

The Mathematical Association of Victoria
MATHEMATICAL METHODS (CAS)
2008 Trial written examination 2 – Solutions – Multiple choice

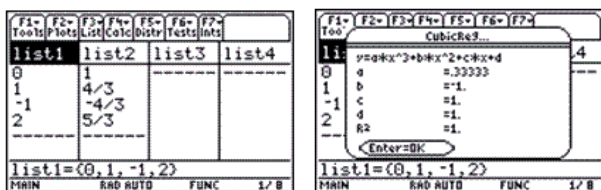
SECTION 1
Solutions

1. D 2. E 3. A 4. C 5. A 6. B
 7. E 8. C 9. E 10. B 11. B 12. A
 13. D 14. E 15. B 16. B 17. A 18. C
 19. A 20. B 21. B 22. C

Question 1

Answer D

Enter data into the calculator and choose cubic regression.



The exact answer is required.

$$y = \frac{1}{3}x^3 - x^2 + x + 1$$

$$3y = x^3 - 3x^2 + 3x + 3$$

Question 2

Answer E

$h(x)$ has factors x , $x(x - b)$ and $(x - c)$.

The coefficient of x^4 is negative.

Since $a > 0$, the rule is $h(x) = -ax^2(x - b)(x - c)$.

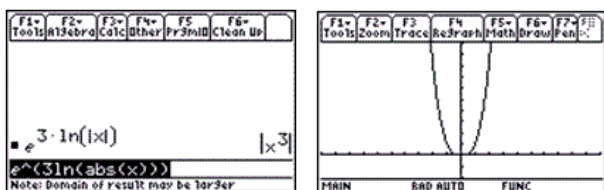
Question 3

Answer A

$$f(g(x)) = e^{3\log_e|x|} = e^{\log_e|x|^3} = |x|^3$$

The domain of $f(g(x))$ is the same as the domain of $g(x) : \mathbb{R} \setminus \{0\}$.

The range is $\mathbb{R}^+$.



Question 4**Answer C**

$f: (-\infty, 3] \rightarrow \mathbb{R}$ where $f(x) = (x-3)^{\frac{2}{3}} + 1$

Let $y = (x-3)^{\frac{2}{3}} + 1$

Inverse swap x and y

$$x = (y-3)^{\frac{2}{3}} + 1$$

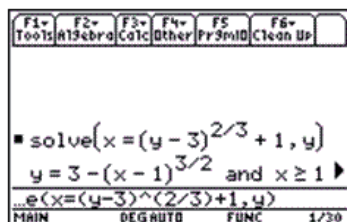
$$(y-3)^{\frac{2}{3}} = x-1$$

$y-3 = -(x-1)^{\frac{3}{2}}$, take the negative square root

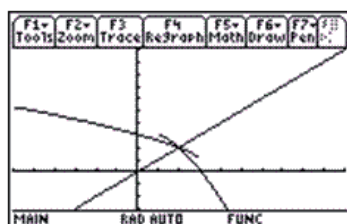
$$y = 3 - (x-1)^{\frac{3}{2}}$$

OR

Solve $x = (y-3)^{\frac{2}{3}} + 1$ for y on the CAS



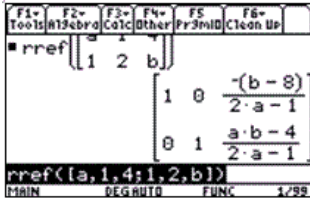
$f: [1, \infty) \rightarrow \mathbb{R}$, where $f^{-1}(x) = 3 - (x-1)^{\frac{3}{2}}$



Question 5**Answer A**

Solve the matrix equation to obtain the result

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{8-b}{2a-1} \\ \frac{ab-4}{2a-1} \end{bmatrix}$$

ORDeterminant = $2a - 1 \neq 0$ Hence there will be a unique solution for $a \in \mathbb{R} \setminus \left\{ \frac{1}{2} \right\}$ and $b \in \mathbb{R}$ 

Note also that if:

 $a = \frac{1}{2}$ and $b = 8$, there are infinite solutions (the solutions are identical) $a = \frac{1}{2}$ and $b \in \mathbb{R} \setminus \{8\}$, there is no solution (the solutions are inconsistent)**Question 6****Answer B**

$$-h(2x - 4) = -h(2(x - 2))$$

The transformation from $h(x)$ to $-h(2(x - 2))$ involvesA reflection in the x -axis, a dilation by a scale factor of $\frac{1}{2}$ from the y -axis and a translation 2 units right.

Question 7**Answer E**

The rule will be of the form $f(x) = a \sin(n(x - \epsilon)) + k$.

Amplitude = 3, therefore $a = 3$.

Translation down 2 units, therefore $k = -2$.

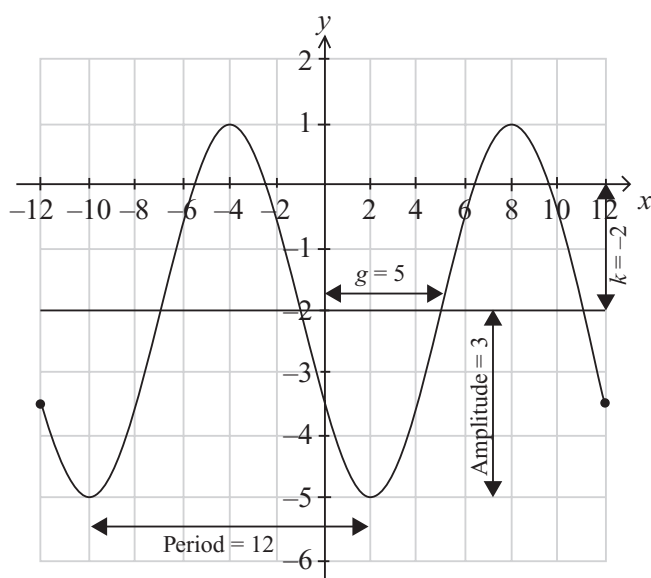
Phase shift, $\epsilon = 5$.

$$\text{Period} = \frac{2\pi}{n}$$

$$12 = \frac{2\pi}{n}$$

$$n = \frac{2\pi}{12}$$

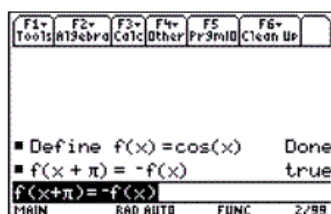
$$n = \frac{\pi}{6}$$

**Question 8****Answer C**

$\cos(x + \pi) = -\cos(x)$ (this can be seen from a unit circle or the graph of $y = \cos(x)$).

Hence $f(x + \pi) = -f(x)$

This can be verified using a CAS.



Question 9**Answer E**

Let x' and y' be the images of x and y , respectively, under T . The transformation of $y = e^x$ to $y = e^{-2(x+1)}$ involves:

$$x = -2(x' + 1) \text{ and } y = y'$$

$$x' + 1 = -\frac{1}{2}x$$

$$x' = -\frac{1}{2}x - 1 \text{ and } y' = y$$

Question 10**Answer B**

$$e^{2x} + be^x + 1 = 0$$

$$\text{Let } e^x = p$$

$$p^2 + bp + 1 = 0$$

$$\Delta < 0 \text{ for no real solutions}$$

$$b^2 - 4 = 0$$

$$\text{Hence } \{b : -2 < b < 2\}$$

Also there is no solution at $b = 2$ as

$$e^{2x} + 2e^x + 1 = (e^x + 1)^2 = 0$$

$$e^x \neq -1$$

$$\text{Thus } \{b : -2 < b \leq 2\}$$

Question 11**Answer B**

$$f(x) = \begin{cases} x^2 + 4x + a & \text{for } x \geq 0 \\ bx + c & \text{for } x < 0 \end{cases}$$

For $f(x)$ to be differentiable at $x = 0$, $f(x)$ has to be continuous at $x = 0$.

$$x^2 + 4x + a = bx + c$$

$$\text{Hence at } x = 0, a = c$$

$$\text{Also } \lim_{x \rightarrow 0^-} f'(x) = \lim_{x \rightarrow 0^+} f'(x)$$

$$2x + 4 = b$$

$$\text{At } x = 0 \quad b = 4$$

Question 12**Answer A**

$$f(x+h) \approx hf'(x) + f(x)$$

$$f(x+h) - f(x) \approx hf'(x)$$

$$\approx -0.1 \times \frac{2}{2x+1}$$

$$\approx -0.1 \times \frac{2}{3} = -\frac{1}{15}$$

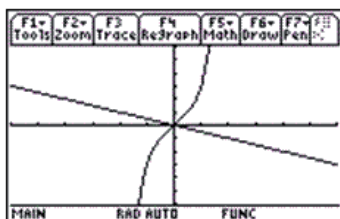
Question 13**Answer D**

$$y = x^5 + 2x$$

$$\frac{dy}{dx} = 5x^4 + 2 = 2 \text{ at } x = 0.$$

The equation of the normal is

$$y = -\frac{1}{2}x.$$

**Question 14****Answer E**

$$\frac{x^2}{y} = 1, y = x^2$$

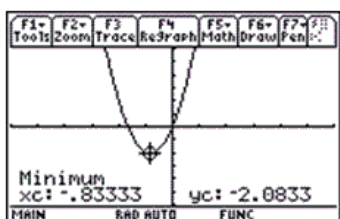
Substitute $y = x^2$ into p

$$p = 5x + 3y = 5x + 3x^2$$

$$\frac{dp}{dx} = 5 + 6x = 0 \text{ or } x = -\frac{b}{2a} = -\frac{5}{6}$$

$$x = -\frac{5}{6}$$

p has a minimum value when $x = -\frac{5}{6}$



Question 15**Answer B**

$$x = \sqrt{1-t}$$

$$v = \frac{dx}{dt} = -\frac{1}{2\sqrt{1-t}}$$

$$a = \frac{dv}{dt} = -\frac{1}{4(1-t)^{\frac{3}{2}}}$$

$$a = -\frac{1}{4x^3}$$

TI-89 calculator screen showing the derivation of $a = -\frac{1}{4x^3}$ from $x = \sqrt{1-t}$. The screen displays the expression $\frac{d(\sqrt{1-t})}{dt}$ and the result $-\frac{1}{4 \cdot x^3}$.

Question 16**Answer B**

$$\text{average value} = \frac{1}{b-a} \int_a^b g(x) dx$$

Apply the formula to this case

$$8 = \frac{1}{a-0} \int_0^a (x^2 - 4) dx$$

Solving for a gives the result

$$a = 6$$

Reject the negative solution because the domain of g requires that $a > 0$.

TI-89 calculator screen showing the definition of $g(x) = x^2 - 4$ and the solution of the equation $\frac{1}{a} \int_0^a g(x) dx = 8$. The screen displays the result $a = -6 \text{ or } a = 6$.

OR

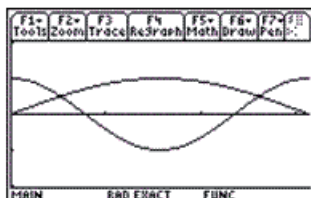
TI-89 calculator screen showing the definition of $g(x) = x^2 - 4$ and the calculation of the average value $\frac{1}{6} \int_0^6 g(x) dx$. The screen displays the result 8 .

Question 17**Answer A**

$y = \sin(x)$ is the top curve and $y = \cos(2x)$ is the bottom curve.

$$\sin\left(\frac{\pi}{6}\right) = \cos\left(\frac{\pi}{3}\right) \text{ and } \sin\left(\frac{5\pi}{6}\right) = \cos\left(\frac{5\pi}{3}\right)$$

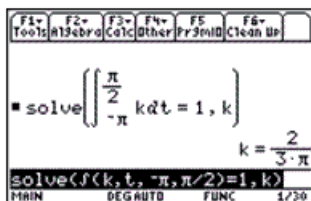
$$\text{Area} = \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (\sin(x) - \cos(2x)) dx$$

**Question 18****Answer C**

For a probability density function, $\int_{-\infty}^{\infty} f(t) dt = 1$.

Solve $\int_{-\pi}^{\frac{\pi}{2}} (k) dt = 1$ for k using CAS

$$k = \frac{2}{3\pi}$$

**OR**

$$0 + \int_{-\pi}^{\pi/2} k dt = 1$$

$$[kt]_{-\pi}^{\pi/2} = 1$$

$$\left[\frac{\pi}{2}k - (-\pi k) \right] = 1$$

$$\frac{3\pi}{2}k = 1$$

$$k = \frac{2}{3\pi}$$

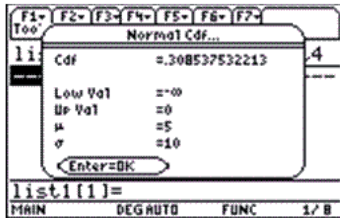
Question 19**Answer A**

Let X be the percentage increase in the value of stocks.

$$X \sim N(\mu = 5, \sigma^2 = 10^2)$$

Stocks decrease in value if $X < 0$.

$$\Pr(X < 0) \approx 0.31$$

**Question 20****Answer B**

This is a problem of selection without replacement. Let S denote dialling Stewart's phone number.

$$\Pr(S', S', S) = \frac{7}{8} \times \frac{6}{7} \times \frac{1}{6}$$

Note from this pattern that the probability that the first, second, third, ..., eighth number dialled is Stewart's number, is $\frac{1}{8}$ each time.

Question 21**Answer B**

$$\sum p(x) = 1$$

$$h + h^2 + \frac{1}{2} - h = 1$$

$$h^2 = \frac{1}{2}$$

$$h = \frac{1}{\sqrt{2}} \text{ (Reject negative solution as } h \in [0,1])$$

$$\begin{aligned} \mu &= \sum x p(x) \\ &= 0 + \left(1 \times \left(\frac{1}{\sqrt{2}} \right)^2 \right) + \left(\sqrt{2} \times \left(\frac{1}{2} - \frac{1}{\sqrt{2}} \right) \right) \\ &= \frac{1}{2} + \frac{\sqrt{2}}{2} - 1 \\ &= \frac{\sqrt{2}}{2} - \frac{1}{2} \\ &= \frac{\sqrt{2} - 1}{2} \end{aligned}$$

OR

Use CAS

$\text{solve}(h + h^2 + 1/2 - h = 1, h)$
 $h = -\frac{\sqrt{2}}{2} \text{ or } h = \frac{\sqrt{2}}{2}$
 $h^2 + \sqrt{2} \cdot (1/2 - h) | h = \frac{\sqrt{2}}{2}$
 $\frac{\sqrt{2}-1}{2}$

$h = -\frac{\sqrt{2}}{2} \text{ or } h = \frac{\sqrt{2}}{2}$
 $h^2 + \sqrt{2} \cdot (1/2 - h) | h = \frac{\sqrt{2}}{2}$
 $\frac{\sqrt{2}-1}{2}$
 $h^2 + f(2) \cdot (1/2 - h) | h = f(2)/2$

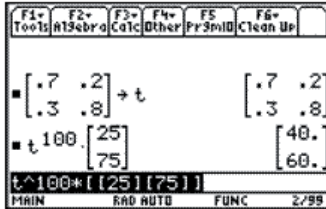
Question 22

Answer C

Method 1

Consider, say, the 100th state.

$$S_{100} = T^{100} S_0 = \begin{bmatrix} 40 \\ 60 \end{bmatrix}$$



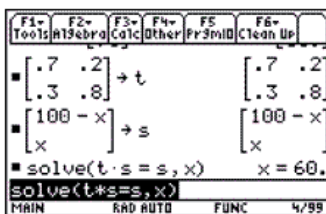
Method 2

The cardinal number of elements in this case is 100. The steady state matrix will therefore be of

the form $\begin{bmatrix} 100-x \\ x \end{bmatrix}$. When the steady-state has

been reached, $T \times \begin{bmatrix} 100-x \\ x \end{bmatrix} = \begin{bmatrix} 100-x \\ x \end{bmatrix}$.

Solving the matrix equation for x gives the result $x = 60$, $100 - x = 40$



Method 3

If Method 2 above is applied to a general

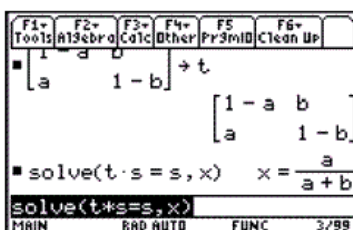
transition matrix, $T = \begin{bmatrix} 1-a & b \\ a & 1-b \end{bmatrix}$ and the steady state matrix is written as the proportions

$\begin{bmatrix} 1-x \\ x \end{bmatrix}$, the solution is

$$x = \frac{a}{a+b} = \frac{0.3}{0.3+0.2} = 0.6.$$

As a **proportion**, the steady-state matrix is $\begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix}$. However, there are 100 elements.

Using the cardinal numbers, the steady-state matrix is $\begin{bmatrix} 40 \\ 60 \end{bmatrix}$.



The Mathematical Association of Victoria
MATHEMATICAL METHODS (CAS)
2008 Trial written examination 2: Solutions – Extended Answer Questions

SECTION 2**Question 1**

a. Solve $h^2 + \left(\frac{b}{2}\right)^2 = b^2$ for h

$$h = \frac{\sqrt{3}b}{2}$$

[1M]

SA = area of the 2 triangles + 3 rectangles

$$bh + 3lb = 120$$

$$\frac{\sqrt{3}b^2}{2} + 3lb = 120$$

[1M]

$$l = \frac{240 - \sqrt{3}b^2}{6b}, 3 \leq b \leq 9$$

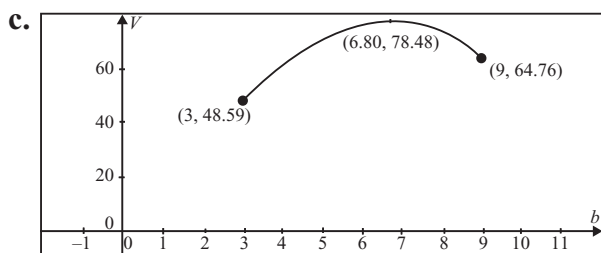
b. $V = \text{area of the triangle} \times \text{length}$

[1M]

$$= \frac{b}{2} \times \frac{\sqrt{3}b}{2} \times \frac{240 - \sqrt{3}b^2}{6b}$$

[1M]

$$= \frac{b(80\sqrt{3} - b^2)}{8}, 3 \leq b \leq 9 \text{ as required}$$



Correct shape [1A]

Correct turning point (6.80, 78.48) [1A]

Correct end points with closed circles (3, 48.59) and (9, 64.76) [1A]

- d. The maximum occurs at the turning point not an end point.

$$\frac{dV}{db} = 10\sqrt{3} - \frac{3b^2}{8}$$

$$\text{Solve } 10\sqrt{3} - \frac{3b^2}{8} = 0 \text{ for } b$$

[1M]

$$b = \frac{4\sqrt{5}}{\sqrt[4]{3}} \text{ as req'd}$$

[1A]

$$h = \frac{\sqrt{3}}{2}b = 2\sqrt{5}\sqrt[4]{3} \text{ cm}$$

[1A]

$$l = \frac{240 - \sqrt{3}b^2}{6b} = \frac{4\sqrt{5}}{3^{3/4}} \text{ cm}$$

[1A]

The first screenshot shows the equation $\frac{d}{db} \left(\frac{-b(b^2 - 80\sqrt{3})}{8} \right)$ and the solution $b = \frac{-4\sqrt{5} \cdot 3^{3/4}}{3}$ or $b = \frac{4\sqrt{5}}{3^{1/4}}$. The second screenshot shows the calculation of $h = \frac{\sqrt{3}}{2} \cdot \frac{4\sqrt{5} \cdot 3^{3/4}}{3} = 2\sqrt{5} \cdot 3^{1/4}$. The third screenshot shows the calculation of $l = \frac{-b(b^2 - 80\sqrt{3})}{6b} = \frac{4\sqrt{5}}{3^{3/4}}$.

e. $V = \frac{b(80\sqrt{3} - b^2)}{8} = \frac{80\sqrt{5}}{3^{3/4}} \text{ cm}^3$

[1A]

The screenshot shows the calculation of $V = \frac{b(80\sqrt{3} - b^2)}{8} = \frac{80\sqrt{5}}{3^{3/4}}$.

f. Average value = $\frac{1}{9-3} \int_3^9 \left(\frac{b(80\sqrt{3} - b^2)}{8} \right) db$

$$= \frac{15(16\sqrt{3} - 9)}{4} \text{ cm}^3$$

[1M]

[1A]

The screenshot shows the calculation of the average value: $\frac{1}{6} \cdot \int_3^9 \left(\frac{-b(b^2 - 80\sqrt{3})}{8} \right) db = \frac{15(16\sqrt{3} - 9)}{4}$.

Question 2

a. $N(t) = 200e^{-mt}$

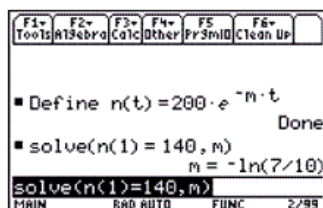
Solve for m , $N(1) = 140$

$$140 = 200 e^{-m} \quad [1M]$$

$$m = -\log_e \left(\frac{7}{10} \right)$$

$$m = \log_e \left(\frac{10}{7} \right), \text{ as required.} \quad [1M]$$

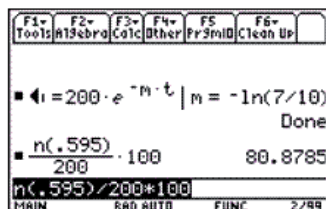
Checking with CAS



b. percentage $= \frac{n(0.595)}{200} \times \frac{100}{1} \quad [1M]$

$$= 100 \times e^{-\log_e \left(\frac{10}{7} \right) \times 0.595}$$

$$= 81\% \quad [1A]$$



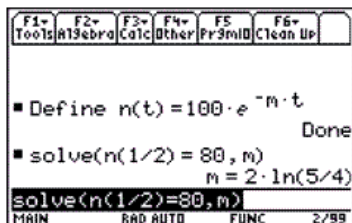
c. $N(t) = 100e^{-mt}$

To find m, solve $N\left(\frac{1}{2}\right) = 80$

$$80 = 100e^{-\frac{1}{2}m}$$

$$m = 2\log_e\left(\frac{5}{4}\right)$$

[1M]



To find when 75% of words are replaced,

$$25 = 100e^{-2\log_e\left(\frac{5}{4}\right) \times t}$$

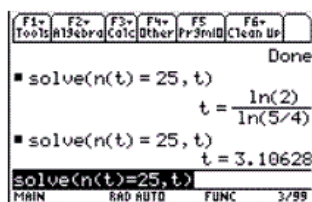
$$t = \frac{\log_e(2)}{\log_e\left(\frac{5}{4}\right)}$$

[1M]

$$t = 3.106$$

It will take 3106 years for 75% replacement.

[1A]



d. To find a , $B(0) = 15$

$$\frac{a}{1 + 3e^0} = 15$$

$$a = 15 \times 4$$

$$B(11) = 22$$

$$\frac{60}{1 + 3e^{-k \times 11}} = 22$$

Solve for k

$$k = -\frac{1}{11} \log_e \left(\frac{19}{33} \right) = \frac{1}{11} \log_e \left(\frac{33}{19} \right)$$

[1M]

[1M]

TI-89 calculator screen showing the definition of $b(t) = \frac{60}{1 + 3 \cdot e^{-k \cdot t}}$ and the solution for k from the equation $b(11) = 22$. The result is $k = \frac{-\ln(19/33)}{11}$.

Solve for t , $B(t) = 40$

$$t \approx 35.70$$

The buzz word count achieved at 36 months.

[1A]

TI-89 calculator screen showing the solution for t from the equation $b(t) = 40$. The result is $t = 35.7009$.

e. For a probability density function,

$$\int_{-\infty}^{\infty} f(x) dx = 1$$

$$\text{Solve for } k, \int_0^{\infty} k e^{-\frac{t}{200}} dt = 1$$

$$k = 0.005, \text{ as required}$$

[1M]

TI-89 calculator screen showing the integration of $k \cdot e^{-\frac{t}{200}}$ from 0 to infinity, resulting in $200 \cdot k$. The screen then shows the solution for k from the equation $200 \cdot k = 1$, which is $k = .005$.

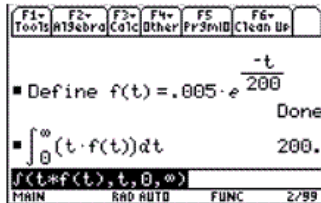
f. $\mu = \int_{-\infty}^{\infty} x f(x) dx$ for a probability density function

$$\mu = 0.005 \int_0^{\infty} t e^{-\frac{t}{200}} dt \quad [1M]$$

$$= 200$$

The mean 200 years.

[1A]

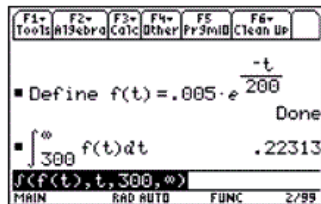


g. $\Pr(T > 300) = 0.005 \int_{300}^{\infty} e^{-\frac{t}{200}} dt$

[1M]

$$\Pr(T > 300) = 0.223$$

[1A]



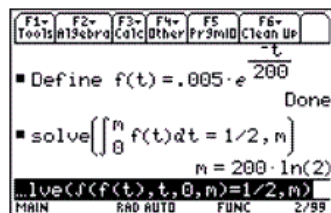
h. To find the median, m , solve

$$0.005 \int_0^m e^{-\frac{t}{200}} dx = \frac{1}{2}$$

[1M]

$$m = 200 \log_e(2)$$

[1A]

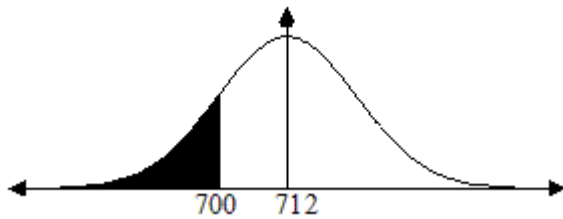


Question 3

a. $X \sim N(\mu = 712, \sigma^2 = 20^2)$

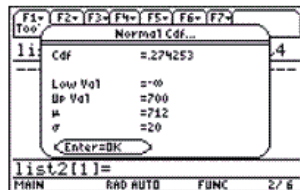
$$\Pr(X < 700) = 0.274$$

[1M]

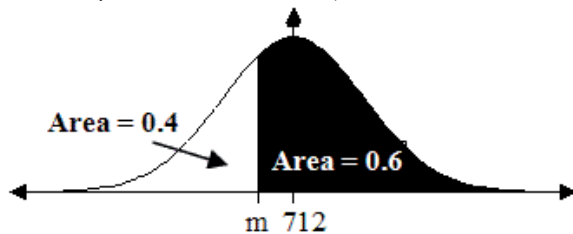


The proportion of cartons less than 700 g is 0.274.

[1A]



b. $X \sim N(\mu = 712, \sigma^2 = 20^2)$



$$\Pr(X > m) = 0.6$$

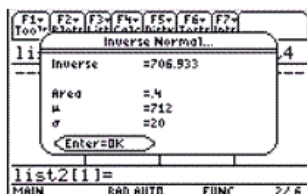
$$\Pr(X \leq m) = 0.4$$

$$m = 706.9$$

60% of cartons exceed 706.9 grams.

[1M]

[1A]



c. $\Pr(X < 695 | X < 700)$

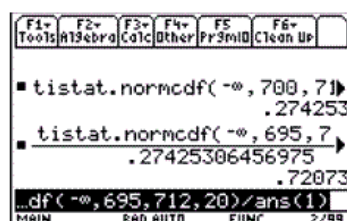
$$= \frac{\Pr(X < 695 \cup X < 700)}{\Pr(X < 700)}$$

$$= \frac{\Pr(X < 695)}{\Pr(X < 700)}$$

$$\Pr(X < 695 | X < 700) = 0.721$$

[1M]

[1A]



d. Let Y be the number of underweight cartons.

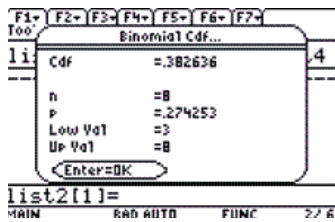
$$Y \sim Bi(8, 0.274)$$

[1M]

(The probability of a carton being underweight was found in part a. to be 0.274)

$$\Pr(Y \geq 3) = 0.383$$

[1A]



e. The transition matrix is as follows:

	Large today	Medium today
Large Tomorrow	0.45	0.75
Medium Tomorrow	0.55	0.25

The probability that Jamie orders medium-sized today and large-sized tomorrow is 0.75.

[1A]

f. i. $\Pr(L,L,M) = 0.75 \times 0.45 \times 0.55$

$$\Pr(L,L,M) = 0.1856$$

[1A]

ii. The n^{th} state is given by $S_n = T^n \times S_0$

$$\text{This Tuesday, } S_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{matrix} \leftarrow \text{Large} \\ \leftarrow \text{Medium} \end{matrix}$$

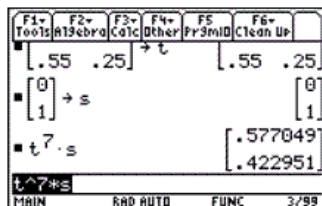
$$\text{Next Tuesday, } S_7 = T^7 \times S_0$$

$$S_7 = \begin{bmatrix} 0.5770 \\ 0.4230 \end{bmatrix}$$

[1M]

The probability of large-sized is 0.5770.

[1A]



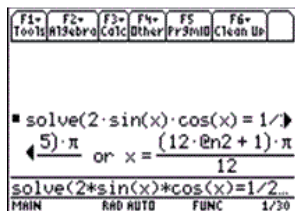
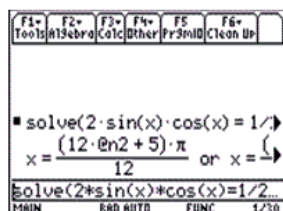
Question 4

a. $2 \sin(x) \cos(x) = \frac{1}{2}$
Using CAS

$$x = \frac{(12n+5)\pi}{12}$$

[1A]

$$x = \frac{(12n+1)\pi}{12} \text{ where } n \in \mathbb{Z}$$

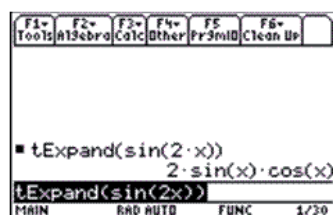
[1A]**OR**

$$2 \sin(x) \cos(x) = \frac{1}{2}$$

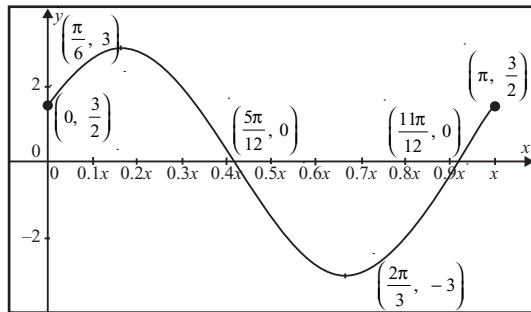
$$\sin(2x) = \frac{1}{2}$$

$$x = \dots, \frac{\pi}{12}, \frac{5\pi}{12}, \dots$$

$$x = \frac{\pi}{12} + n\pi \text{ or } x = \frac{5\pi}{12} + n\pi, \text{ where } n \in \mathbb{Z}$$



- b. i.** A dilation of a factor of 3 from the x -axis
and a translation of $\frac{\pi}{12}$ units to the left.
(The order does not matter.)

[1A]**[1A]****ii.**

Correct shape

[1A]Correct coordinates for the x -intercepts

$$\left(\frac{5\pi}{12}, 0\right) \text{ and } \left(\frac{11\pi}{12}, 0\right)$$

[1A]

Correct end points with closed circles

$$\left(0, \frac{3}{2}\right) \text{ and } \left(\pi, \frac{3}{2}\right)$$

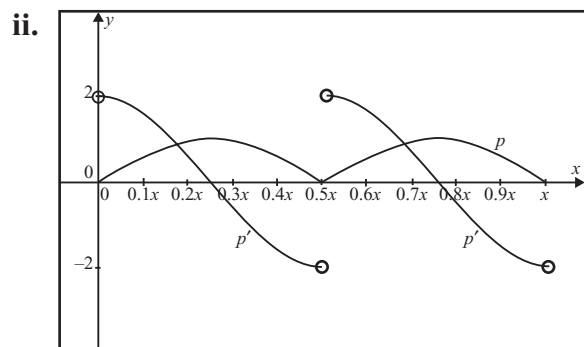
[1A]

Correct turning points

$$\left(\frac{\pi}{6}, 3\right) \text{ and } \left(\frac{2\pi}{3}, -3\right)$$

[1A]

c. i. $p'(x) = \begin{cases} 2 \cos(2x) & \text{if } \sin(2x) \geq 0 \\ -2 \cos(2x) & \text{if } \sin(2x) < 0 \end{cases}$ [2A]



Correct graph for p [1A]
 Correct graph for p' [1A]

x -intercepts of p' : $\frac{\pi}{4}, \frac{3\pi}{4}$

Open circles

[1A]

iii. $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\sin(2x) - 2 \cos(2x)) dx$

[1M]

$= 1.5 \text{ unit}^2$

[1A]