Test 6

Section A: Technology free. 51 marks
Section B: CAS technology assumed. 43 marks
Suggested time: 90 minutes

Section A: Short answer and extended response questions. Technology free.

Specific instructions to students

- Answer **all** questions in the spaces provided.
- A decimal approximation will not be accepted if an **exact** answer is required to a question.
- In questions where more than one mark is available, appropriate working **must** be shown.

QUESTION 1 — **Total 5 marks**

a Simplify $\frac{2x^{-2}y^2}{(2xy^{-2})^{-1}}$, expressing the answer with positive indices. **2 marks**

$\frac{2x^{-2}y^2}{(2xy^{-2})^{-1}}$

$= \frac{2x^{-2}y^2}{2^{-1}x^{-1}y^2}$

$= 2^{1+1}x^{-2+1}y^{2-2}$

$= 2^2x^{-1}y^0$

$= \frac{4}{x}$

b Evaluate $256^{\frac{3}{4}}$. **2 marks**

$(2^8)^{\frac{3}{4}} = 2^6$

$= 64$

c Solve $2^{x+3} = 32$ for x. **1 mark**

$2^{x+3} = 2^5$

equate indices

$x + 3 = 5$

$x = 2$

QUESTION 2 — **Total 5 marks**

a Evaluate $\log_2 32$. **1 mark**

$\log_2(2)^5 = 5\log_2 2$

$= 5 \times 1$

$= 5$

b Solve the following equations for x.

i $\log_2 x = -3$ **2 marks**

Using $\log_a x = y \Rightarrow x = a^y$,

$x = 2^{-3}$

$= \frac{1}{8}$

ii $\log_{10} x + \log_{10} 2 - \log_{10}(x + 2) = 0$ **2 marks**

$\log_{10}\left(\frac{2x}{x+2}\right) = \log_{10} 1$

Equate logarithms

$\frac{2x}{x+2} = 1$

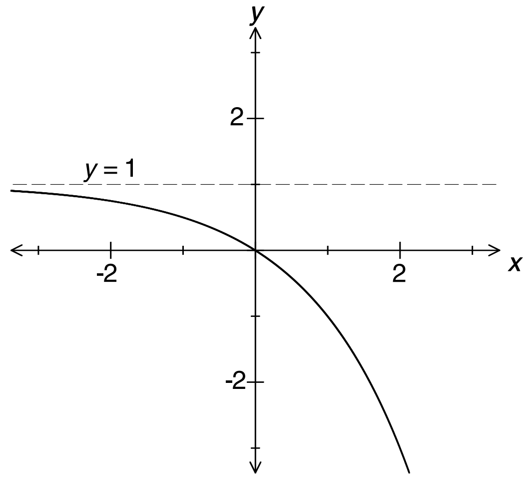
$2x = x + 2$

$x = 2$

QUESTION 3 — **Total 8 marks**

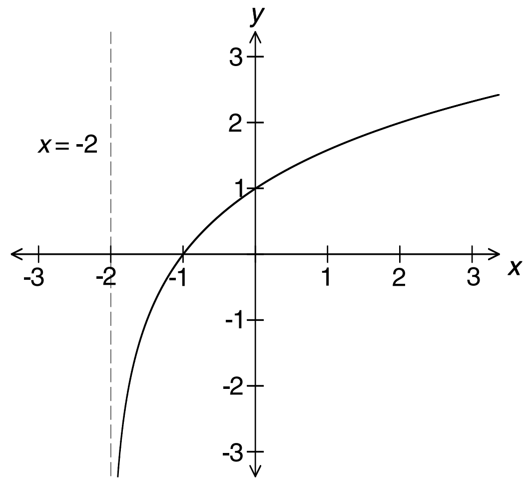
Sketch the graph of each of the following. Label any x and y intercepts. Write the equations of any asymptotes. State the domain and range of each graph.

a $y = 1 - 2^x$ **4 marks**



Domain: $x \in R$ Range: $x < 1$ or $x \in (-\infty, 1)$

b $y = \log_2(x + 2)$ **4 marks**



The y intercept is $\log_2(2) = 1$

The x intercept is

$\log_2(x + 2) = 0$

$x + 2 = 2^0$

$x + 2 = 1$

$x = -1$

Domain: $x > -2$ or $x \in (-2, \infty)$

Range: $y \in R$

QUESTION 4 **Total 3 marks**

The number of insects in a particular experiment is given by $N = N_0 10^{kt}$, where N is the number of insects at any time t days.

i If the number present at the start is 200, find the value of N_0. **1 mark**

$(t = 0, N = 200); 200 = N_0 10^0$

$N_0 = 200$

ii If $k = 0.01$, find the number present after 300 days. **2 marks**

$N = 200 \times 10^{0.01t}$

$t = 300, N = 200 \times 10^3$

$= 200000$

QUESTION 5 **Total 8 marks**

a Solve the simultaneous equations $x - 2y + 2 = 0$ and $y = \frac{3}{4}x - \frac{3}{2}$. **2 marks**

$x - 2y = -2$ Equation 1

$4y = 3x - 6 \Rightarrow 3x - 4y = 6$ Equation 2

Equation 1 × −2: $-2x + 4y = 4$

$3x - 4y = 6$

$x = 10$

$10 - 2y = -2$

$2y = 12$

$y = 6$

b Solve $\frac{3 - 2x}{3} + \frac{9 - 2x}{6} < 2$ for x. **2 marks**

$2(3 - 2x) + 9 - 2x < 12$

$6 - 4x + 9 - 2x < 12$

$15 - 6x < 12$

$-6x < -3$

$x > \frac{1}{2}$

c The area of an annulus is $A = \pi R^2 - \pi r^2$, where R is the radius of the outer circle and r is the radius of the inner circle.

i Transpose the formula to make R the subject. **3 marks**

$\pi R^2 = A + \pi r^2$

$R^2 = \frac{A + \pi r^2}{\pi}$

$R = \pm\sqrt{\frac{A + \pi r^2}{\pi}}$

$R = \sqrt{\frac{A + \pi r^2}{\pi}}$, as $R > 0$

ii Find the exact value of R when $A = 1000$ and $r = 2$. **1 mark**

$R = \sqrt{\frac{1000 + 4\pi}{\pi}}$

QUESTION 6 **Total 4 marks**

a If $A = \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}$:

i A^2 **1 mark**

$\begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}$

$= \begin{bmatrix} 1 \times 1 + 3 \times -1 & 1 \times 3 + 3 \times 2 \\ -1 \times 1 + 2 \times -1 & -1 \times 3 + 2 \times 2 \end{bmatrix} = \begin{bmatrix} -2 & 9 \\ -3 & 1 \end{bmatrix}$

ii A^{-1} **1 mark**

Determinant = $(1 \times 2) - (-1 \times 3) = 5$.

$A^{-1} = \frac{1}{5}\begin{bmatrix} 2 & -3 \\ 1 & 1 \end{bmatrix}$

b Solve the matrix equation, $A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ for x and y. **2 marks**

$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 2 & -3 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix}$

$= \frac{1}{5}\begin{bmatrix} 2 \times 3 + -3 \times 2 \\ 1 \times 3 + 1 \times 2 \end{bmatrix}$

$= \frac{1}{5}\begin{bmatrix} 0 \\ 5 \end{bmatrix}$

$= \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

QUESTION 7 **Total 8 marks**

a By completing the square, show that $y = -x^2 + 4x - 1$ can be expressed as $y = -(x - 2)^2 + 3$. **2 marks**

$y = -(x^2 - 4x + 1)$

$= -[(x - 2)^2 - 3]$

$= -(x - 2)^2 + 3$

b Find the exact value of the x and y intercepts. **3 marks**

y intercept:

when $x = 0$, $y = -1$

x intercepts:

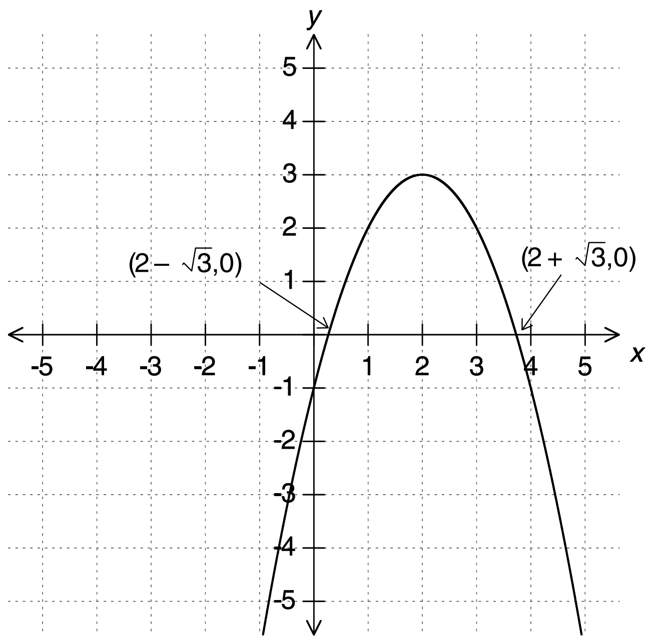
$-(x - 2)^2 + 3 = 0$

$(x - 2)^2 = 3$

$x - 2 = \pm\sqrt{3}$

$x = 2 \pm\sqrt{3}$

c Sketch the graph of $y = -x^2 + 4x - 1$ on the axes provided. **3 marks**



QUESTION 8

Total 3 marks

a Use the factor theorem to show that the factors of $2x^3 - 5x^2 - 4x + 3$ are $(2x - 1)(x + 1)(x - 3)$. **1 mark**

$$P\left(\frac{1}{2}\right) = 2 \times \frac{1}{8} - 5 \times \frac{1}{4} - 4 \times \frac{1}{2} + 3$$
$$= \frac{1}{4} - \frac{5}{4} - 2 + 3 = 0$$

$(2x - 1)$ is a factor.

$$P(3) = 54 - 45 - 12 + 3$$
$$= 0$$

$(x - 3)$ is a factor.

$P(-1) = -2 - 5 + 4 + 3 = 0$

$(x + 1)$ is a factor.

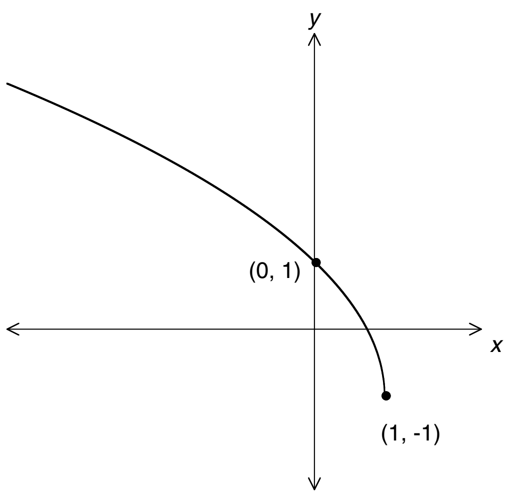
b Hence, find the x and y intercepts for the graph of $y = 2x^3 - 5x^2 - 4x + 3$. **2 marks**

The x intercepts are $\frac{1}{2}$, -1, 3; the y intercept is 3.

QUESTION 9

Total 7 marks

The graph of $y = a\sqrt{b - x} + c$ is shown.



a State the values of b and c. **2 marks**

$y = a\sqrt{-(x - b)} + c$. Hence, $b = 1$ and $c = -1$.

b Show that $a = 2$. **2 marks**

$$(0,\ 1): 1 = a\sqrt{-(0 - 1)} - 1$$
$$a\sqrt{1} = 2$$
$$a = 2$$

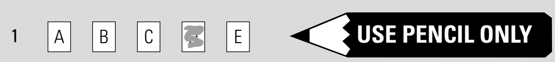
c State the transformations on $y = \sqrt{x}$ that give $y = a\sqrt{b - x} + c$ as its image. **3 marks**

Dilation by 2 from the x axis, reflection in the y axis, translation of 1 in the x direction and a translation of -1 in the y direction.

Section B: Multiple-choice questions. CAS technology assumed.

Specific instructions to students

- A correct answer scores 1, and an incorrect answer scores 0.
- Marks are not deducted for incorrect answers.
- No marks are given if more than one answer is given.
- Choose the alternative which most correctly answers the question and mark your choice on the multiple-choice answer section at the bottom of each page, as shown in the example below.



- Use pencil only.

QUESTION 10

The temperature, T°C, of a cooling liquid is given by the formula $T = 76(10)^{-kt} + 20$, where t is the time in minutes and $k = 0.156$. The temperature of the liquid after 5 minutes is closest to:

A 13°C
B 21°C
C 33°C
D 35°C
E 53°C

QUESTION 11

The range of the function $f: R \to R, f(x) = 2 \times 10^{-x} - 1$ is:

A R

B $R \backslash \{2\}$

C $R \backslash \{-1\}$

D $(-1, \infty)$

E $[-1, \infty)$

QUESTION 12

For $3 \times 3^{2x} = 9$, the value of x is:

A $\frac{1}{3}$

B $\frac{1}{2}$

C $\frac{1}{3}\log_9 2$

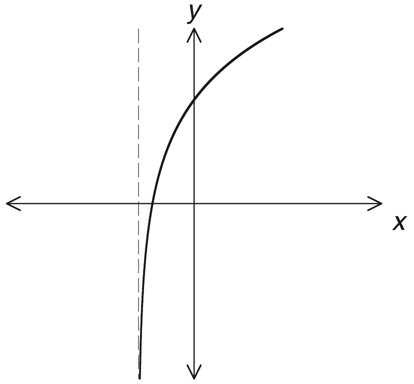
D $2\log_3 3$

E $\log_3 9 - 1$

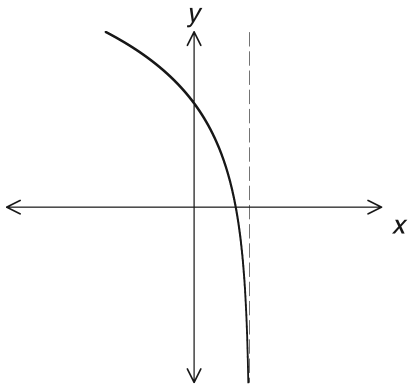
QUESTION 13

Which of the following graphs could be the graph of $f(x) = \log_2 (x - a) + b$, where a and b are positive real numbers?

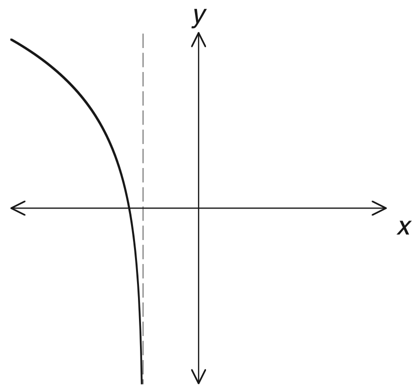
A



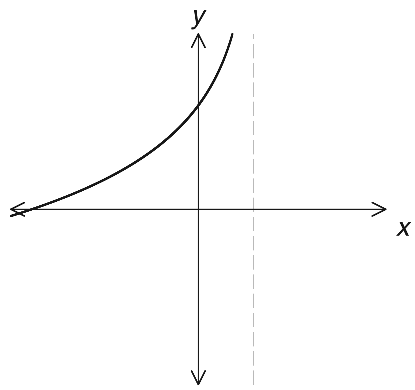
B



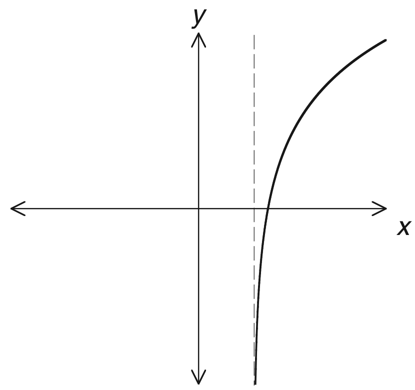
C



D



E



QUESTION 14

The value of the x intercept for the graph $g(x) = 3 - \log_2 (1 - x)$ is:

A 2

B -2

C $-\frac{7}{8}$

D -7

E 9

ONE ANSWER PER LINE **USE PENCIL ONLY**

10	A	B	C	D	E	13	A	B	C	D	E
11	A	B	C	D	E	14	A	B	C	D	E
12	A	B	C	D	E						

Section B: Extended response questions. CAS technology assumed.

Specific instructions to students

- Answer **all** questions in the spaces provided.
- In questions where more than one mark is available, appropriate working **must** be shown.

QUESTION 15 **Total 8 marks**

It is observed that over a two-week period the number of a certain organism grows according to the rule $N(t) = 10 \times 2^{0.35t}$, where N is the number of organisms (measured in thousands) present after t days.

a What is the domain of the function? **1 mark**

$t \in [0, 14]$

b Find the increase in the number of organisms from $t = 4$ to $t = 7$, correct to four decimal places. **2 marks**

$$N(7) - N(4) = 54.6416 - 26.3902$$
$$= 28.2515 \text{ thousand organisms}$$

c Determine the average daily increase in the weight of the organisms over this period, correct to two decimal places. **3 marks**

$$\text{Average daily increase} = \frac{N(7) - N(4)}{7 - 4}$$
$$= \frac{28.2515}{3}$$
$$= 9.42 \text{ thousand/day}$$

d Find the number of days, correct to the nearest day, when the number of organisms is 100 000. **2 marks**

CAS: SOLVE $N(t) = 100$ for t.

$t = 9.5$

After 9 days.

QUESTION 16 **Total 7 marks**

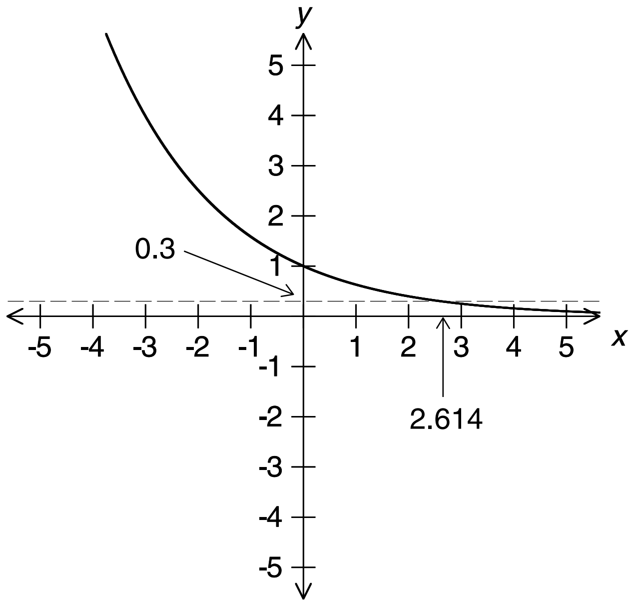
a Solve $10^{-0.2x} = 0.3$, giving the answer in the form $a \log_{10}\left(\frac{10}{b}\right)$. **3 marks**

$$-0.2x = \log_{10} 0.3$$
$$x = -\frac{1}{0.2}\log_{10}\left(\frac{3}{10}\right)$$
$$= 5\log_{10}\left(\frac{3}{10}\right)^{-1}$$
$$= 5\log_{10}\left(\frac{10}{3}\right)$$

b Give an approximate value for this answer, correct to three decimal places. **1 mark**

Using CAS: 2.614

c Sketch the graph of $y = 10^{-0.2x}$ on the axes provided. Locate the solution to part **a** on the graph. **2 marks**



d Hence, find $\{x : 10^{-0.2x} > 0.3\}$, correct to three decimal places. **1 mark**

$x < 2.614$

QUESTION 17 **Total 11 marks**

Consider the function $f: R \to R$ where $f(x) = 2^x - 2$.

a State the domain and range of f^{-1}, the inverse of f. **2 marks**

Domain of f^{-1} : $x > -2$ or $x \in (-2, \infty)$;
range of f^{-1}: $x \in R$

b Find the rule of f^{-1}. **2 marks**

$$x = 2^y - 2$$
$$x + 2 = 2^y$$
$$y = \log_2(x + 2)$$

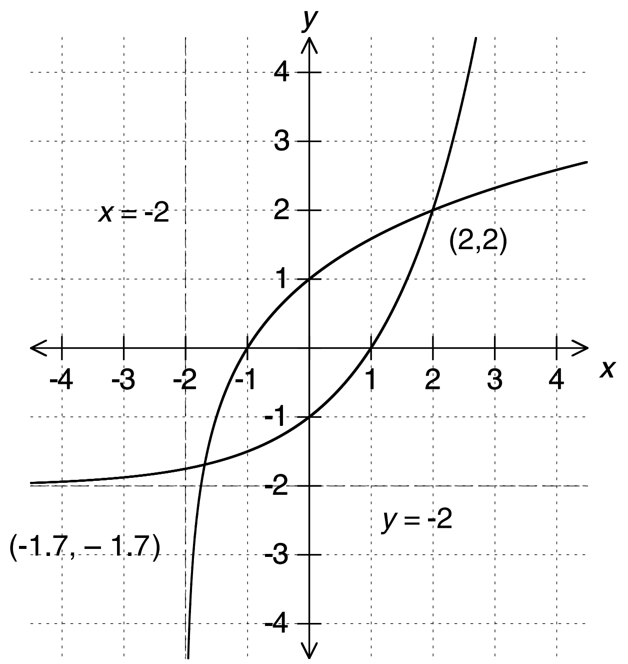
c Write an equation to find the points of intersection between f and f^{-1}. Solve the equation, correct to one decimal place. **3 marks**

Three choices: $f(x) = x$ or $f^{-1}(x) = x$ or $f(x) = f^{-1}(x)$

Using CAS: As f^{-1} includes log to the base 2, it may be easier to use $f(x) = x$, so solve $2^x - 2 = x$ for x.

$x = -1.7, 2.0$

d Sketch the graph of f and f^{-1} on the set of axes provided. Label any x and y intercepts. Draw and write the equation of any asymptotes. **4 marks**



QUESTION 18 **Total 9 marks**

A family of parabolas has the equation $y = (x + 1)(x - a)$, where a is a positive number.

a Expand the brackets. Hence, write the equation in the form $y = x^2 + bx + c$. **1 mark**

$y = x^2 + x - ax - a$
$= x^2 + (1 - a)x - a$

b Using $y = (x + 1)(x - a)$, find the coordinates of the turning point and the values of the x intercepts in terms of a. **5 marks**

The x intercepts are $x = -1, a$.
The x value of the turning point is $\frac{a-1}{2}$ (midpoint of x intercepts).
The y value is:
$y = \left(\frac{a-1}{2} + 1\right)\left(\frac{a-1}{2} - a\right)$
$= -\frac{(a+1)^2}{4}$ (using CAS).

c Write the equation of the family of graphs when $a = \{0, 1, 2\}$. Which one of these is an even function? **3 marks**

$a = 0;\ y = x(x + 1)$
$a = 1;\ y = (x - 1)(x + 1) = x^2 - 1 \therefore$ even function
$a = 2;\ y = (x - 2)(x + 1)$