	STUDENT NUMBER								LETTER
Figures									
Words									

MATHEMATICAL METHODS (CAS)

Written Examination Unit 2

SAMPLE

Reading time: 15 minutes
Section A writing time : 40 minutes
Section B writing time : 50 minutes

Note: At the end of the 40 minutes, Section A will be collected. Students may use a CAS calculator and annotated notes for the remainder of the examination only after Section A has been collected.

Multiple choice, short answer and extended answer questions

QUESTION AND ANSWER BOOK

Section	*Number of questions*	*Number of questions to be answered*	*Number of marks*
A	9 technology-free questions	9	45
B	10 multiple choice questions	10	10
	3 extended questions	3	35

- Students are to write their names in the spaces provided on the front of Section A and on the front page of Section B.
- Section A is technology free and no reference material can be used.
- Section B: an approved CAS calculator and one bound reference text (which may be annotated) or lecture pad, may be used.
- Unless otherwise stated, the diagrams in this booklet are not drawn to scale.
- All written responses must be in English.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

Student Name:____

Section A: Short answer and extended response questions. Technology free.

Specific instructions to students

- Answer **all** questions in the spaces provided.
- A decimal approximation will not be accepted if an **exact** answer is required to a question.
- In questions where more than one mark is available, appropriate working **must** be shown.
- Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

QUESTION 1 **Total 4 marks**

a Simplify $\left(\frac{x^2 y}{x^{-1} y^{-2}}\right)^{-1} \div \frac{(3x^2y^3)^4}{81xy}$, giving the answer with positive indices. **2 marks**

$$\frac{x^{-2}y^{-1}}{xy^2} \div \frac{81x^8y^{12}}{81xy}$$
$$= \frac{x^{-2}y^{-1}}{xy^2} \times \frac{81xy}{81x^8y^{12}}$$
$$= 81^{1-1}x^{-2+1-1-8}y^{-1+1-2-12}$$
$$= 81^0x^{-10}\,y^{-14} = \frac{1}{x^{10}y^{14}}$$

b Solve for x: $2^{3x-1} = \frac{1}{32}$. **2 marks**

$$2^{3x-1} = \frac{1}{2^5}$$
$$= 2^{-5}$$
$$3x - 1 = -5$$
$$x = -\frac{4}{3}$$

QUESTION 2 **Total 5 marks**

a Express $\log_2(24) - \frac{1}{2}\log_2(16) + \log_2\left(\frac{2}{15}\right)$ in the form $a + b\log_2(c)$. State the values of a, b and c. **3 marks**

$$\log_2(24) - \log_2(16)^{\frac{1}{2}} + \log_2\left(\frac{2}{15}\right)$$
$$= \log_2\left(\frac{24 \times 2}{4 \times 15}\right)$$
$$= \log_2\left(\frac{4}{5}\right)$$
$$= \log_2(4) - \log_2(5)$$
$$= 2\log_2(2) - \log_2(5)$$
$$= 2 - \log_2(5)$$
$$a = 2,\ b = -1,\ c = 5$$

b Solve for x: $\log_5(x + 3) = 2$. **2 marks**

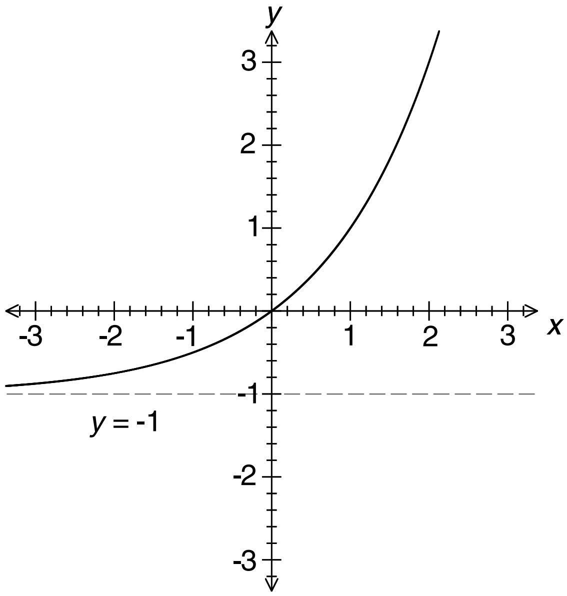
$$x + 3 = 5^2$$
$$x + 3 = 25$$
$$x = 22$$

QUESTION 3

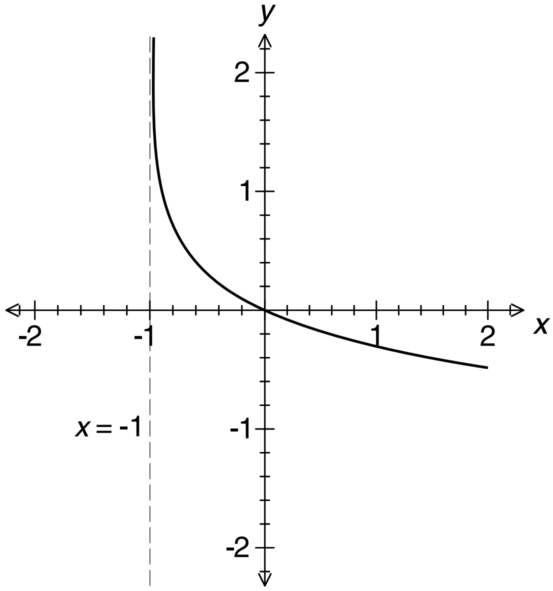
Total 6 marks

Sketch the graph of the following, labelling any asymptotes and any x or y intercepts.

a $y = 2^x - 1$ **3 marks**



b $y = -\log_{10}(x + 1)$ **3 marks**



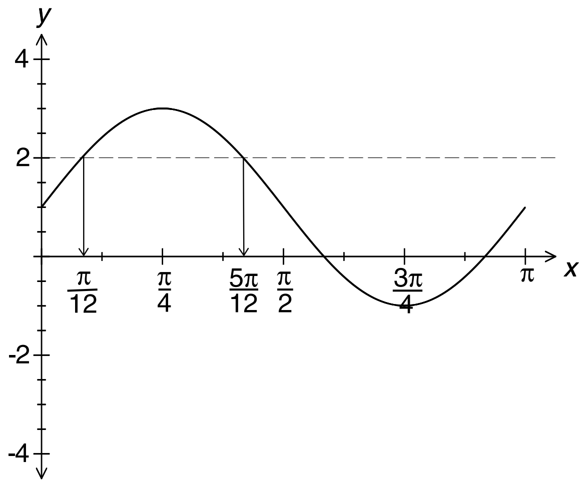
QUESTION 4

Total 6 marks

a Solve $2\sin(2x) + 1 = 2, 0 \le x \le \pi$. **3 marks**

$$\sin(2x) = \frac{1}{2},\ 0 \le 2x \le 2\pi$$
$$2x = \sin^{-1}\left(\frac{1}{2}\right)$$
$$= \frac{\pi}{6}, \pi - \frac{\pi}{6}$$
$$= \frac{\pi}{6}, \frac{5\pi}{6}$$
$$x = \frac{\pi}{12}, \frac{5\pi}{12}$$

b Sketch the graph of $y = 2\sin(2x) + 1, 0 \leq x \leq \pi$. **2 marks**



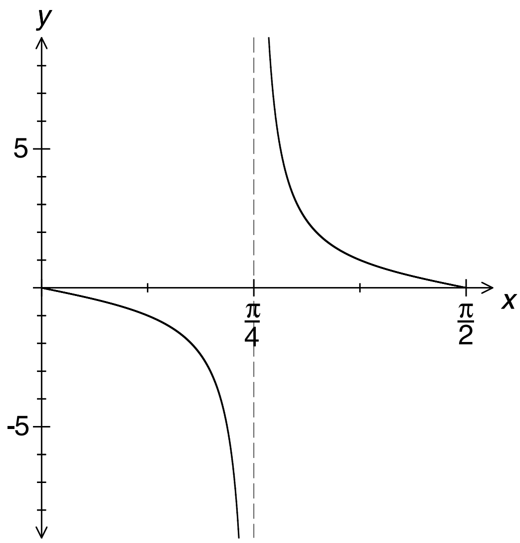
c Hence find $\{x : 2\sin(2x) + 1 > 2, 0 < x < \pi\}$. **1 mark**

See graph in part **b**: $\frac{\pi}{12} < x < \frac{5\pi}{12}$

QUESTION 5

Total 4 marks

a Draw the graph of $y = -\tan(2x)$, for one period, starting at $x = 0$. **2 marks**



b State the transformations on $y = \tan(x)$ to give $y = -\tan(2x)$ as its image. **2 marks**

Dilation of $\frac{1}{2}$ in the x axis direction, followed by a reflection in the x axis.

QUESTION 6

Total 6 marks

a Differentiate $y = 2x^3 - \frac{2}{x^2} - \frac{1}{2}\sqrt{x} + 1$. **3 marks**

$$y = 2x^3 - 2x^{-2} - \frac{1}{2}x^{\frac{1}{2}} + 1$$
$$\frac{dy}{dx} = 6x^2 + 4x^{-3} - \frac{1}{4}x^{-\frac{1}{2}}$$
$$= 6x^2 + \frac{4}{x^3} - \frac{1}{4\sqrt{x}}$$

b If $f(x) = x(x - 1)(x - 2)$, find $f'(x)$. **1 mark**

$$f(x) = x^3 - 3x^2 + 2x$$
$$f'(x) = 3x^2 - 6x + 2$$

c Hence, find the x values where $f'(x) = 0$. **2 marks**

$$3x^2 - 6x + 2 = 0$$

$$x = \frac{6 \pm \sqrt{36 - 24}}{6}$$

$$= \frac{6 \pm \sqrt{12}}{6}$$

$$= \frac{3 \pm \sqrt{3}}{3}$$

QUESTION 7

Total 5 marks

a Write the antiderivative of $x^2 + 2x + 2$. **3 marks**

$$\frac{1}{3}x^3 + x^2 + 2x + c$$

b Evaluate $\int_1^2 (x + 2)\,dx$. **2 marks**

$$\left[\frac{x^2}{2} + 2x\right]_1^2 = [2 + 4] - \left[\frac{1}{2} + 2\right] = 6 - 2\frac{1}{2} = 3\frac{1}{2}$$

QUESTION 8

Total 4 marks

a Five species of trees are to be planted along the boundary fence of a property. If one of the trees is a eucalyptus gum and another is a red river gum, what is the probability they are placed first and second, respectively, in the row, given that the plantings are done randomly? **2 marks**

The eucalyptus and red river gums must be placed in the first and second positions respectively. This can be done in 1×1 ways. The remainder can be placed in 3! ways. If there are no restrictions on the 5 trees, they can be arranged in 5! ways.

Probability $= \frac{1 \times 1 \times 3!}{5!} = \frac{1}{20}$

b If John is late for school today there is a 5% chance he will be late tomorrow. If he is on time today, there is an 8% chance he will be late tomorrow.

i Write a transition matrix to represent this data. **1 mark**

		today	
		early	late
tomorrow	early	0.92	0.95
	late	0.08	0.05

Matrix is: $T = \begin{bmatrix} 0.92 & 0.95 \\ 0.08 & 0.05 \end{bmatrix}$

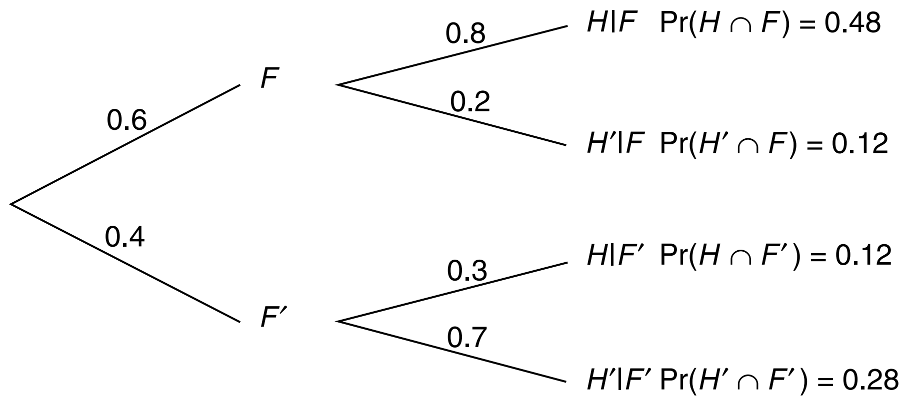
ii John is on time today. What is the probability he will be on time in 2 days' time? (Set up a matrix expression but do not evaluate.) **1 mark**

$$T^2 \times \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.92 & 0.95 \\ 0.08 & 0.05 \end{bmatrix}^2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

QUESTION 9 **Total 5 marks**

Wheat farmers need follow-up rains to achieve a good harvest. In a particular wheat-farming region, the probability of adequate follow-up rains is 0.6. If there are little follow-up rains, the probability of a good harvest is 0.3, while adequate follow-up rains give a probability of 0.8 of a good harvest. F represents the probability of adequate follow-up rains and H represents the probability of a good harvest.

a Set up a tree diagram to represent these probabilities. **2 marks**



b What is the overall probability of a good harvest? **1 mark**

$\Pr(H) = \Pr(H \cap F) + \Pr(H \cap F') = 0.48 + 0.12 = 0.6$

c There has been a good harvest. What is the probability there were poor follow-up rains for that harvest? **2 marks**

$\Pr(F' | H) = \dfrac{\Pr(F' \cap H)}{\Pr(H)} = \dfrac{0.12}{0.6} = 0.2$

Student Name:____

Section B: Multiple-choice and extended response questions.
CAS technology assumed.

Specific instructions to students

- A correct answer scores 1, and an incorrect answer scores 0.
- Marks are not deducted for incorrect answers.
- No marks are given if more than one answer is given.
- Choose the alternative which most correctly answers the question and mark your choice on the multiple-choice answer section at the bottom of the page.

1 A B C D E — USE PENCIL ONLY

- Use pencil only.

QUESTION 10

The transformations on $y = 2^x$ to give the image $y = 1 - 2^{-x}$ are:

A reflection in the x and y axes, followed by a translation of 1 in the x axis direction

B reflection in the x and y axes, followed by a translation of 1 in the y axis direction

C reflection in the x axis, followed by a translation of -2 in the x axis direction and 1 in the y axis direction

D reflection in the y axis, followed by a translation of -2 in the x axis direction and 1 in the y axis direction

E reflection in the x and y axes, followed by a translation of -2 in the y axis direction.

QUESTION 11

If $\log_a (x - 1) = b$, then x is equal to:

A $1 + a^b$

B $1 + b^a$

C $1 - \log_a b$

D $\log_a (b + 1)$

E 1 000 001

QUESTION 12

The inverse function of $y = \log_a (b - x)$ is:

A $b - \log_a y$

B $b + a\log_a y$

C $b - 10^x$

D $b - e^x$

E $b - a^x$

ONE ANSWER PER LINE — USE PENCIL ONLY

10 A B C D E

11 A B C D E

12 A B C D E

QUESTION 13

If $\sin(\theta) = \frac{1}{3}$ and $\frac{\pi}{2} \leq \theta \leq \pi$, $\cos(\theta)$ is equal to:

A $\frac{2}{3}$

B $-\frac{2}{3}$

C $\frac{8}{9}$

D $-\frac{2\sqrt{2}}{3}$

E $\frac{2\sqrt{2}}{3}$

QUESTION 14

The number of solutions to the equation $2^{-x} - 1 = \sin(4x)$, $0 \leq x \leq 2\pi$ is:

A 5

B 6

C 7

D 8

E 9

QUESTION 15

For the graph of $y = f(x)$, the instantaneous rate of change of y with respect to x is given by:

A $\frac{f(x+h) - f(x)}{h}$

B $\frac{f(x) - f(x-h)}{h}$

C $\frac{f(x+h) - f(x-h)}{h}$

D $\lim\limits_{h \to 0} \frac{f(x+h) - f(x)}{h}$

E $\lim\limits_{x \to 0} \frac{f(x+h) - f(x)}{h}$

QUESTION 16

Let $f(x) = x^2 - \frac{1}{x^2}$. Evaluate $f'(-1)$.

A $-2\frac{1}{2}$

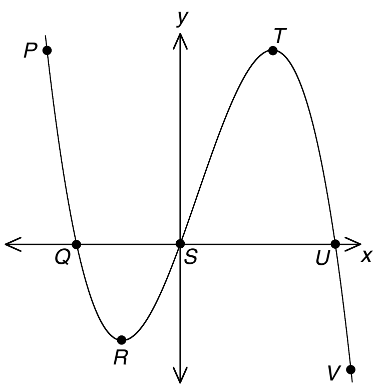
B $2\frac{1}{2}$

C -4

D 0

E 4

The graph of $y = f(x)$ refers to questions 17 and 18.



QUESTION 17

Between which points does y increase as x increases?

A From S to T.

B From R to T.

C From R to V.

D From P to Q and from S to U.

E From P to R and from T to V.

QUESTION 18

The area enclosed between the graph of $f(x)$ and the x axis is given by which integral using the x values at the points indicated?

A $\int_{Q}^{U} f(x)\,dx$

B $\int_{S}^{Q} f(x)\,dx + \int_{S}^{U} f(x)\,dx$

C $\int_{Q}^{S} f(x)\,dx - \int_{S}^{U} f(x)\,dx$

D $-\int_{S}^{Q} f(x)\,dx + \int_{S}^{U} f(x)\,dx$

E $\int_{Q}^{S} f(x)\,dx + \int_{S}^{U} f(x)\,dx$

QUESTION 19

It is found that 85% of students who pass a weekly calculator test this week will also pass the test next week, and that 65% of students who fail the test this week will fail the test next week. Of 100 students, 80 pass the test this week. How many are expected to pass the test in 4 weeks' time?

A 30

B 67

C 71

D 57

E 52

ONE ANSWER PER LINE **USE PENCIL ONLY**

13	A	B	C	D	E	17	A	B	C	D	E
14	A	B	C	D	E	18	A	B	C	D	E
15	A	B	C	D	E	19	A	B	C	D	E
16	A	B	C	D	E						

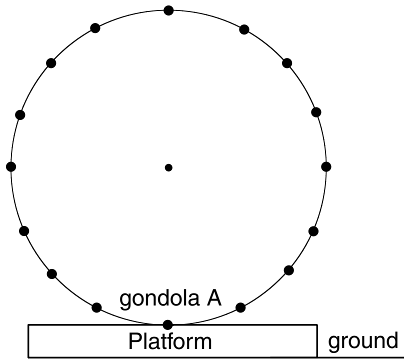
Section B: Extended response questions. CAS technology assumed.

Specific instructions to students

- Answer **all** questions in the spaces provided.
- In questions where more than one mark is available, appropriate working **must** be shown.

QUESTION 20 **Total 12 marks**

A scenic Ferris wheel has gondolas arranged around its circumference. The positions of some of the gondolas are shown in the diagram. Gondola A is stationed at the bottom, where people can board and disembark from the gondolas at a raised platform.



The equation $y = 50 - 48\cos\left(\frac{\pi}{10}\right)t$ represents the vertical height, y metres, of gondola A above the ground at time t minutes.

a State the period and amplitude of the graph of $y = 50 - 48\cos\left(\frac{\pi}{10}t\right)$. **2 marks**

Period is 20; amplitude is 48.

b Hence, state:

i the time it takes for one revolution of gondola A. **1 mark**

20 minutes

ii the radius of the Ferris wheel. **1 mark**

48 m

iii the maximum and minimum heights above the ground reached by the gondolas as the Ferris wheel rotates. **2 marks**

Maximum height is 98 m; minimum height is 2 m.

c After it has left the platform, at what times, to the nearest minute, is gondola A at 74 m above the ground during its first revolution? **3 marks**

When $t = 0$, gondola A is at the platform. ($y = 50 - 48\cos(0) = 50 - 48 = 2$.)
Using CAS: SOLVE $50 - 48\cos\left(\frac{\pi}{10}t\right) = 74$, $0 \leq t \leq 20$. $t = 6.66\ldots, 13.333\ldots$
Gondola A is 74 m above the ground $6\frac{2}{3}$ min and $13\frac{1}{3}$ min after it leaves the platform.

d Write an equation for $\frac{dy}{dt}$, the vertical velocity of gondola A. **1 mark**

Using CAS: $\frac{dy}{dt} = \frac{24\pi}{5}\sin\left(\frac{\pi}{10}t\right)$

e Find the vertical velocity of gondola A when $t = 8$ and $t = 12$. Comment on these answers. **2 marks**

$t = 8$, $\frac{dy}{dt} = 8.9$ m/minutes. $t = 12$, $\frac{dy}{dt} = -8.9$ m/minutes.

The positive velocity, when $t = 8$, means the gondola is moving upwards, while the negative value, when $t = 12$, means the gondola is moving downwards.

QUESTION 21 **Total 9 marks**

A radioactive substance decays according to the rule $M = A \times 2^{-k \times t}$, where M is the mass of the substance in grams at time t years and k is a positive constant.

a Initially the mass of the substance is 3.5 g. Find A. **1 mark**

$3.5 = A \times 2^0$

$A = 3.5$

b After 50 years, the mass of the substance is 2.5 grams. Use an algebraic method to show that $k = \frac{1}{50} \log_2\left(\frac{7}{5}\right)$. Hence give an approximate value for k, correct to four decimal places. **4 marks**

$$2.5 = 3.5 \times 2^{-50k}$$

$$2^{-50k} = \frac{2.5}{3.5} = \frac{5}{7}$$

$$-50k = \log_2\left(\frac{5}{7}\right)$$

$$k = -\frac{1}{50} \log_2\left(\frac{5}{7}\right)$$

$$= \frac{1}{50} \log_2\left(\frac{5}{7}\right)^{-1}$$

$$= \frac{1}{50} \log_2\left(\frac{7}{5}\right)$$

Using CAS: using change of base rule $k = \frac{1}{50} \times \frac{\log_{10}\left(\frac{7}{5}\right)}{\log_{10}(2)} = 0.0097$

c Using this approximate value for k, find the mass of substance remaining after 30 years, correct to one decimal place. **1 mark**

$$M = 3.5 \times 2^{-0.0097 \times 30}$$

$$= 2.86$$

$$= 2.9 \text{ grams}$$

d Find the time it takes for the substance to decay to 1.75 g, correct to the nearest year. **1 mark**

$$1.75 = 3.5 \times 2^{-0.0097 \times t}$$

$$2^{-0.0097t} = \frac{1}{2} = 2^{-1}$$

$$-0.0097t = -1$$

$$t = \frac{1}{0.0097}$$

$$= 103.09$$

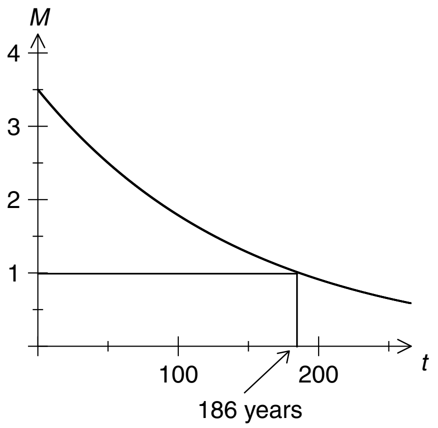
$$= 103 \text{ years}$$

Or using CAS: SOLVE $1.75 = 3.5 \times 2^{-0.0097 \times t}$

e Find the values of t such that $3.5 \times 2^{-0.0097t} > 1$. Give answers correct to the nearest whole number. **2 marks**

Using CAS: SOLVE $1 = 3.5 \times 2^{-0.0097t}$

$t = 186$ years



Or from the graph, find the point of intersection between the graphs of $y = 3.5 \times 2^{-0.0097t}$ and $y = 1$, $t < 186$ years.

QUESTION 22

Total 14 marks

A family of graphs have the equation $y = ax^2(x + k)(x - 1)$, where a and k are real numbers.

Let $f(x)$ be the particular equation where $a = 10$ and $k = 1$.

a Write the equation of $f(x)$. **1 mark**

$f(x) = 10x^2(x + 1)(x - 1)$

b Write the equation of $f'(x)$, the derivative of $f(x)$, and hence find the exact values of any stationary points of the graph of $f(x)$. State the nature of these stationary points and sketch the graph of $f(x)$. **4 marks**

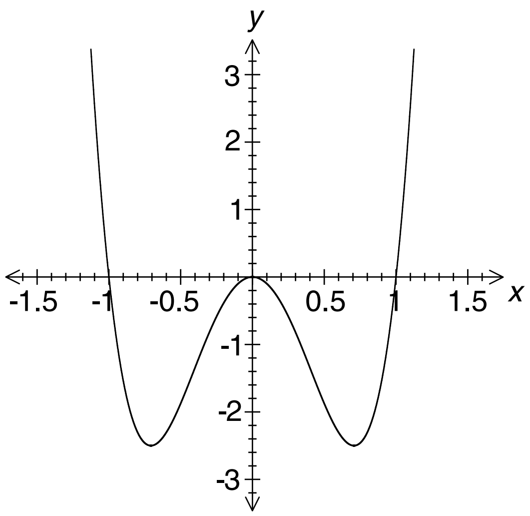
Equation of derivative: Using CAS: $f'(x) = 20x(2x^2 - 1)$

Stationary points: Using CAS: SOLVE $20x(2x^2 - 1) = 0 \Rightarrow x = 0, \pm\frac{\sqrt{2}}{2}$

CAS: $f(0) = 0$, local maximum

$f\left(\frac{\sqrt{2}}{2}\right) = -2\frac{1}{2}$, minimum

$f\left(-\frac{\sqrt{2}}{2}\right) = -2\frac{1}{2}$, minimum



c Find the area enclosed between the graph of $f(x)$, the x axis and $x = 0$ to $x = 2$. **2 marks**

Using CAS: $\int_1^2 10x^2(x + 1)(x - 1)dx - \int_0^1 10x^2(x + 1)(x - 1)dx = 40$ units^2

Let $g(x)$ be the family of graphs of $y = ax^2(x + k)(x - 1)$ where $a = 10$.

d Write the equation of $g'(x)$ in terms of x and k. **1 mark**

Using CAS: $g'(x) = 10x(4x^2 + 3(k - 1)x - 2k)$

e In terms of k, write the values of x for the stationary points of $g(x)$. **1 mark**

Using CAS: solve $10x(4x^2 + 3(k - 1)x - 2k) = 0$

$x = 0, \ x = \dfrac{3(1 - k) \pm \sqrt{9k^2 + 14k + 9}}{8}$

f Write the x values of the stationary points when $k = 0$. **1 mark**

Stationary points: $x = 0, \frac{3}{4}$

g State the nature of the stationary points. **2 marks**

$x = 0$ stationary point of inflection $x = \frac{3}{4}$ minimum.

h Find the stationary points for $g(x)$ when $k = -1$. **2 marks**

If $k = -1$, $x = 0$ and $x = \dfrac{(3 + 1) \pm \sqrt{4}}{8} = 1$. Coordinates are $(0, 0)$, $\left(\frac{1}{2}, \frac{5}{8}\right)$, $(1, 0)$.

Mathematical Methods (CAS) Formulas

Mensuration

area of trapezium:	$\frac{1}{2}(a + b)h$
curved surface area of a cylinder:	$2\pi rh$
volume of a cylinder:	$\pi r^2 h$
volume of a cone:	$\frac{1}{3}\pi r^2 \mathrm{h}$
volume of a pyramid:	$\frac{1}{3}Ah$
volume of a sphere:	$\frac{4}{3}\pi r^3$
area of a triangle:	$\frac{1}{2}bc \sin A$

Matrix algebra

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$A^{-1} = \frac{1}{\triangle}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\triangle = ad - bc$$

Calculus

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$$\int x^n \, dx = \frac{1}{n+1}x^{n+1} + c, n \neq -1$$

Probability

$$\Pr(A) = 1 - \Pr(A')$$

$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$$