

MATHEMATICAL METHODS (CAS)

Units 3 & 4 – Written examination 2



2008 Trial Examination

SOLUTIONS

SECTION 1: Multiple-choice questions (1 mark each)

Question 1

Answer: D

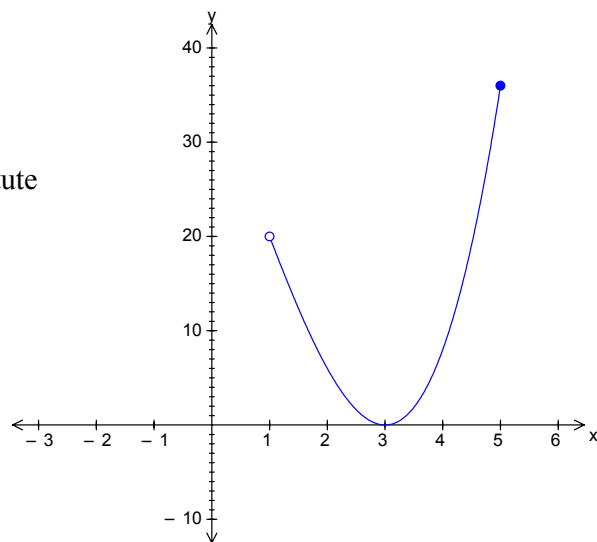
Explanation:

Factorised $y = x^2(4 - x^2) = x^2(2 - x)(2 + x)$ giving TP at $x = 0$, intercepts at $x = -2, 2$ curve reflected in x axis

Question 2

Answer: B

Explanation: From graph min range = 0 and substitute endpoint $x = 5$ for max = 36



Question 3*Answer:* B*Explanation:*

Solving this equation yields the two general solutions:

$$x = \frac{\sqrt{3}\pi(6n+1)}{9} \quad (1) \quad \text{and} \quad x = \frac{\sqrt{3}\pi(6n-1)}{9} \quad (2), \text{ where } n \in \mathbb{Z}.$$

$$\text{Therefore, using (1): } x = \frac{\sqrt{3}\pi}{9}, \frac{7\sqrt{3}\pi}{9}, \frac{13\sqrt{3}\pi}{9}, \frac{19\sqrt{3}\pi}{9}$$

$$\text{Also, using (2): } x = \frac{5\sqrt{3}\pi}{9}, \frac{11\sqrt{3}\pi}{9}, \frac{17\sqrt{3}\pi}{9}, \frac{23\sqrt{3}\pi}{9}.$$

Arranging the solutions in order, the seventh solution must be $\frac{19\sqrt{3}\pi}{9}$.

Question 4*Answer:* D*Explanation:*

To solve, set up the matrix equation:

$$\begin{bmatrix} 2 & 1 & -3 & 0.5 & -0.2 \\ 1 & -1 & 1 & -1 & 1 \\ 1 & -3 & 2 & -0.3 & 0.1 \\ 0.3 & -2 & 7 & -1 & 0.1 \\ 1 & -3 & -4 & -5 & -2 \end{bmatrix} \times \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} = \begin{bmatrix} -14.24 \\ 9.3 \\ 12.39 \\ 32.3 \\ -3.3 \end{bmatrix}.$$

$$a = 0.1$$

$$b = -1.2$$

This yields the solution: $c = 3.9$. Therefore, $c = 39a$ is correct.

$$d = -2.4$$

$$e = 1.7$$

Question 5*Answer:* B*Explanation:*

When $f(x) = 3x^2 - 7x - 0.5$ and $g(x) = 3x^2 + 0.25x - \sqrt{2}$:

$g(x-1) \approx 3x^2 - 5.75x + 1.34$, so $\sqrt{3}g(x-1) \approx 5.2x^2 - 10.0x + 2.31$.

Therefore, $f(\sqrt{3}g(x-1)) \approx 81x^4 - 310.5x^3 + 333.32x^2 - 68.54x - 0.64$

Question 6*Answer:* A*Explanation:*

Rearrange equation $y = e^{2(x-2)} - 2$ makes dilation factor and translations more obvious

Question 7*Answer:* B*Explanation:*

$$x = 0, y = \log_e(3) - 1 \therefore (0, \log_e(3) - 1)$$

$$y = 0, 1 = \log_e(x + 3)$$

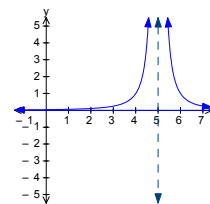
$$\dots\dots\dots e = x + 3$$

$$\dots\dots\dots x = e - 3 \therefore (e - 3, 0)$$

Question 8*Answer:* C*Explanation:*

Need to restrict domain to make function one-to-one

From options given $x < 5$ or $x \in (-\infty, 5)$



Question 9

Answer: D

Explanation:

Reflected in x axis, amplitude 4 therefore $a = -4$, period = 4 therefore $n = \frac{\pi}{2}$, translated down 3 units so $c = -3$

Question 10

Answer: E

Explanation:

Using CAS technology (or the product rule) you find that $f'(x) = \frac{e^{4hx} x^{\frac{1}{2}}}{3} \left(\frac{3}{2} + 4hx \right)$.

Question 11

Answer: D

Explanation:

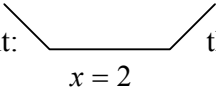
Solving for $\int_{\frac{\pi}{2}}^{\sqrt{7}} (x^3 - 7x^2 + xe^x - 2) dx$, we get (approximately) -5.143.

Therefore, the average value is found by $\frac{1}{\sqrt{7} - \frac{\pi}{2}} \times -5.143 \approx -4.784$.

Question 12

Answer: E

Explanation:

(2, 6) co-ordinate where gradient:  therefore a local minimum

Question 13*Answer:* C*Explanation:*

Graph not smooth at $x = 0$, $x = 3$ gradient negative $x < 0$, gradient changes from positive to zero to negative in $0 < x < 3$, gradient positive $x > 3$ therefore must be C.

Question 14*Answer:* A*Explanation:*

$$(6x + 1) \div (2x - 3)$$

$$(2x - 3) \overline{) \frac{3}{6x + 1}}$$

$$\dots\dots\dots 6x - 9$$

$$\dots\dots\dots 10$$

$$f(x) = \int \left(3 + \frac{10}{2x - 3} \right) dx$$

$$\dots\dots\dots 3x + \frac{10}{2} \log_e(2x - 3) + c$$

$$\therefore 3x + 5 \log_e(2x - 3) + c$$

Question 15*Answer:* E*Explanation:*

$$\frac{dr}{dt} = \frac{dr}{dV} \times \frac{dV}{dt}$$

$$\frac{dV}{dr} = \frac{4(3)\pi r^2}{3}$$

$$\dots\dots\dots = 4\pi r^2$$

$$\frac{dr}{dV} = \frac{1}{4\pi r^2}$$

$$\therefore \frac{dr}{dt} = -\frac{3}{4\pi r^2}$$

Question 16*Answer:* B*Explanation:*

The only correct option provided for dealing with negative areas. Must integrate between x values and graphs swap at $x = 0$

Question 17*Answer:* D*Explanation:*

	A	A'	
B	0.35	0.3	0.65
B'	0.25	0.1	0.35
	0.6	0.4	1

The following is also suggested:

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$$

$$\dots\dots\dots 0.9 = 0.6 + 0.65 - \Pr(A \cap B)$$

$$\dots\dots\dots 0.9 = 1.25 - \Pr(A \cap B)$$

$$\dots\dots\dots 0.35 = \Pr(A \cap B)$$

Question 18*Answer:* B*Explanation:* (1) $np = 12$, (2) $npq = 4$

$$\frac{npq}{np} = \frac{4}{12}$$

$$(2) \div (1) = q = \frac{1}{3} \therefore p = \frac{2}{3}$$

$$n \left(\frac{2}{3} \right) = 12 \therefore n = 18$$

Question 19*Answer:* C*Explanation:*

The graph of the derivative has three x -intercepts, so the original function's graph must have three stationary points. Also, $f'(x) < 0$ for $x < -1$, so the graph of $f(x)$ must have a negative gradient for $x < -1$. Therefore, **C** is the appropriate option.

Question 20

Answer: C

Explanation:

$$T \sim \text{Bi}(150, k) \text{ and } q = 1 - k \therefore {}^{150}C_{15} (k)^{15} (1 - k)^{135}$$

Question 21

Answer: A

Explanation:

$$\int_0^{\frac{\pi}{4}} (x^2 \sqrt{2} \cos(x)) dx - \left(\int_0^{\frac{\pi}{4}} (x \sqrt{2} \cos(x)) dx \right)^2 = 0.0499$$

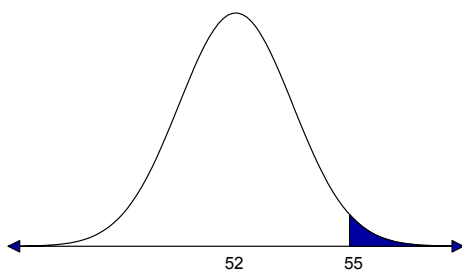
Question 22

Answer: B

Explanation:

$$\text{normalcdf}(55, 10^{99}, 52, 2) = 0.0668$$

Therefore 7%



SECTION 2: Analysis Questions**Question 1**

- a. From the y -intercept, $d = 20$.

M1

Set up three simultaneous equations with the three x -intercepts:

$$x = -5.550 \Rightarrow -170.954a + 30.803b - 5.550c = -20$$

When $x = 3.097 \Rightarrow 29.705a + 9.591b + 3.097c = -20$. Therefore, set up the

$$x = 17.453 \Rightarrow 5316.310a + 304.607b + 17.453c = -20$$

matrix equation:
$$\begin{bmatrix} -170.954 & 30.803 & -5.550 \\ 29.705 & 9.591 & 3.097 \\ 5316.310 & 304.607 & 17.453 \end{bmatrix} \times \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -20 \\ -20 \\ -20 \end{bmatrix}$$
 and solve, yielding

$$a = 0.067 \approx \frac{1}{15}, b = -1 \text{ and } c = -4.$$

M1

Therefore, $f(x) = \frac{x^3}{15} - x^2 - 4x + 20$

A1

- b. $(-5.802, -3.478), (3.246, -1.239), (17.941, 11.348)$

A3

$$\begin{aligned} \text{c. } A &= \int_{-5.802}^{3.246} \left(\frac{x^3}{15} - x^2 - x + 20 - (e^{0.1(x+10)} - 5) \right) dx - \int_{3.246}^{17.941} \left(\frac{x^3}{15} - x^2 - x + 20 - (e^{0.1(x+10)} - 5) \right) dx \\ &= \left[\frac{x^4}{60} - \frac{x^3}{3} - \frac{4x^2}{2} + 25x - 10e^{0.1(x+10)} \right]_{-5.802}^{3.246} - \left[\frac{x^4}{60} - \frac{x^3}{3} - \frac{4x^2}{2} + 25x - 10e^{0.1(x+10)} \right]_{3.246}^{17.941} \\ &= 156.521 + 569.806 \\ &= 726.327 \text{ km}^2 \end{aligned}$$

M2 + A1

- d.

$$V = 156.521 \times 0.025$$

$$\dots = 3.913025 \text{ km}^3$$

$$\dots = 3.913025 \times 10^6$$

$$= 3\,913\,025 \text{ ML}$$

A1

- e. Find point on curve $f(5) = \frac{125}{15} - 25 - 20 + 20 = -16\frac{2}{3}$

$$f'(x) = \frac{3x^2}{15} - 2x - 4$$

Find gradient $f'(5) = \frac{25}{5} - 10 - 4$

$$\dots\dots\dots = -9 \therefore m_n = \frac{1}{9}$$

$$y + 16\frac{2}{3} = \frac{1}{9}(x - 5)$$

Find equation of line:

$$y = \frac{1}{9}x - 17\frac{2}{9} \text{ or } .9y = x - 155$$

M2 + A1

- f. points of intersection on calculator $(5, -16.67), (16.69, -15.37)$

$$d = \sqrt{1.3^2 + 11.69^2}$$

distance between .. = $\sqrt{138.35}$

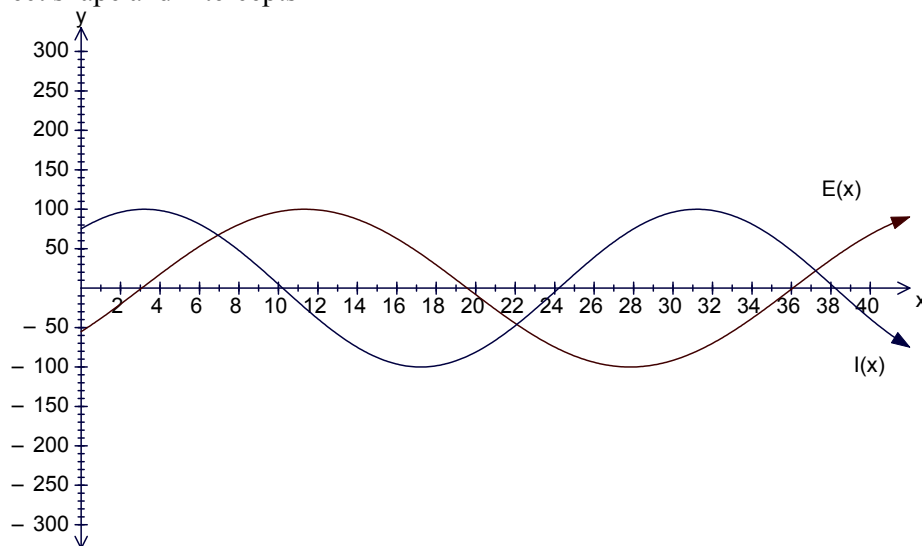
$$\dots = 11.76\text{km}$$

M1 + A1

Total 15 marks

Question 2

- a. Correct shape and intercepts

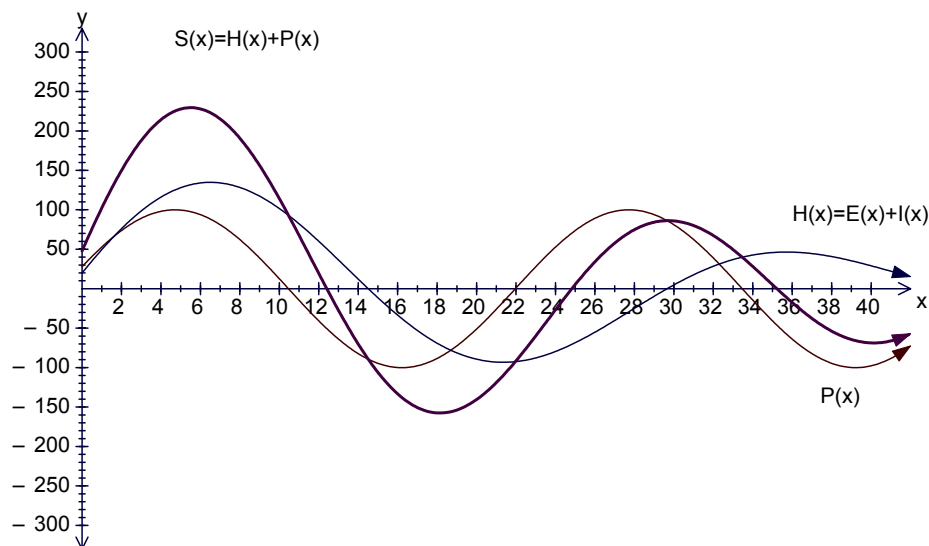


A2

- b. Critical days $I(x) = 0, x = 3, 19, 36$ therefore 3rd Oct, 19th Oct
 High days $I(x) = 100, x = 11$ therefore 11th October
 Low days $I(x) = -100, x = 27$ therefore 27th October
 May be slight discrepancies if values read from graphs

A3

- c. Correct shape, show important points, $x = 0$, points of intersection



A2

- d. Best Date maximum of $S(x)$ 5th October, Worst Date minimum of $S(x)$ 18th October
Again, may vary if read from graphs

A2

- e. Find point of intersection between $y = S(x)$ and $y = 105$ giving (1.051, 105) and (10.205, 105) therefore best dates are between 1st and 10th of October.

M1 + A1

Total 11 marks

Question 3

$$9.2500 = Ae^0 \therefore A = 9.2500$$

$$5.9724 = 9.2500e^{-9k}$$

a. $0.64566 = e^{-9k}$

$$-0.4375 = -9k$$

$$\therefore k = 0.0486 \text{ \& } A = 9.2500$$

M2 + A1

$$R = 9.2500e^{-0.0486 \frac{1}{24}}$$

b. $\dots = 9.2313$

$$\therefore \text{amount..decayed} = 9.2500 - 9.2313 = 0.0187g$$

M1 + A1

$$4.625 = 9.2500e^{-0.0486t}$$

c. $0.5 = e^{-0.0486t}$

$$-0.6931 = -0.0486t$$

$$t = 14.2622 \therefore 14 \text{ days..} 6 \text{ hours}$$

M1 + A1

d. $D_0 = 5.2700g$

$$D = 5.2700e^{-0.0309t}$$

Find point of intersection of $y = D(t)$ and $y = R(t)$, giving (31.7849, 1.9736). Therefore,...

- i. After 31 days 19 hours
- ii. 1.9736g of both Raybon and Decabon

M1 + A1
Total 9 marks

Question 4

a. $\Pr(NNNWW) = \frac{7}{8} \times \frac{7}{8} \times \frac{7}{8} \times \frac{1}{8} \times \frac{1}{8} = 0.0105$

M1 + A1

b. $C \sim \text{Bi}(50, \frac{1}{8})$

$$\Pr(X \geq 1) = 1 - \Pr(X < 0)$$

$$= 1 - \text{binompdf}(50, \frac{1}{8}, 0)$$

$$= 0.9987$$

M1 + A1

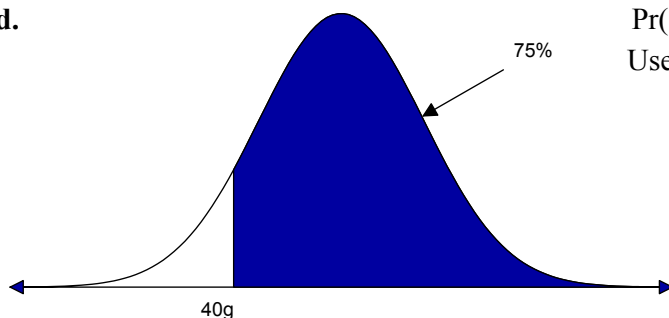
$$\Pr(W \leq 10 | W \geq 1) = \frac{\Pr(1 \leq W \leq 10)}{\Pr(W \geq 1)}$$

c. $\dots = \text{binomcdf}(50, \frac{1}{8}, 10) - \text{binompdf}(50, \frac{1}{8}, 0)$

$$\dots = 0.9579$$

M2 + A1

d.



$$\Pr(C < 40) = 0.25$$

Use standard normal $\text{Invnorm}(0.25, 0, 1)$

$$z = -0.6745$$

$$-0.6745 = \frac{40 - \mu}{0.15}$$

$$\mu = 40 + 0.1011$$

$$\therefore \mu = 40.10g$$

M2 + M1

e. $\Pr(C < 40) = \text{normalcdf}(-1\text{EXP}(99), 40, 40.1, 0.15)$

$$\dots = 0.2525$$

M1 + A1
Total 12 marks

Question 5**a.**

$$\int_0^2 (k(x^2 + 1))dx = 1$$

$$k \left[\frac{x^3}{3} + x \right]_0^2 = 1$$

$$k \left[\frac{8}{3} + 2 \right] = 1$$

$$k \left(\frac{14}{3} \right) = 1$$

$$k = \frac{3}{14}$$

M1 + A1

b.

$$\int_0^m \left(\frac{3}{14} (x^2 + 1) \right) dx = 0.5$$

$$\frac{m^3}{3} + m = 2 \frac{1}{3}$$

Graph $y = 2\frac{1}{3}$ and $y = \frac{m^3}{3} + m$ find the points of intersection (1.4063, 2.3333). Therefore, the median is 1.4063

M1 + A1

c. mode is max of graph at $x = 2$

A1

$$\text{d. } E(X) = \int_0^2 \left(\frac{3}{14} x(x^2 + 1) \right) dx$$

$$\dots\dots\dots = 1.2857$$

M1 + A1

$$\text{e. } \Pr(X < 1.2857) = \int_0^{1.2857} \left(\frac{3}{14} (x^2 + 1) \right) dx$$

$$= 0.4273$$

A1

$$\text{f. } \text{var}(\mathbf{X}) = \int_0^2 \left(\frac{3}{14} x^2 (x^2 + 1) \right) dx - (1.2857)^2$$

$$= 1.9429 - 1.6530$$

$$= 0.2899$$

M1

A1

$$\text{g. } \sigma_x = \sqrt{\text{var}(\mathbf{X})} = \sqrt{0.2899} \approx 0.5384$$

A1

Total 11 marks