatrix} \\ &= -\frac{1}{4} \begin{bmatrix} 4(x' - 2) \\ -1(y' + 3) \end{bmatrix} \\ &= \begin{bmatrix} \frac{2 - x'}{4} \\ \frac{y' + 3}{4} \end{bmatrix} \end{aligned}$$

So $y = x^2 + 1$

becomes $\frac{y' + 3}{4} = (2 - x')^2 + 1$

$$y' + 3 = 4(2 - x')^2 + 4$$

$$y' = 4(2 - x')^2 + 1$$

The equation of the image is $y = 4(2 - x)^2 + 1$.

The answer is E.

Question 14

$$\frac{dV}{dt} = 5$$

$$V = \frac{4}{3}\pi r^3 \text{ (from formula sheet)}$$

$$\frac{dV}{dr} = 4\pi r^2$$

$$\frac{dr}{dt} = \frac{dr}{dV} \cdot \frac{dV}{dt}$$

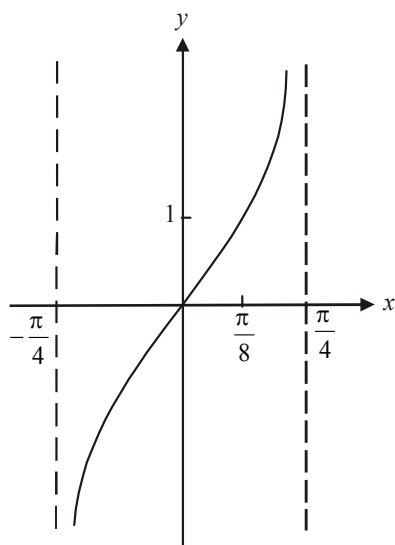
$$= \frac{1}{4\pi r^2} \cdot 5$$

$$= \frac{5}{4\pi r^2}$$

The answer is B.

Question 15

Do a quick sketch.



A dilation by a factor of 3 from the x -axis changes the rule $y = \tan(2x)$ to become

$$\frac{y}{3} = \tan(2x)$$

$$y = 3\tan(2x)$$

□

A reflection in the x -axis changes the rule to

$$-y = 3\tan(2x)$$

$$y = -3\tan(2x)$$

Note that the domain is not affected by these two transformations.

The answer is C.

Question 16

Since we have a probability density function,

$$\int_{-\infty}^{\infty} f(x) dx = 1$$

$$\int_0^1 \frac{x^2}{k} dx = 1$$

$$\frac{1}{k} \left[\frac{x^3}{3} \right]_0^1 = 1$$

$$\left[\frac{x^3}{3} \right]_0^1 = k$$

$$\frac{1}{3} - 0 = k$$

$$k = \frac{1}{3}$$

The answer is A.

Question 17

$$f(x) = \log_e(3x)$$

$$2f(x) = 2\log_e(3x)$$

$$= \log_e(3x)^2$$

$$= \log_e(9x^2)$$

$$\text{So } f(y) = \log_e(9x^2)$$

$$= \log_e(3 \times 3x^2)$$

$$\text{So } y = 3x^2$$

The answer is D.

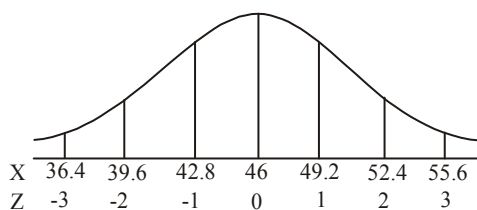
Question 18

Let w = minimum weight required.

$$\Pr(X < w) = 0.1$$

$$w = 41.899$$

The answer is A.

**Question 19**

If n is small, there will be few rectangles and the approximation would be not as accurate as it could be.

If a is small, b could be large so this does not guarantee an increase in accuracy.

Similarly if b is small a could be very much smaller and an increase in accuracy is not guaranteed.

If h is small then the width of the rectangles will be small which ensures a lot of rectangles, which increases the accuracy of the approximation.

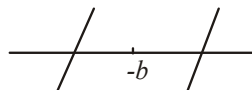
The answer is D.

Question 20

For a stationary point of inflection to occur $f'(x) = 0$. This only occurs at

$x = -b$ and at $x = d$.

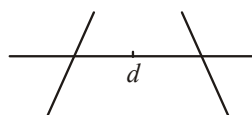
To the left of $x = -b$, $f'(x) > 0$ and to the right of $x = -b$, $f'(x) < 0$ so we have a stationary point of inflection at $x = -b$.



Note that to the left of $x = d$, $f'(x) > 0$ and to the right of $x = d$, $f'(x) < 0$

So we have a local maximum at $x = d$.

The answer is B.

**Question 21**

$f(x) = \frac{1}{\sqrt{x-1}}$, has a maximal domain.

That maximal domain is given by

$$x - 1 > 0$$

$$x > 1$$

So $d_f = (1, \infty)$.

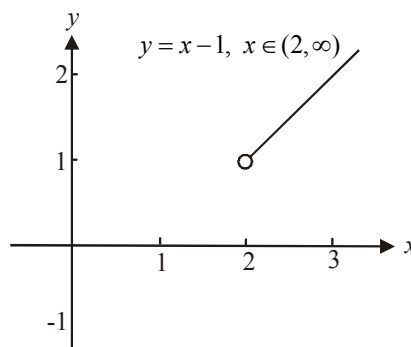
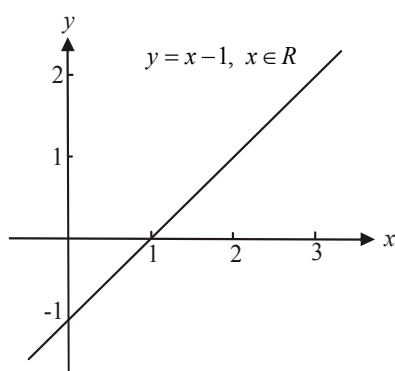
$f(g(x))$ exists iff $r_g \subseteq d_f$

So we require $r_g \subseteq (1, \infty)$

Method 1

The graph of $y = x - 1$ for $x \in \mathbb{R}$ is shown in the diagram on the left below.

In order to restrict the range to $(1, \infty)$, we are going to have to restrict the domain to $x \in (2, \infty)$ as shown on the graph on the right below.



Since $r_g \subseteq (1, \infty)$, $d_g \subseteq (2, \infty)$.

So $a \neq -2, -1, 0$ or 1

So $a = 2$ is the only possible answer.

The answer is E.

Method 2

We require $r_g \subseteq (1, \infty)$

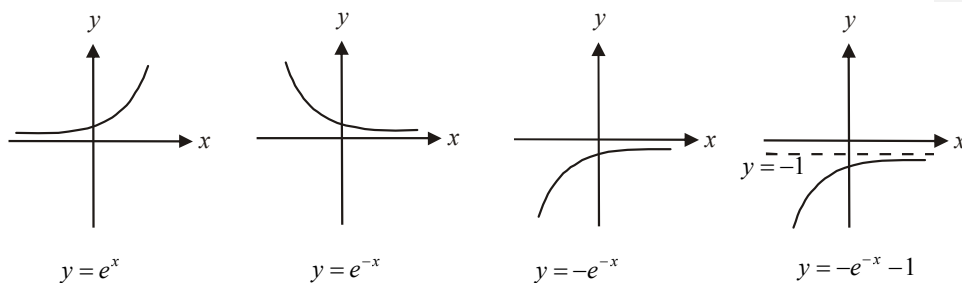
$$r_g = (a - 1, \infty)$$

$$a - 1 \geq 1$$

$$a \geq 2$$

So $a = 2$ is the only possible answer.

The answer is E.

Question 22

The only possible graph is E.

The answer is E.

SECTION 2

Question 1

- a. The graph of $y = f(x)$ passes through $(0,0)$

$$\text{so, } 0 = e^{ax \cdot 0} - b$$

$$b = e^0$$

$$b = 1 \text{ as required}$$

(1 mark)

- b. $f(x) = e^{ax} - 1$

$$f'(x) = ae^{ax}$$

$$f'(1) = ae^a$$

$$\text{Since } f'(1) = 2e^2 \quad (\text{given})$$

$$\text{then } a = 2$$

(1 mark)

(1 mark)

- c. i. Note that the equation of the asymptote of $y = e^{2x} - 1$ is $y = -1$.

Method 1

The graph of $y = f(x)$ has undergone a reflection in the x -axis followed by a translation of 1 unit down. (1 mark) – reflection (1 mark) – translation

Method 2

The graph of $y = f(x)$ has undergone a translation of 1 unit up followed by a reflection in the x -axis. (1 mark) – reflection (1 mark) – translation

- ii. Method 1

After the reflection in the x -axis the rule $y = e^{2x} - 1$ becomes $-y = e^{2x} - 1$ so

$$y = 1 - e^{2x}.$$

(1 mark)

After the translation of 1 unit down, $y = 1 - e^{2x}$ becomes

$$y = 1 - e^{2x} - 1 = -e^{2x}. \text{ So } g(x) = -e^{2x} \text{ as required.}$$

(1 mark)

Method 2

After the translation of 1 unit up, the rule $y = e^{2x} - 1$ becomes $y = e^{2x} - 1 + 1$

$$\text{so } y = e^{2x}.$$

(1 mark)

After the reflection in the x -axis, $y = e^{2x}$ becomes $-y = e^{2x}$ so $y = -e^{2x}$.

$$\text{So } g(x) = -e^{2x} \text{ as required.}$$

(1 mark)

- d. Solve $f(x) = g(x)$

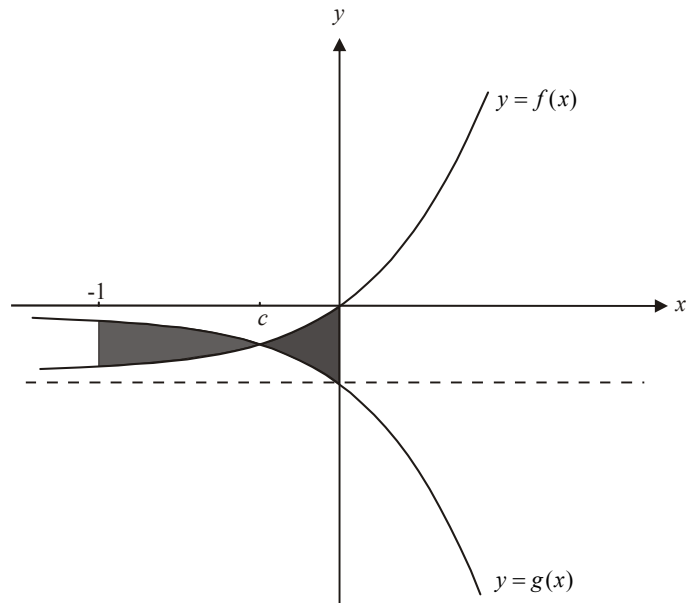
(1 mark)

$$x = -\frac{1}{2} \log_e(2)$$

$$\text{So } c = -\frac{1}{2} \log_e(2) \text{ as required.}$$

(1 mark)

e.



$$\text{Area required} = \int_{-1}^c (g(x) - f(x)) dx + \int_c^0 (f(x) - g(x)) dx$$

$$\text{OR} \quad \int_{-1}^{-\frac{1}{2} \log_e(2)} (g(x) - f(x)) dx + \int_{-\frac{1}{2} \log_e(2)}^0 (f(x) - g(x)) dx$$

(1 mark) for first integrand
and terminals

(1 mark) – for second integrand
and terminals

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f. i. $f(x) = e^{2x} - 1$
 $f\left(\frac{u+v}{2}\right) = e^{2\left(\frac{u+v}{2}\right)} - 1$
 $= e^{(u+v)} - 1$ as required

(1 mark)

ii. To show

$$\left(f\left(\frac{u+v}{2}\right)\right)^2 = f(u+v) - 2f\left(\frac{u+v}{2}\right)$$

$$\left(f\left(\frac{u+v}{2}\right)\right)^2 = (e^{(u+v)} - 1)^2 \quad \text{from part i.}$$

(1 mark)

$$= e^{2(u+v)} - 2e^{(u+v)} + 1$$

$$= f(u+v) + 1 - 2\left\{f\left(\frac{u+v}{2}\right) + 1\right\} + 1$$

(1 mark)

$$= f(u+v) + 1 - 2f\left(\frac{u+v}{2}\right) - 2 + 1$$

$$= f(u+v) - 2f\left(\frac{u+v}{2}\right)$$

as required

(1 mark)

Total 15 marks

Question 2

- a. i. $\Pr(\text{rides on next 5 days})$
 $= 0.6^5$
 $= 0.0778$ (correct to 4 decimal places) (1 mark)

- ii. This is a binomial distribution with $n = 5$, $x = 2$ and $p = 0.6$.
 $\Pr(\text{rides 2 out of next 5 days})$
 $= \Pr(X = 2)$
 $= {}^5C_2 (0.6)^2 (0.4)^3$ (1 mark)
 $= 0.2304$ (1 mark)

- b. i. $\Pr(wwww) = 0.4^4$
 $= 0.0256$ (1 mark)

- ii. $\Pr(wcc) + \Pr(cwc) + \Pr(ccw)$ (1 mark)
 $= 0.4 \times 0.6 \times 0.7 + 0.6 \times 0.3 \times 0.6 + 0.6 \times 0.7 \times 0.3$ (1 mark)
 $= 0.402$ (1 mark)

- iii. Use the transition matrix

$$\begin{array}{cc} & \text{one day} \\ & \begin{array}{cc} w & c \end{array} \\ \begin{bmatrix} 0.4 & 0.3 \\ 0.6 & 0.7 \end{bmatrix} & \begin{array}{c} w \\ c \end{array} \text{ next day} \end{array}$$

(1 mark) – for transition matrix

$\Pr(\text{walked on 10th day of term})$

$= \Pr(\text{walks next 9 days})$

$$= \begin{bmatrix} 0.4 & 0.3 \\ 0.6 & 0.7 \end{bmatrix}^9 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

(1 mark) – for giving matrix expression including the power of 9

$$= \begin{bmatrix} 0.3333... \\ 0.6666... \end{bmatrix}$$

So the probability that Jordan walked on the tenth day of term is 0.3333 (correct to 4 decimal places).

(1 mark) – correct answer

- iv. Method 1
 $\Pr(\text{walks over long term})$

$$= \frac{0.3}{0.6 + 0.3}$$

$$= \frac{1}{3}$$

Over the long term Jordan will walk on $\left(\frac{1}{3} \times \frac{100}{1}\right)\% = 33.33\%$ (correct to 2 decimal places).

(1 mark)

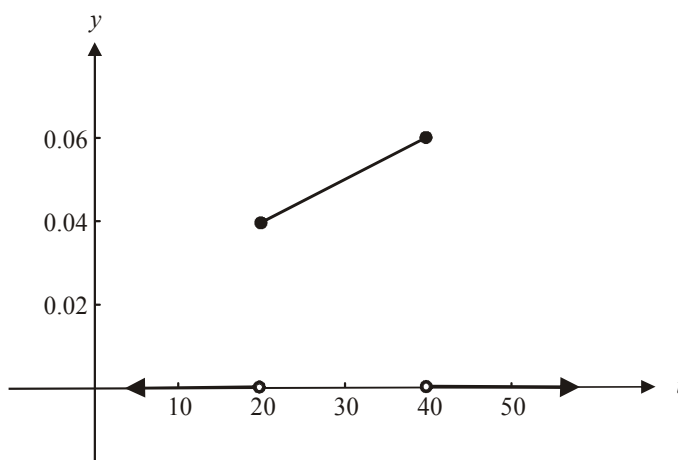
Method 2

Find the steady state.

Let $n = 20$

$$\begin{bmatrix} 0.4 & 0.3 \\ 0.6 & 0.7 \end{bmatrix}^{20} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ = \begin{bmatrix} 0.3333... \\ 0.6666... \end{bmatrix}$$

Using our result from part **iii.**, we see that a steady state has been reached so the probability that Jordan will walk over the long term is 33.33% (correct to 2 decimal places). **(1 mark)**

c. i.

(1 mark) correct linear function and included endpoints for $20 \leq t \leq 40$

(1 mark) correct marking of function along t -axis

- ii.** From the graph, we see that the mode is 40; that is, the value of t with the highest probability; that is, the highest value of $f(t)$.

(1 mark)

- iii.** Let $m = \text{median}$

$$\text{Solve } \int_{20}^m f(t) dt = 0.5$$

(1 mark)

$$m = -70.9902$$

(1 mark)

$$\text{or } m = 30.9902$$

Since $20 \leq m \leq 40$, $m = 31$ minutes (to the nearest minute).

(1 mark) correct answer**Total 17 marks**

Question 3

a. $f(x) = \cos(2x)$
 $f'(x) = -2\sin(2x)$

(1 mark)

- b. The gradient of the tangent is -1 when

$$f'(x) = -1$$

$$-2\sin(2x) = -1$$

$$\sin(2x) = \frac{1}{2}$$

$$2x = \frac{\pi}{6}, \frac{5\pi}{6} \quad x \in \left[0, \frac{\pi}{2}\right] \quad \text{so } 2x \in [0, \pi]$$

$$x = \frac{\pi}{12}, \frac{5\pi}{12}$$

(1 mark)

$$f\left(\frac{\pi}{12}\right) = \cos\left(\frac{\pi}{6}\right) \quad f\left(\frac{5\pi}{12}\right) = \cos\left(\frac{5\pi}{6}\right)$$

$$= \frac{\sqrt{3}}{2}$$

$$= -\frac{\sqrt{3}}{2}$$

$$\text{Required points are } \left(\frac{\pi}{12}, \frac{\sqrt{3}}{2}\right) \text{ and } \left(\frac{5\pi}{12}, -\frac{\sqrt{3}}{2}\right).$$

(1 mark)

- c. i. Tangent passes through $\left(\frac{\pi + \sqrt{3}}{6}, 0\right)$ and $\left(0, \frac{\sqrt{3}\pi + 3}{6}\right)$.

Method 1

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \left(\frac{\sqrt{3}\pi + 3}{6} - 0\right) \div \left(0 - \frac{\pi + \sqrt{3}}{6}\right)$$

$$= -\left(\frac{\sqrt{3}(\pi + \sqrt{3})}{\pi + \sqrt{3}}\right)$$

$$= -\sqrt{3} \text{ as required}$$

(1 mark)Method 2

Note that the gradient must be negative since the tangent is sloping up to the left.

Gradient of tangent is given by

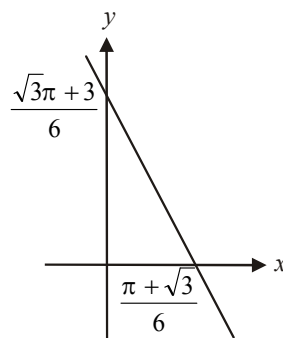
$$\frac{\text{rise}}{\text{run}}$$

$$= -\left(\frac{\frac{\sqrt{3}\pi + 3}{6}}{\frac{\pi + \sqrt{3}}{6}}\right)$$

$$= -\left(\frac{\sqrt{3}\pi + 3}{6} \times \frac{6}{\pi + \sqrt{3}}\right)$$

$$= -\left(\frac{\sqrt{3}(\pi + \sqrt{3})}{\pi + \sqrt{3}}\right)$$

$$= -\sqrt{3} \text{ as required}$$

(1 mark)

ii. $f'(x) = -2\sin(2x) = -\sqrt{3} \quad x \in \left[0, \frac{\pi}{2}\right] \quad (1 \text{ mark})$

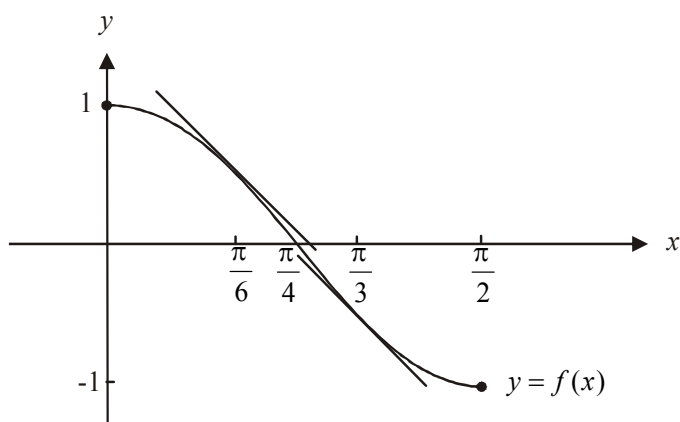
$$\sin(2x) = \frac{\sqrt{3}}{2} \quad 2x \in [0, \pi]$$

$$2x = \frac{\pi}{3}, \frac{2\pi}{3}$$

$$x = \frac{\pi}{6}, \frac{\pi}{3}$$

(1 mark)

iii.



The x -intercept of the tangent we require is $\frac{\pi + \sqrt{3}}{6} = 0.8122\dots$

Now $\frac{\pi}{4} = 0.7853\dots$

From the diagram above, we can see that the tangent we require must pass

through the point where $x = \frac{\pi}{6}$. The other possible tangent which passes

through the point where $x = \frac{\pi}{3}$, has an x -intercept which is less than $\frac{\pi}{4}$ and

therefore less than $\frac{\pi + \sqrt{3}}{6}$.

(1 mark)

$$f(x) = \cos(2x)$$

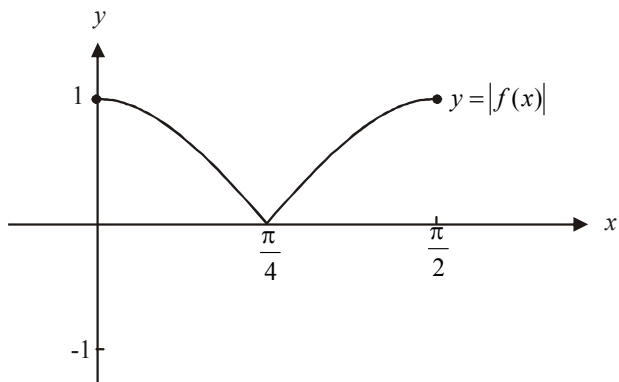
$$f\left(\frac{\pi}{6}\right) = \cos\left(\frac{\pi}{3}\right)$$

$$= \frac{1}{2}$$

Point of tangency is $\left(\frac{\pi}{6}, \frac{1}{2}\right)$.

(1 mark)

- d. Do a quick sketch of the graph of $y = |f(x)|$.



The minimum value of $|f(x)| = 0$.

(1 mark)

The maximum value of $|f(x)| = 1$.

(1 mark)

- e. $g: \mathbb{R} \rightarrow \mathbb{R}, g(x) = \cos(2x)$
 $g(x) = 0.5$

☐ Method 1 – using CAS
☐ Solve $\cos(2x) = 0.5$

$$x = \frac{(6n+1)\pi}{6} \text{ or } \frac{(6n-1)\pi}{6}$$

(1 mark)

☐ When $n = 0$, $x = \frac{\pi}{6}, \frac{-\pi}{6}$

When $n = -1$, $x = \frac{-5\pi}{6}, \frac{-7\pi}{6}$

When $n = 1$, $x = \frac{7\pi}{6}, \frac{5\pi}{6}$

When $n = -2$, $x = \frac{-11\pi}{6}, \frac{-13\pi}{6}$

The values of x start to repeat.

$$x = \frac{\pi}{6} + n\pi \text{ or } x = \frac{5\pi}{6} + n\pi, n \in \mathbb{Z}$$

Alternatively, $x = n\pi \pm \frac{\pi}{6}, n \in \mathbb{Z}$.

(1 mark)

Method 2

Solve $\cos(2x) = 0.5$ for $x \in [0, \pi]$ that is, for one period of the graph of $y = \cos(2x)$.
 $\cos 2x = 0.5 \quad x \in [0, \pi]$

$$\square \quad 2x = \frac{\pi}{3}, \frac{5\pi}{3} \quad 2x \in [0, 2\pi]$$

$$x = \frac{\pi}{6}, \frac{5\pi}{6} \quad \text{for } x \in [0, \pi] \quad \textbf{(1 mark)}$$

For $x \in R$,

$$x = \frac{\pi}{6} + n\pi \quad \text{or} \quad x = \frac{5\pi}{6} + n\pi, \quad n \in Z.$$

\square Alternatively,

$$x = n\pi \pm \frac{\pi}{6}, \quad n \in Z.$$

S	A
T	C

(1 mark)

Total 12 marks

Question 4

- a. Solve $f(x) = 0$
 $5 \log_e(x-10) = 0$
 $e^0 = x-10$
 $1 = x-10$
 $x = 11$
 So $a = 11$ as required.

(1 mark)

- b. Method 1 – using CAS
 Find the inverse function of $y = 5 \log_e(x-10)$

$$\text{So } f^{-1}(x) = e^{\frac{x}{5}} + 10$$

(1 mark) – correct rule

- Method 2 – by hand
 $f(x) = 5 \log_e(x-10)$
 Let $y = 5 \log_e(x-10)$
 Swap x and y for inverse.
 $x = 5 \log_e(y-10)$

$$\frac{x}{5} = \log_e(y-10)$$

$$e^{\frac{x}{5}} = y-10$$

$$y = e^{\frac{x}{5}} + 10$$

$$f^{-1}(x) = e^{\frac{x}{5}} + 10$$

(1 mark) – correct rule

$$d_f = [11, 50]$$

$$r_f = [0, f(50)]$$

$$= [0, 18.44\dots]$$

$$\text{So } d_{f^{-1}} = r_f$$

$$= [0, 18.4] \text{ (correct to 1 decimal place)}$$

(1 mark) – correct domain

- c. Since the graph of $y = f^{-1}(x)$ is a reflection of the graph of $y = f(x)$ in the line $y = x$, the floodlit areas to the north and to the east are the same.

$$\text{Total area} = 2 \int_{11}^{50} f(x) dx$$

(1 mark)

$$= 10 \int_{11}^{50} \log_e(x-10) dx$$

$$= 10 \cdot 86 \text{ m}^2 \text{ (to nearest square metre)}$$

(1 mark)

- d. Find the x -coordinates of the points of intersection between $y = 15$ and $y = f(x)$ and between $y = 15$ and $y = f^{-1}(x)$.

Method 1 – using CAS

$y = 15$ and $y = 5 \log_e(x - 10)$ intersect when $x = 30.09$ (to 2 decimal places)

$y = 15$ and $y = e^{\frac{x}{5}} + 10$ intersect when $x = 8.05$ (to 2 decimal places)

So $b \in (8.05, 30.09)$ or $8.05 < b < 30.09$

(1 mark) correct values

(1 mark) correct brackets or inequality signs

Method 2 – by hand

$$15 = 5 \log_e(x - 10) \qquad 15 = e^{\frac{x}{5}} + 10$$

$$3 = \log_e(x - 10) \qquad 5 = e^{\frac{x}{5}}$$

$$e^3 = x - 10 \qquad \log_e(5) = \frac{x}{5}$$

$$x = 10 + e^3 \qquad x = 5 \log_e(5)$$

$$= 30.09 \text{ (to 2 dec. places)} \qquad = 8.05 \text{ (to 2 dec. places)}$$

So $b \in (8.05, 30.09)$ or $8.05 < b < 30.09$

(1 mark) correct values

(1 mark) correct bracket or inequality signs

e. i. $T = \sqrt{x^2 + 2500} + \frac{50 - x}{2}$

$$\frac{dT}{dx} = \frac{2x - \sqrt{x^2 + 2500}}{2\sqrt{x^2 + 2500}}$$

$$\frac{dT}{dx} = 0 \text{ for minimum.}$$

$$x = 28.87 \text{ (correct to 2 decimal places)}$$

(1 mark)

(1 mark)

ii. $T = 68.3$ seconds correct to 1 decimal place.

(1 mark)

- f. To take the quickest path, Victoria runs to $P(28.87, 50)$ from part e. i.

The straight line from $O(0, 0)$ to $P(28.87, 50)$ is given by $y = \frac{50}{28.87}x$. (1 mark)

Check whether this intersects with the perimeter of the floodlit area to the north given

by $f^{-1}(x) = e^{\frac{x}{5}} + 10$.

$$\text{Solve } e^{\frac{x}{5}} + 10 = \frac{50}{28.87}x.$$

There are two solutions $x = 10.3476$ or $x = 11.2262$.

So Victoria does enter the floodlit area.

(1 mark)

g. From part e. ii. the shortest time it takes is 68.3 seconds (to 1 decimal place).

The dogs have to run $\sqrt{50^2 + 50^2} = 50\sqrt{2}\text{m}$.

They run at 7m/sec.

It takes the dogs

$$50\sqrt{2}\text{m} \div \frac{7\text{m}}{\text{sec}}$$

$$= \frac{50\sqrt{2}}{7}\text{secs}$$

$$= 10 \cdot 1015 \dots \text{secs}$$

to get to H from O .

(1 mark)

The dogs leave 60 secs after Victoria so they arrive 70.1015...secs after she leaves

$O(0,0)$.

□

So Victoria escapes the dogs; but only just!

(1 mark)

Total 14 marks

