

**THE
HEFFERNAN
GROUP**

P.O. Box 1180
Surrey Hills North VIC 3127
Phone 03 9836 5021
Fax 03 9836 5025
info@theheffernangroup.com.au
www.theheffernangroup.com.au

**MATHS METHODS (CAS) 3 & 4
TRIAL EXAMINATION 1
SOLUTIONS
2009**

Question 1

$$f: R \rightarrow R, f(x) = x + 1$$

$$g: (0, \infty) \rightarrow R, g(x) = \log_e(2x)$$

a.
$$\begin{aligned} f(g(x)) &= f(\log_e(2x)) \\ &= \log_e(2x) + 1 \end{aligned}$$

(1 mark)

b. $g(f(x))$ exists iff $r_f \subseteq d_g$.
Now $r_f = R$ and $d_g = (0, \infty)$
Since $R \not\subseteq (0, \infty)$, $g(f(x))$ does not exist.

(1 mark)

Question 2

a.
$$\begin{aligned} f(x) &= x \log_e(x^2 + 5) \\ f'(x) &= x \times \frac{2x}{x^2 + 5} + \log_e(x^2 + 5) \\ &= \frac{2x^2}{x^2 + 5} + \log_e(x^2 + 5) \end{aligned}$$

(1 mark) – use of product rule

(1 mark) – correct derivative

b.
$$y = \frac{\tan(x)}{e^{2x}}$$

$$\frac{dy}{dx} = \frac{e^{2x} \times \sec^2(x) - 2e^{2x} \tan(x)}{e^{4x}}$$

(1 mark) use of quotient rule

When $x = 0$,
$$\frac{dy}{dx} = \frac{e^0 \times \sec^2(0) - 2e^0 \tan(0)}{e^0}$$

(1 mark) substituting $x = 0$

$$\begin{aligned} &= \frac{1 \times \frac{1}{\cos^2(0)} - 2 \times 1 \times 0}{1} \\ &= 1 \end{aligned}$$

(1 mark) – correct answer

Question 3

$$\sqrt{3} \tan(2x) = 1 \quad 0 \leq x \leq 2\pi$$

$$\tan(2x) = \frac{1}{\sqrt{3}} \quad 0 \leq 2x \leq 4\pi$$

$$2x = \frac{\pi}{6}, \frac{7\pi}{6}, \frac{13\pi}{6}, \frac{19\pi}{6}$$

$$x = \frac{\pi}{12}, \frac{7\pi}{12}, \frac{13\pi}{12}, \frac{19\pi}{12}$$

S	A
T	C

(1 mark) – for $\frac{\pi}{12}$

(1 mark) – for remaining 3 correct answers

Question 4

- a. i. Method 1 – using a probability table or Karnaugh map.

This is what is given.

	A	A'	
B	0.15		0.4
B'			
	0.3		1

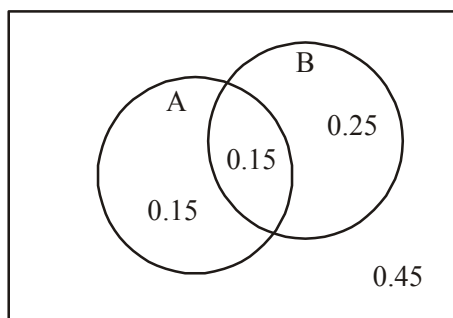
$$\Pr(A' \cap B') = 0.45$$

This is what we can work out.

	A	A'	
B	0.15	0.25	0.4
B'	0.15	0.45	0.6
	0.3	0.7	1

(1 mark)

Method 2 – using a Venn Diagram



$$\Pr(A' \cap B') = 0.45$$

(1 mark)

Method 3 – using Addition rule

$$\begin{aligned} \Pr(A \cup B) &= \Pr(A) + \Pr(B) - \Pr(A \cap B) \\ &= 0.3 + 0.4 - 0.15 \\ &= 0.55 \end{aligned}$$

$$\begin{aligned} \Pr(A' \cap B') &= \Pr(A \cup B)' \\ &= 1 - \Pr(A \cup B) \\ &= 1 - 0.55 \\ &= 0.45 \end{aligned}$$

(1 mark)

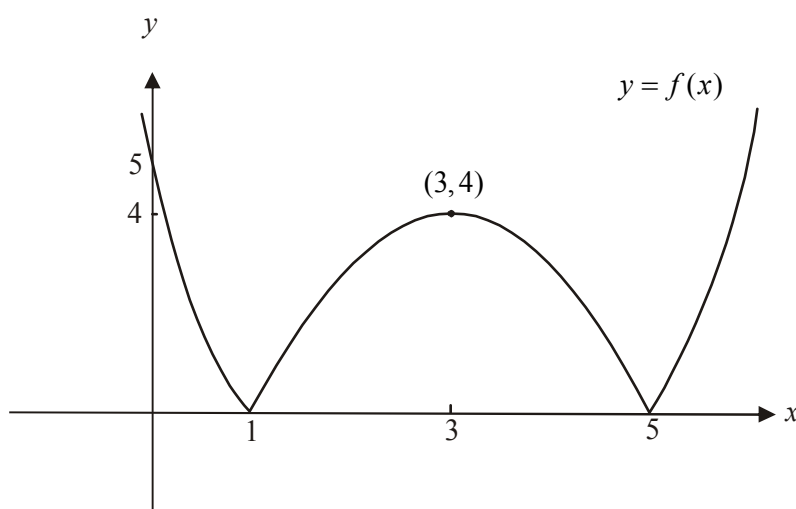
ii.

$$\begin{aligned}
 \Pr(A|B) &= \frac{\Pr(A \cap B)}{\Pr(B)} \\
 &= \frac{0.15}{0.4} \\
 &= \frac{3}{8}
 \end{aligned}
 \quad (1 \text{ mark})$$

b. If A and B are mutually exclusive then $\Pr(A \cap B) = 0$ so $\Pr(A|B) = 0$. (1 mark)

Question 5

a.



(1 mark) – correct shape including cusps at $x=1$ and $x=5$
 (1 mark) correct labelling of intercepts and turning point

b. $d_{f'} = \mathbb{R} \setminus \{1, 5\}$ (1 mark)

c. $f'(x) > 0$ for $x \in (1, 3) \cup (5, \infty)$ (1 mark)

Question 6

a. $\Pr(X < 2) = \int_1^2 \frac{1}{2\sqrt{x}} dx$ (1 mark)

$$= \frac{1}{2} \int_1^2 x^{-\frac{1}{2}} dx$$

$$= \frac{1}{2} \left[2x^{\frac{1}{2}} \right]_1^2$$

$$= \frac{1}{2} (2\sqrt{2} - 2\sqrt{1})$$

$$= \sqrt{2} - \sqrt{1}$$

$$= \sqrt{2} - 1$$

(1 mark)

b.

$$\begin{aligned}
 \mu &= \int_{-\infty}^{\infty} xf(x)dx \\
 &= \int_1^4 x \times \frac{1}{2\sqrt{x}} dx \\
 &= \frac{1}{2} \int_1^4 x^{\frac{1}{2}} dx \\
 &= \frac{1}{2} \left[\frac{2x^{\frac{3}{2}}}{\frac{3}{2}} \right]_1^4 \\
 &= \frac{1}{3} \left[x^{\frac{3}{2}} \right]_1^4 \\
 &= \frac{1}{3} (4^{\frac{3}{2}} - 1^{\frac{3}{2}}) \\
 &= \frac{1}{3} (2^3 - 1) \\
 &= \frac{1}{3} (8 - 1) \\
 &= \frac{7}{3}
 \end{aligned}$$

(1 mark)

(1 mark)

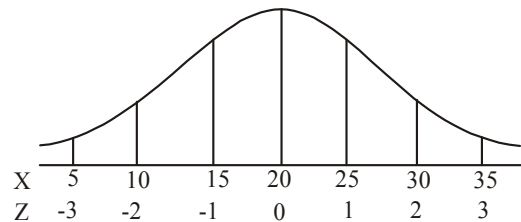
Question 7

a. From the diagram,

$$\Pr(X > 20) = \frac{1}{2}$$

$$\Pr(Z < 0) = \frac{1}{2}$$

$$m = 0$$



(1 mark)

b. $\Pr(X < 18) = \Pr(Z > n)$

Again from the diagram,

$$\Pr(X < 18) = \Pr(X > 22) \text{ by symmetry}$$

(1 mark)

$$\text{Since } z = \frac{x - \mu}{\sigma}$$

$$z = \frac{22 - 20}{5}$$

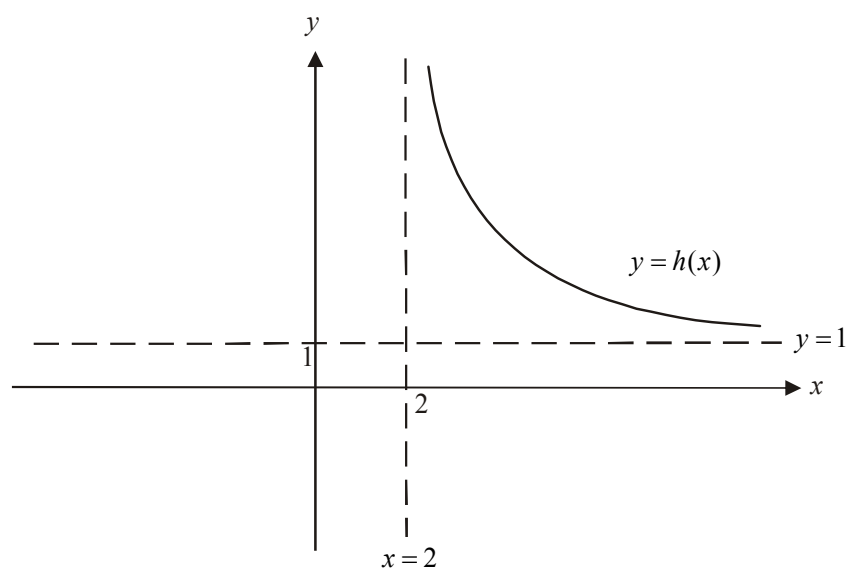
$$= 0.4$$

$$\text{So } n = 0.4$$

(1 mark)

Question 8

a.



(1 mark) – correct shape of graph
 (1 mark) – correct asymptotes

b. $h(x) = \frac{1}{x-2} + 1$

Let $y = \frac{1}{x-2} + 1$

Swap x and y for inverse

$$x = \frac{1}{y-2} + 1$$

Rearrange

$$x - 1 = \frac{1}{y-2}$$

$$(x-1)(y-2) = 1$$

$$y-2 = \frac{1}{x-1}$$

$$y = \frac{1}{x-1} + 2$$

So $h^{-1}(x) = \frac{1}{x-1} + 2$

(1 mark) – correct rule

$r_h = (1, \infty)$ (from graph)

So $d_{h^{-1}} = r_h = (1, \infty)$

(1 mark) – correct domain

Question 9

a. $y = \frac{2}{x}$
 $= 2x^{-1}$
 $\frac{dy}{dx} = -2x^{-2}$
 $= \frac{-2}{x^2}$

When $x = 2$

$$\frac{dy}{dx} = \frac{-2}{4}$$

$$= -\frac{1}{2}$$

The gradient of the tangent to f at $x = 2$ is $-\frac{1}{2}$.

Therefore the gradient of the normal to f at $x = 2$ is 2.

(1 mark)

$$\text{Also } f(2) = \frac{2}{2}$$

$$= 1$$

The equation of the normal through $(2, 1)$ with gradient of 2 is

$$y - 1 = 2(x - 2)$$

$$y = 2x - 3$$

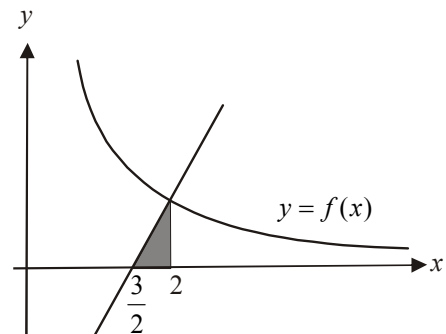
(1 mark)

b. The normal crosses the x -axis when $y = 0$.

$$y = 2x - 3$$

$$0 = 2x - 3$$

$$x = \frac{3}{2}$$

(1 mark)Method 1

$$\text{Area} = \int_{\frac{3}{2}}^2 (2x - 3) dx$$

$$= [x^2 - 3x]_{\frac{3}{2}}^2$$

$$= \left\{ (4 - 6) - \left(\frac{9}{4} - \frac{9}{2} \right) \right\}$$

$$= -2 - \frac{-9}{4}$$

$$= \frac{-8}{4} + \frac{9}{4}$$

$$= \frac{1}{4} \text{ units}^2$$

(1 mark)Method 2

$$\text{Area} = \frac{1}{2} \times \text{base} \times \text{height}$$

$$= \frac{1}{2} \times \left(2 - \frac{3}{2} \right) \times f(2)$$

$$= \frac{1}{2} \times \frac{1}{2} \times 1$$

$$= \frac{1}{4} \text{ units}^2$$

(1 mark)

Question 10Method 1

Given $f(x) = x\sqrt{1-x}$ and $f'(x) = \frac{-x}{2\sqrt{1-x}} + \sqrt{1-x}$

Therefore $\int \left(\frac{-x}{2\sqrt{1-x}} + \sqrt{1-x} \right) dx = x\sqrt{1-x} + c$ **(1 mark)**

$$\begin{aligned} \text{so } -\frac{1}{2} \int \frac{x}{\sqrt{1-x}} dx + \int \sqrt{1-x} dx &= x\sqrt{1-x} + c \\ -\frac{1}{2} \int \frac{x}{\sqrt{1-x}} dx &= x\sqrt{1-x} - \int \sqrt{1-x} dx + c \\ \int \frac{x}{\sqrt{1-x}} dx &= 2 \int (1-x)^{\frac{1}{2}} dx - 2x\sqrt{1-x} - 2c \end{aligned}$$

$$= 2 \times \frac{1}{-1 \times \frac{3}{2}} (1-x)^{\frac{3}{2}} - 2x\sqrt{1-x} - 2c$$

$$= \frac{-4}{3} \sqrt{(1-x)^3} - 2x\sqrt{1-x} \quad \text{where } c = 0 \text{ for an antiderivative}$$

$$\text{or } = \frac{-4}{3} (1-x)^{\frac{3}{2}} - 2x(1-x)^{\frac{1}{2}}$$

(1 mark) correct antiderivative of $\sqrt{1-x}$ **(1 mark)** correct answer

Method 2

Given $f(x) = x\sqrt{1-x}$ and $f'(x) = \frac{-x}{2\sqrt{1-x}} + \sqrt{1-x}$

So, $\frac{x}{2\sqrt{1-x}} = \sqrt{1-x} - f'(x)$

$$\frac{x}{\sqrt{1-x}} = 2\sqrt{1-x} - 2f'(x)$$

$\int \frac{x}{\sqrt{1-x}} dx = 2 \int \sqrt{1-x} dx - 2 \int f'(x) dx$ **(1 mark)**

$$= 2 \times \frac{1}{-1 \times \frac{3}{2}} (1-x)^{\frac{3}{2}} - 2 \times x\sqrt{1-x} + c$$

$$= \frac{-4}{3} \sqrt{(1-x)^3} - 2x\sqrt{1-x} \quad \text{where } c = 0 \text{ for an antiderivative}$$

$$\text{or } = \frac{-4}{3} (1-x)^{\frac{3}{2}} - 2x(1-x)^{\frac{1}{2}}$$

(1 mark) correct antiderivative of $\sqrt{1-x}$ **(1 mark)** correct answer

Question 11

a. Perimeter = $2x + 2y + \frac{1}{2} \times 2\pi x$ (1 mark)

So $100 = 2x + 2y + \pi x$

$$2y = 100 - 2x - \pi x$$

$$2y = 100 - x(\pi + 2)$$

$$y = \frac{100 - x(\pi + 2)}{2}$$

(1 mark)

b. Surface area = $2xy + \frac{1}{2} \times \pi x^2$

$$= 2x \frac{(100 - x(\pi + 2))}{2} + \frac{\pi x^2}{2}$$

from part a.

$$= 100x - x^2(\pi + 2) + \frac{\pi x^2}{2}$$

$$= 100x - \pi x^2 - 2x^2 + \frac{\pi x^2}{2}$$

$$= 100x - \frac{\pi x^2}{2} - 2x^2$$

So $A = 100x - \frac{x^2}{2}(\pi + 4)$

(1 mark)

c. $A = 100x - \frac{x^2}{2}(\pi + 4)$

Max/min occur when $\frac{dA}{dx} = 0$.

(1 mark)

$$\frac{dA}{dx} = 100 - x(\pi + 4) = 0$$

$$100 = x(\pi + 4)$$

$$x = \frac{100}{\pi + 4} \text{ m}$$

(1 mark)

d. Method 1 – using part b.

$$A = 100x - \frac{x^2}{2}(\pi + 4)$$

This is the equation of an inverted parabola which has a local maximum and hence there will be a maximum rather than a minimum value to be found.

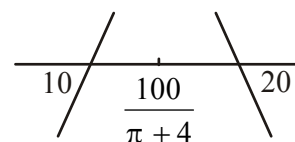
(1 mark) for reference to inverted parabola

Method 2 – using a sign diagram

$$x = \frac{100}{\pi + 4} = 14.0...$$

At $x = 10$, $\frac{dA}{dx} = 28.5... > 0$.

At $x = 20$, $\frac{dA}{dx} = -42.8... < 0$



From the sign diagram we see that we have a maximum surface area.

(1 mark)

Total 40 marks