Test 8

Section A: Technology free. 53 marks
Section B: CAS technology assumed. 37 marks
Suggested time: 90 minutes

Section A: Short answer and extended response questions. Technology free.

Specific instructions to students

- Answer **all** questions in the spaces provided.
- A decimal approximation will not be accepted if an **exact** answer is required to a question.
- In questions where more than one mark is available, appropriate working **must** be shown.

QUESTION 1 — Total 7 marks

a Evaluate $\lim_{x \to -1} \frac{2x^2 - 2}{x + 1}$. — 2 marks

$$\lim_{x \to -1} \frac{2(x^2 - 1)}{x + 1}$$
$$= \lim_{x \to -1} \frac{2(x - 1)(x + 1)}{x + 1}$$
$$= \lim_{x \to -1} 2(x - 1),\ x \neq -1$$
$$= -4$$

b The curve with equation $y = -\frac{1}{2}x^2$ has points P and Q, where $x = 2$ and $x = 2 + h$, respectively.

i Find the y values of P and Q. — 2 marks

$$y_P = -\frac{1}{2} \times 2^2 = -2$$
$$y_Q = -\frac{1}{2}(2 + h)^2$$
$$= -\frac{1}{2}(4 + 4h + h^2)$$
$$= -2 - 2h - \frac{1}{2}h^2$$

ii Find the gradient of the line joining P and Q. — 2 marks

$$m_{PQ} = \frac{-2 - 2h - \frac{1}{2}h^2 + 2}{2 + h - 2}$$
$$= \frac{h\left(-2 - \frac{1}{2}h\right)}{h}$$
$$= -2 - \frac{1}{2}h,\ h \neq 0$$

iii Hence, find the gradient at P. — 1 mark

Gradient at $P = \lim_{h \to 0} \left(-2 - \frac{1}{2}h\right) = -2$

QUESTION 2 — 3 marks

Find the derivative of $y = 3x^2 + \frac{2}{x} - \sqrt{x} + 1$.

$$y = 3x^2 + 2x^{-1} - x^{\frac{1}{2}} + 1$$
$$\frac{dy}{dx} = 6x - 2x^{-2} - \frac{1}{2}x^{-\frac{1}{2}}$$
$$= 6x - \frac{2}{x^2} - \frac{1}{2\sqrt{x}}$$

QUESTION 3 — 2 marks

For the curve with equation $f(x) = \frac{x^3}{3} - \frac{3}{2}x^2 - 4x + 2$, find $f'(2)$.

$$f'(x) = x^2 - 3x - 4$$
$$f'(2) = 4 - 6 - 4 = -6$$

QUESTION 4 — Total 4 marks

a Differentiate $y = \frac{1}{2}(x - 2)^2$. — 1 mark

$$y = \frac{1}{2}x^2 - 2x + 2$$
$$\frac{dy}{dx} = x - 2$$

b Find the coordinates of the point on the graph of $y = \frac{1}{2}(x - 2)^2$ whose tangent is parallel to the line with equation $4x - y = 16$. — 3 marks

Equation of line: $y = 4x - 16$. Gradient of line: $m = 4$.

$$\frac{dy}{dx} = x - 2 = 4$$
$$x = 6$$
$$y = \frac{1}{2}(6 - 2)^2 = 8$$

The gradient of the curve is 4 at the point (6, 8).

QUESTION 5 — Total 13 marks

a Expand $(x - 1)^2 (x + 1)$. — 1 mark

$(x - 1)(x^2 - 1) = x^3 - x - x^2 + 1 = x^3 - x^2 - x + 1$

b For $f(x) = (x - 1)^2 (x + 1)$, find when $f'(x) = 0$. Hence find the coordinates of any stationary points of the graph of $f(x)$. — 5 marks

$$f'(x) = 3x^2 - 2x - 1$$
$$3x^3 - 2x - 1 = 0$$
$$(3x + 1)(x - 1) = 0$$
$$x = -\frac{1}{3}, 1.$$

When

$x = -\frac{1}{3}$, $f(x) = \left(-\frac{4}{3}\right)^2\left(\frac{2}{3}\right) = \frac{32}{27}$

$x = 1$, $f(x) = 0$

stationary points are $\left(-\frac{1}{3}, \frac{32}{27}\right)$ and (1, 0).

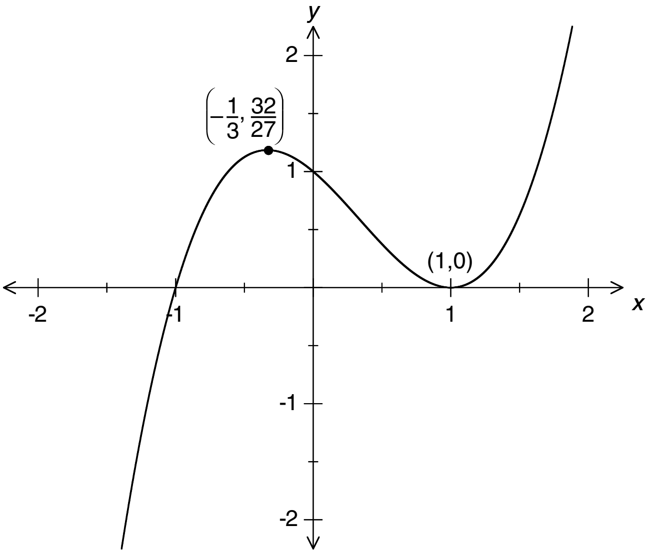
c Find the nature of the stationary points. **2 marks**

$x < -\frac{1}{3}$, $f'(x)$ is positive
$-\frac{1}{3} < x < 1$, $f'(x)$ is negative
$x > 1$, $f'(x)$ is positive
at $x = -\frac{1}{3}$, $f(x)$ is a local maximum
at $x = -1$, $f(x)$ is a local minimum

d State the values of the x and y intercepts. **2 marks**

Solve $f(x) = 0$. x intercepts are -1 and 1;
y intercept is 1.

e Sketch the graph of $f(x)$ on the axes provided. **3 marks**



QUESTION 6

Find the absolute maximum and absolute minimum for $f: [-1, 4] \to R$, $f(x) = 3x^2 - x^3$. **4 marks**

$f'(x) = 6x - 3x^2 = 0$
$3x(2 - x) = 0$
$x = 0, 2$
$f(0) = 0$
$f(2) = 4$ Stationary points are (0, 0) and (2, 4).
$f(-1) = 4$ 4 is the absolute maximum.
$f(4) = -16$ -16 is the absolute minimum.

QUESTION 7 — Total 5 marks

a Find the antiderivative of $\sqrt{x} + 1$. **3 marks**

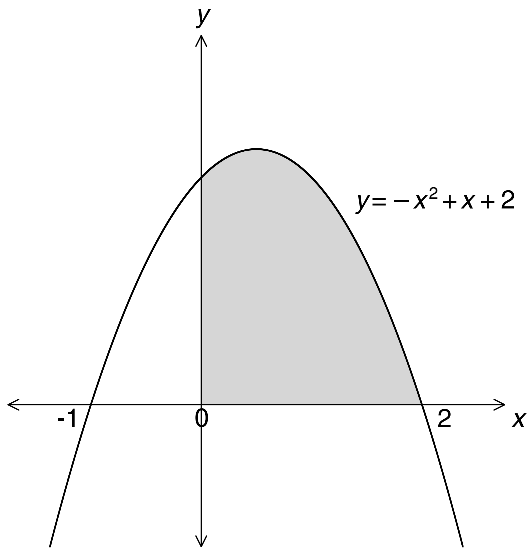
$\int \left(x^{\frac{1}{2}} + 1\right)dx = \frac{2}{3}x^{\frac{3}{2}} + x + c$

b Evaluate $\int_1^2 (3 - x)\,dx$. **2 marks**

$\left[3x - \frac{1}{2}x^2\right]_1^2 = \left[3 \times 2 - \frac{1}{2} \times 2^2\right] - \left[3 \times 1 - \frac{1}{2} \times 1^2\right]$
$= 4 - 2\frac{1}{2} = 1\frac{1}{2}$

QUESTION 8 — Total 6 marks

a Calculate the shaded area shown in the diagram. **3 marks**



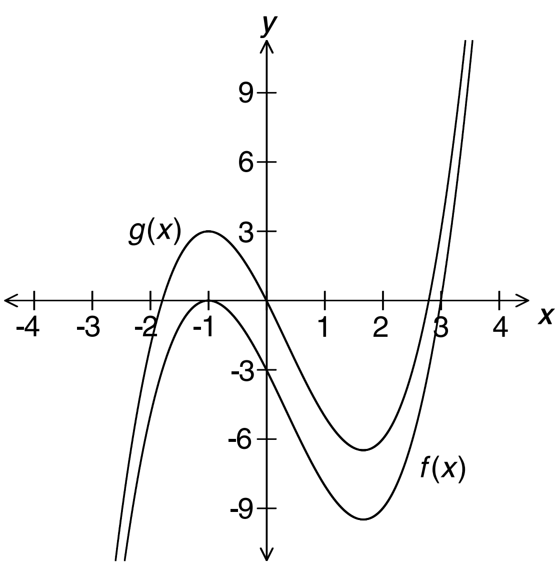
$\int_0^2 (-x^2 + x + 2)dx = \left[-\frac{x^3}{3} + \frac{x^2}{2} + 2x\right]_0^2$
$= \left[-\frac{8}{3} + 2 + 4\right] - [0]$
$= \frac{10}{3}$

b Find the equation of the curve, $y = F(x)$, where $f(x) = 2x - 3$ and where $F(x)$ passes through the point (1, 4). **3 marks**

$F(x) = \int (2x - 3)dx$
$F(x) = x^2 - 3x + c$
$F(1) = 1 - 3 + c = 4$
$c = 6$
$F(x) = x^2 - 3x + 6$

QUESTION 9 — Total 9 marks

a Sketch the graph of $y = f(x) = (x + 1)^2 (x - 3)$. (Do not find stationary points.) **3 marks**



b Hence, solve $(x + 1)^2 (x - 3) \geq 0$. **3 marks**

From the graph: $x = 1$ and $x \geq 3$

c $f(x)$ is translated 3 units in the y direction to give $g(x)$. Write the equation of $g(x)$, the image of $f(x)$. **1 mark**

$g(x) = f(x) + 3$

$= (x + 1)^2 (x - 3) + 3$

d Sketch the graph of $g(x)$ on the set of axes in part **a**. **2 marks**

Section B: Multiple-choice questions. CAS technology assumed.

Specific instructions to students

- A correct answer scores 1, and an incorrect answer scores 0.
- Marks are not deducted for incorrect answers.
- No marks are given if more than one answer is given.
- Choose the alternative which most correctly answers the question and mark your choice on the multiple-choice answer section at the bottom of each page, as shown in the example below.

1 A B C D E **USE PENCIL ONLY**

- Use pencil only.

QUESTION 10

The values of x where $\frac{dy}{dx} \geq 0$ for the graph of $y = \frac{1}{2}(x - 4)(1 - x)$ are:

A $x \leq 2\frac{1}{2}$

B $x > 2\frac{1}{2}$

C $x \geq 2\frac{1}{2}$

D $1 < x < 4$

E $1 \leq x \leq 4$

QUESTION 11

During a brief storm, water flows into a storage tank according to the formula $V = t^2 (10 - t)$, $0 \leq t \leq 10$, where V is the volume in litres at time t minutes. The instantaneous rate of change of the volume of water entering the tank when $t = 5$ is:

A $20t + 3t^2$ L/min

B 125 L/min

C 200 L/min

D 25 L/min

E $\frac{10}{3}t^3 - \frac{1}{4}t^4$ L/min

QUESTION 12

The graph $f(x) = x^3 - bx + c$ has stationary points when x is:

A 0 or $\pm\sqrt{b}$

B $\frac{b}{3}$

C $\frac{\sqrt{3b}}{3}$

D $\pm\frac{\sqrt{3b}}{3}$

E 0

QUESTION 13

An object moving along a straight line has its displacement, x m, from a fixed point O, given by the equation $x = 2t^2 - t$, $t \geq 0$. The velocity of this object would then have the equation:

A $v = 3t^2$

B $v = 3t$

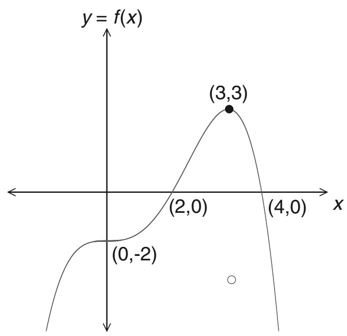
C $v = 4t - 1$

D $v = 2t - 1$

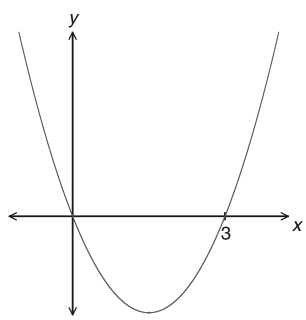
E $v = \frac{2t^3}{3} - \frac{t^2}{2} + c$

QUESTION 14

The graph of $y = f(x)$ is shown. Which graph best represents $\frac{dy}{dx}$, the derivative function?



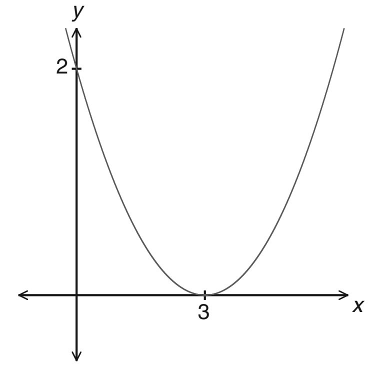
A



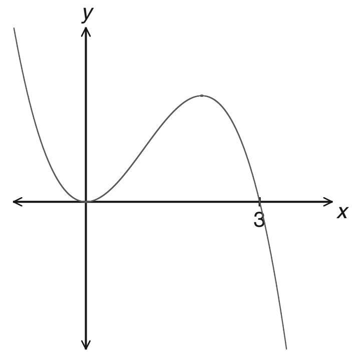
ONE ANSWER PER LINE **USE PENCIL ONLY**

10 A B C D E
11 A B C D E
12 A B C D E
13 A B C D E

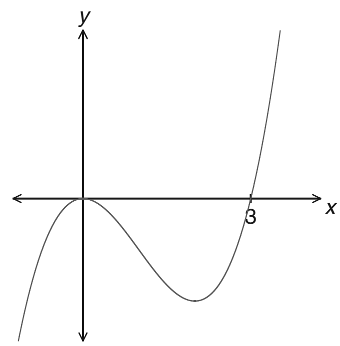
B



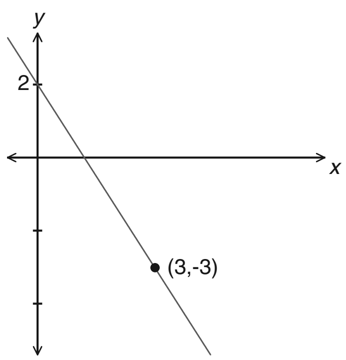
C



D



E



ONE ANSWER PER LINE **USE PENCIL ONLY**

14 A B C D E

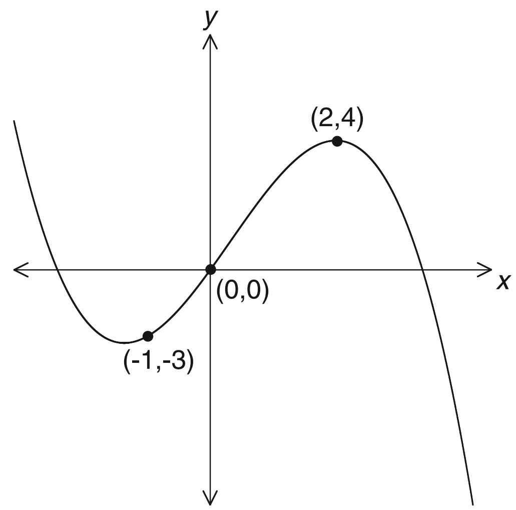
Section B: Extended response questions. CAS technology assumed.

Specific instructions to students

- Answer **all** questions in the spaces provided.
- In questions where more than one mark is available, appropriate working **must** be shown.

QUESTION 15 **Total 10 marks**

The graph shown in the diagram is of the form $f(x) = ax^3 + bx^2 + cx + d$. The point (2, 4) is a stationary point of the graph, which also passes through $(-1, -3)$ and the origin.



a Write an equation for $f'(x)$. **1 mark**

$f'(x) = 3ax^2 + 2bx + c$

b List four simultaneous equations to evaluate a, b, c and d. **4 marks**

$f(0) = 0$: $d = 0$

$f(2) = 4$: $8a + 4b + 2c + d = 4$

$f(-1) = -3$: $-a + b - c + d = -3$

$f'(2) = 0$: $12a + 4b + c = 0$

c Use these four equations to form a matrix equation. Hence, or otherwise, find the values of a, b, c and d. **3 marks**

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 8 & 4 & 2 & 1 \\ -1 & 1 & -1 & 1 \\ 12 & 4 & 1 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ -3 \\ 0 \end{bmatrix}$$

Using CAS: $a = -\frac{2}{9}$, $b = -\frac{1}{9}$, $c = \frac{28}{9}$, $d = 0$

d Find the exact values of the coordinates of the second stationary point. **2 marks**

Using CAS: SOLVE $f'(x) = 0$. Solutions are $x = 2, -\frac{7}{3}$.
$f\left(-\frac{7}{3}\right) = -5\frac{10}{243}$.
Coordinates of second point: $\left(-2\frac{1}{3}, -5\frac{10}{243}\right)$

QUESTION 16 **Total 12 marks**

The position of a particle moving in a straight line relative to a point O is given by $x(t) = t^3 - 7t^2 + 8t + 16$, where x metres is its position at time t seconds.

a Write the velocity, v, and acceleration, a, in terms of t. **2 marks**

$v(t) = 3t^2 - 14t + 8$
$a(t) = 6t - 14$

b What is the initial position, velocity and acceleration of the particle? **3 marks**

$x(0) = 16$ m
$v(0) = 8$ m/s
$a(0) = -14$ m/s^2

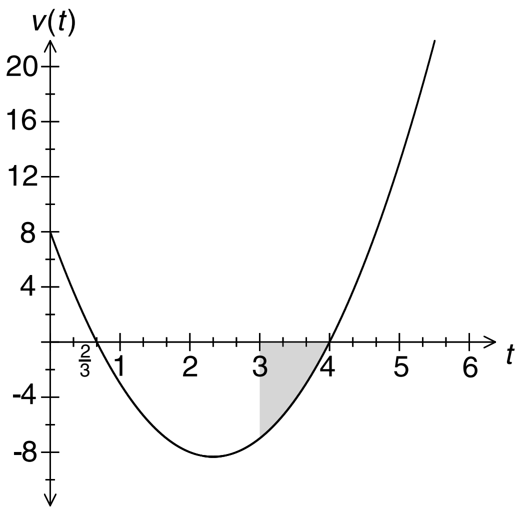
c When is the particle at rest? **2 marks**

Using CAS: SOLVE $3t^2 - 14t + 8 = 0$. Solution: $t = \frac{2}{3}, 4$

d Find the position of the particle when it is at rest. **2 marks**

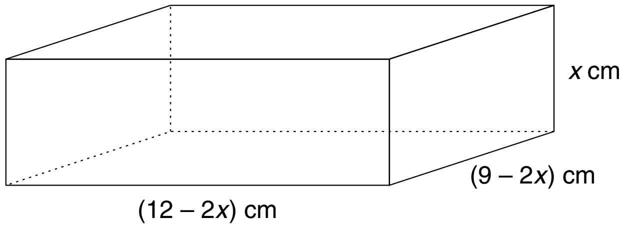
$x\left(\frac{2}{3}\right) = \frac{500}{27} = 18\frac{14}{27}$ m
$x(4) = 4$ m

e Sketch the graph of the velocity against time and mark the displacement of the particle between $x = 3$ and $x = 4$. **3 marks**



QUESTION 17 **Total 10 marks**

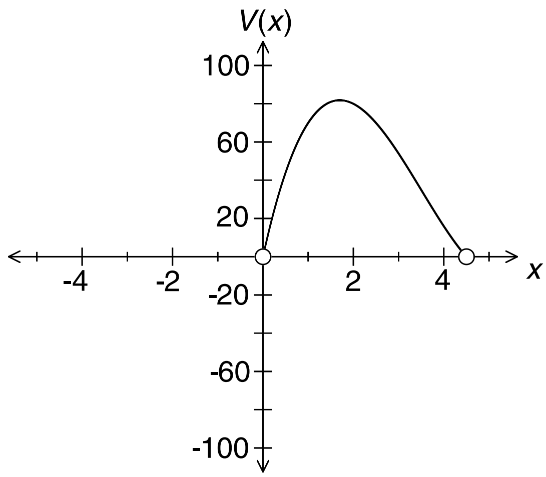
A rectangular box has dimensions as shown.



a Express the volume of the box, V cm^3, in terms of x. **1 mark**

$V(x) = x(12 - 2x)(9 - 2x)$

b Sketch the graph of V against x over a suitable domain. State the domain. **4 marks**



Domain: $x \in \left(0, 4\frac{1}{2}\right)$

c State the maximum volume of the box and the value of x at which it occurs. Give answers correct to two decimal places. **2 marks**

From the graph: $V = 81.87$ cm^3 when $x = 1.70$ cm

d Find $\frac{dV}{dx}$. Hence, find the exact value of the maximum volume. **3 marks**

Using CAS: $\frac{dV}{dx} = 12x^2 - 84x + 108$
Maximum volume when $\frac{dV}{dx} = 0$.
Using CAS: SOLVE $12x^2 - 84x + 108 = 0$.
Solution: $x = \frac{7 - \sqrt{13}}{2}$, as $x \in \left(0, 4\frac{1}{2}\right)$.
$V\left(\frac{7 - \sqrt{13}}{2}\right) = 35 + 13\sqrt{13}$