



THE SCHOOL FOR EXCELLENCE
UNIT 3 & 4 MATHEMATICAL METHODS 2008
COMPLIMENTARY WRITTEN EXAMINATION 1 - SOLUTIONS

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QUESTION 1

Construct a Karnaugh Table:

	A	A'
B	a	b
B'	c	d
	$\frac{1}{2}$	$\frac{1}{2}$

	A	A'	
B	$\Pr(A \cap B)$	$\Pr(A' \cap B)$	$\Pr(B)$
B'	$\Pr(A \cap B')$	$\Pr(A' \cap B')$	$\Pr(B')$
	$\Pr(A)$	$\Pr(A')$	

Required to find: $\Pr(A|B') = \frac{\Pr(A \cap B')}{\Pr(B')} = \frac{c}{c+d} \dots (1)$

$$\Pr(A' | B) = \frac{1}{3}$$

$$\frac{\Pr(A' \cap B)}{\Pr(B)} = \frac{b}{a+b} = \frac{1}{3}$$

$$\therefore 2b = a \dots (2)$$

$$\Pr(A \cup B) = \frac{3}{5}$$

$$1 - d = \frac{3}{5}$$

$$\therefore d = \frac{2}{5}$$

From the Karnaugh Table: $d = \frac{2}{5}$

$$b + \frac{2}{5} = \frac{1}{2}$$

$$\therefore b = \frac{1}{10}$$

Substitute $b = \frac{1}{10}$ into (2): $a = \frac{1}{5}$

From the Karnaugh Table: $a = \frac{1}{5}$
 $\frac{1}{5} + c = \frac{1}{2}$
 $\therefore c = \frac{3}{10}$

Substitute $d = \frac{2}{5}$ and $c = \frac{3}{10}$ into (1):

$$\Pr(A|B') = \frac{\frac{3}{10}}{\frac{3}{10} + \frac{2}{5}} = \frac{3}{3+4} = \frac{3}{7}$$

4 marks

QUESTION 2

(a) $x(x^3 - 2x - 1) = 0$

$$\Rightarrow x(x+1)(x^2 - x - 1) = 0$$

$$x = 0 \quad \text{or} \quad (x+1) = 0 \quad \text{or} \quad (x^2 - x - 1) = 0$$

$$\therefore x = 0 \quad \therefore x = -1 \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\text{Therefore: } x = 0, -1, \frac{1 \pm \sqrt{5}}{2}.$$

2 marks

(b) $f(x) = x^4 - 2x^2 - x$

$$f'(x) = 4x^3 - 4x - 1$$

$$f'(1) = 4(1)^3 - 4(1) - 1 = -1$$

$$f'(-1) = 4(-1)^3 - 4(-1) - 1 = -1$$

2 marks

- (c) If the tangent is common to both points then the gradient to the curve has the same value at both points.

Part (b) suggests that the x-coordinates of the two points might be $x = 1$ and $x = -1$.

When $x = 1$: $y = f(1) = (1)^4 - 2(1)^2 - 1 = -2$

Equation of tangent at $(1, -2)$: $y - y_1 = m(x - x_1)$

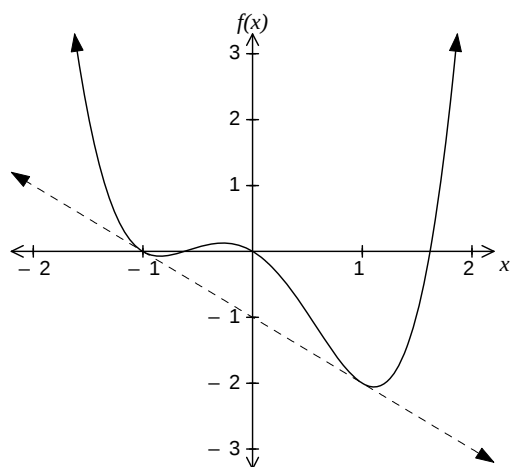
$$\Rightarrow y - (-2) = -1(x - 1)$$

$$\Rightarrow y = -x - 1$$

When $x = -1$: $y = f(-1) = (-1)^4 - 2(-1)^2 - (-1) = 0$

Equation of tangent at $(-1, 0)$: $y - y_1 = m(x - x_1)$

$$\Rightarrow y = -1(x - [-1]) = -x - 1$$



2 marks

QUESTION 3

$$\begin{aligned}
 \text{(a) } \log_b \left(\frac{5\sqrt{3}}{2} \right) &= \log_b(5\sqrt{3}) - \log_b(2) \\
 &= \log_b(5) + \log_b(\sqrt{3}) - \log_b(2) \\
 &= \log_b(5) + \log_b(3^{1/2}) - \log_b(2) \\
 &= \log_b(5) + \frac{1}{2}\log_b(3) - \log_b(2) \\
 &= r + \frac{1}{2}q - p
 \end{aligned}$$

2 marks

(b) The $(r+1)^{th}$ term in the expansion of $(A+B)^n$ is $T_{r+1} = \binom{n}{r} A^{n-r} B^r$.

$$A = 3x, \quad B = -\frac{4}{x^2} = -4x^{-2} \quad \text{and} \quad n = 14.$$

$$T_{r+1} = \binom{14}{r} (3x)^{14-r} (-4x^{-2})^r = \binom{14}{r} (3)^{14-r} (-4)^r x^{14-r} x^{-2r} = \binom{14}{r} (3)^{14-r} (-4)^r x^{14-3r}$$

Since the coefficient of x^2 is required:

$$\begin{aligned} x^{14-3r} &= x^2 \\ 14-3r &= 2 \\ \Rightarrow r &= 4 \end{aligned}$$

Substitute $r = 4$ into $\binom{14}{r} (3)^{14-r} (-4)^r$ to get the coefficient of x^2 :

$$\begin{aligned} \binom{14}{4} (3)^{14-4} (-4)^4 &= \binom{14}{4} (3)^{10} (-4)^4 \\ &= \binom{14}{4} (3^2)^5 ([-4]^2)^2 = \binom{14}{4} (9)^5 (16)^2 \end{aligned}$$

2 marks

QUESTION 4

Multiply both sides of the equation by 2^x : $(2^x)^2 - k = 2^x$
 $\Rightarrow (2^x)^2 - 2^x - k = 0$

Substitute $w = 2^x$: $w^2 - w - k = 0$

Solve for w using the quadratic formula: $w = 2^x = \frac{1 \pm \sqrt{1+4k}}{2}$

Case 1: $1+4k = 0$

$$\Rightarrow k = -\frac{1}{4}$$

Case 2: No real solution if $2^x < 0$.

$$\begin{aligned} \Rightarrow 1 - \sqrt{1+4k} &< 0 \\ \Rightarrow -\sqrt{1+4k} &< -1 \\ \Rightarrow \sqrt{1+4k} &> 1 \\ \Rightarrow 1+4k &> 1 \\ \Rightarrow k &> 0 \end{aligned}$$

Case 3: $k = 0$

4 marks

QUESTION 5

(a) Let $y = x \log_e x$ and use the product rule: $\frac{dy}{dx} = \log_e(x) + 1$.

1 mark

$$(b) \quad I = \int_a^2 |\log_e x| dx = \int_a^1 -\log_e(x) dx + \int_1^2 \log_e(x) dx = -\int_a^1 \log_e(x) dx + \int_1^2 \log_e(x) dx$$

Integration by recognition: $\frac{dy}{dx} = \log_e(x) + 1$

$$\Rightarrow y = \int \log_e(x) + 1 dx$$

$$\Rightarrow x \log_e(x) = \int \log_e(x) dx + \int 1 dx$$

$$\Rightarrow \int \log_e(x) dx = x \log_e(x) - x$$

$$\text{Therefore: } I = -\left[x \log_e(x) - x\right]_a^1 + \left[x \log_e(x) - x\right]_1^2 = a \log_e(a) + 2 \log_e(2) - a$$

Compare $a \log_e(a) + 2 \log_e(2) - a$ with $\log_e(2\sqrt{2}) - \frac{1}{2}$:

$$\log_e(a)^a + \log_e(2)^2 - a = \log_e(2\sqrt{2}) - \frac{1}{2}$$

$$\log_e(a)^a (2)^2 - a = \log_e(2\sqrt{2}) - \frac{1}{2}$$

$$\therefore a = \frac{1}{2}$$

$$\text{Check: } \frac{1}{2} \log_e\left(\frac{1}{2}\right) + 2 \log_e(2) = -\frac{1}{2} \log_e(2) + \log_e(2^2) = \log_e(2^{-1/2}) + \log_e(2^2)$$

$$= \log_e(2^2 \times 2^{-1/2}) = \log_e(2^{3/2}) = \log_e(2 \times 2^{1/2}) = \log_e(2\sqrt{2}), \text{ as required.}$$

4 marks

QUESTION 6

$$\begin{aligned}
& \cos A \sin\left(\frac{\pi}{2} - A\right) + \cos\left(\frac{3\pi}{2} - A\right) \sin A \\
&= \cos A \cos A + (-\sin A) \sin A \\
&= \cos^2 A - \sin^2 A \\
&= \cos^2 A - (1 - \cos^2 A) \\
&= \cos^2 A - 1 + \cos^2 A \\
&= 2\cos^2 A - 1 \\
&= a \cos^2 A + B
\end{aligned}$$

Where $a = 2$ and $b = -1$.

3 marks

QUESTION 7

Solve $t = \frac{2y-6}{2y^2-y+3}$ for y where $y = f^{-1}(t)$:

$$t = \frac{2y-6}{2y^2-y+3}$$

$$\Rightarrow 2ty^2 - ty + 3t = 2y - 6$$

$$\Rightarrow 2ty^2 - (t+2)y + 3t + 6 = 0$$

Solve for y using the quadratic formula:

$$y = \frac{t+2 \pm \sqrt{(t+2)^2 - 4(2t)(3t+6)}}{4t} = \frac{t+2 \pm \sqrt{-23t^2 - 44t + 4}}{4t}.$$

Decide which of the solutions to reject:

$\left(-1, -\frac{4}{3}\right)$ is a simple point on $y = f(t)$ therefore $\left(-\frac{4}{3}, -1\right)$ must satisfy $y = f^{-1}(t)$:

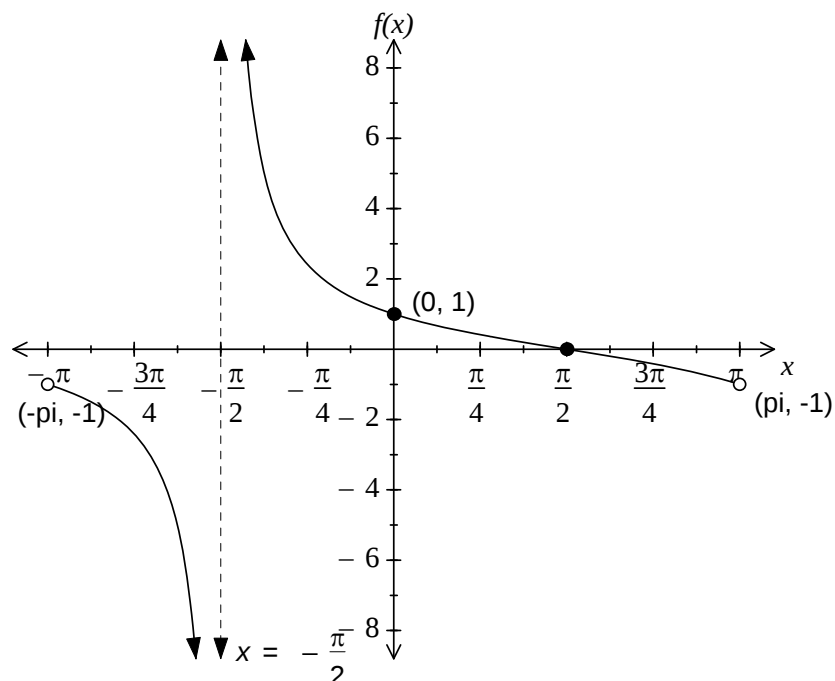
$$-1 = \frac{-\frac{4}{3} + 2 \pm \sqrt{(-23(-\frac{4}{3})^2 - 44(-\frac{4}{3}) + 4)}}{4(-\frac{4}{3})} = \frac{\frac{2}{3} \pm \frac{14}{3}}{-\frac{16}{3}} = \frac{2 \pm 14}{-16}$$

Therefore the negative root solution is rejected and $y = \frac{t+2 + \sqrt{-23t^2 - 44t + 4}}{4t}$.

3 marks

QUESTION 8

Note that $f(x) = \tan\left(\frac{\pi}{4} - \frac{x}{2}\right) = \tan\left(-\left[\frac{x}{2} - \frac{\pi}{4}\right]\right) = -\tan\left(\frac{x}{2} - \frac{\pi}{4}\right)$ which means that the reflection in the y-axis can be treated as a reflection in the x-axis.



3 marks

QUESTION 9

Find r when $\frac{dV}{dt} = 75\pi \text{ cm}^3 / \text{min}$

Chain rule: $\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$

Substitute $V = \frac{4}{3}\pi r^3$: $\Rightarrow \frac{dV}{dr} = 4\pi r^2$

$$\frac{dV}{dt} = 4\pi r^2 \times \frac{dr}{dt}$$

Substitute $\frac{dr}{dt} = 0.75$ and $\frac{dV}{dt} = 75\pi$:

$$75\pi = 4\pi r^2 \times 0.75$$

$$\Rightarrow 4r^2 = \frac{75}{0.75} = 100$$

$$\Rightarrow r^2 = 25$$

$$\Rightarrow r = 5 \text{ cm}$$

3 marks

QUESTION 10

(a) $y = \frac{\cos x}{e^{2x}}$

Apply Quotient Rule:

$$\begin{aligned}\frac{dy}{dx} &= \frac{(e^{2x} \times -\sin x) - (\cos x \times 2e^{2x})}{(e^{2x})^2} = \frac{-e^{2x} \sin x - 2 \cos x e^{2x}}{(e^{2x})^2} \\ &= \frac{-e^{2x}(\sin x + 2 \cos x)}{(e^{2x})^2} = \frac{-(\sin x + 2 \cos x)}{e^{2x}}\end{aligned}$$

2 marks

(b) $f(x+h) \approx f(x) + hf'(x)$

Let $x=1$

Let $(x+h) = 0.9$

$\therefore h = -0.1$

$f(0.9) \approx f(1) - 0.1f'(1)$

$$\begin{aligned}&\approx \frac{\cos(1)}{e^2} - 0.1 \left[\frac{2 \cos(1) - \sin(1)}{e^2} \right] \\ &\approx \frac{\cos(1) - 0.2 \cos(1) - 0.2 \sin(1)}{e^2} \\ &\approx \frac{0.8 \cos(1) - 0.2 \sin(1)}{e^2}\end{aligned}$$

$$f(1) = \frac{\cos(1)}{e^{2(1)}} = \frac{\cos(1)}{e^2} \quad \text{Note: } \cos(1) \neq 0$$

$$f'(1) = -\frac{(\sin 1 - 2 \cos 1)}{e^{2(1)}} = \frac{2 \cos(1) - \sin(1)}{e^2}$$

Change in f is given by: $f_{\text{final}} - f_{\text{initial}} = f(0.9) - f(1)$

$$\Delta f \approx \frac{0.8 \cos(1) - 0.2 \sin(1)}{e^2} - \frac{\cos(1)}{e^2}$$

$$\Delta f \approx \frac{0.2 \cos(1) - 0.2 \sin(1)}{e^2}$$

3 marks