



THE SCHOOL FOR EXCELLENCE
UNITS 3&4 MATHEMATICAL METHODS 2007
COMPLIMENTARY WRITTEN EXAMINATION 2 - SOLUTIONS

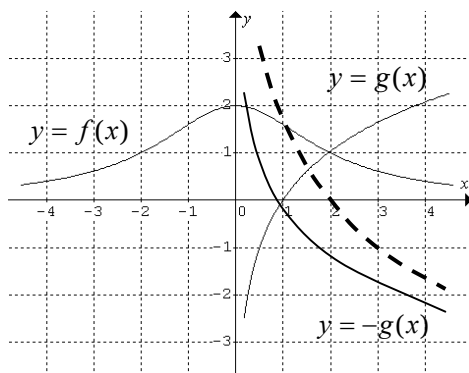
SECTION 1 – MULTIPLE CHOICE QUESTIONS

1	2	3	4	5	6	7	8	9	10	11
A	C	B	E	A	A	C	B	D	E	D

12	13	14	15	16	17	18	19	20	21	22
B	A	D	D	A	C	D	C	B	C	E

QUESTION 1

Subtract ordinates across the common domain i.e. $(0, \infty)$. Alternatively, draw the graph of $y = -g(x)$ then add the ordinates of $y = f(x)$ and $y = -g(x)$ across the common domain.

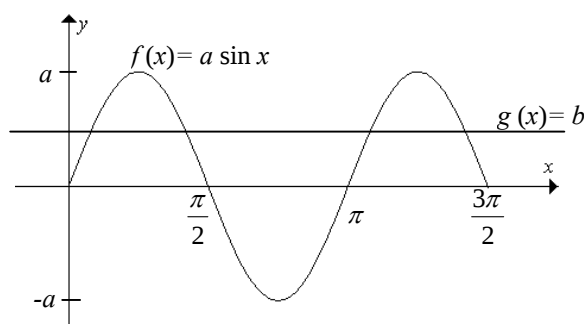


The answer is A.

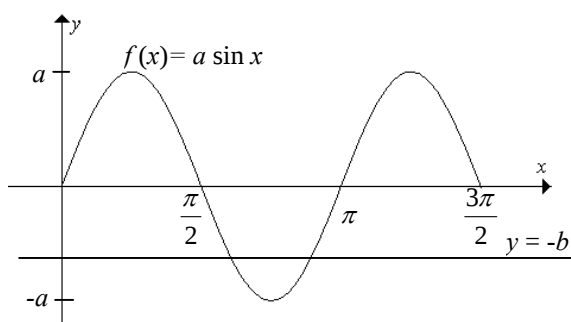
QUESTION 2

Option A: $b > a \Rightarrow \frac{b}{a} > 1$ therefore $\sin(2x) = \frac{b}{a}$ has no real solution (as $\frac{b}{a}$ must lie between -1 and 1 inclusive. Therefore option A is true.

Option B: If $0 < b < a$, it can be seen from the graph below that there are 4 solutions to $\sin(2x) = \frac{b}{a}$ for all possible values of a and b . Therefore option B is true.



Option C: If $0 < b < a$, it can be seen from the graph below that there are 2 solutions to $a \sin(2x) = -b$. Therefore option C is **not** true.



Option D: $f(x) = -g(x) \Rightarrow a \sin(2x) = -b$.

$$a = b \Rightarrow \sin(2x) = -1 \quad \therefore 2x = \frac{3\pi}{2} + 2n\pi, \quad n \in \mathbb{Z} \quad \therefore x = \frac{3\pi}{4} + n\pi$$

There is only 1 solution for $0 < x < \frac{3\pi}{2}$. Therefore option D is true.

Option E: $f(x) = g(x) \Rightarrow a \sin(2x) = b$

$$a = b \Rightarrow \sin(2x) = 1 \quad \therefore 2x = \frac{\pi}{2} + 2n\pi, \quad n \in \mathbb{Z} \quad \therefore x = \frac{\pi}{4} + n\pi$$

There are 2 solutions for $0 < x < \frac{3\pi}{2}$. Therefore option E is true.

The answer is C.

QUESTION 3

To solve $|2x-3| \geq x^2-2$, first note that $|2x-3| = \begin{cases} 2x-3 & \text{when } 2x-3 \geq 0 \Rightarrow x \geq \frac{3}{2} \\ -(2x-3) & \text{when } 2x-3 < 0 \Rightarrow x < \frac{3}{2} \end{cases}$

Case 1:

$$2x-3 \geq x^2-2, \quad x \geq \frac{3}{2},$$

$$\therefore x^2-2x+1 \leq 0$$

$$\therefore (x-1)^2 \leq 0$$

$$\therefore x=1.$$

But $x \geq \frac{3}{2}$ is not satisfied.

Therefore there is no solution.

Case 2:

$$-2x+3 \geq x^2-2, \quad x < \frac{3}{2},$$

$$\therefore x^2+2x-5 \leq 0.$$

$$\text{When } x^2+2x-5=0,$$

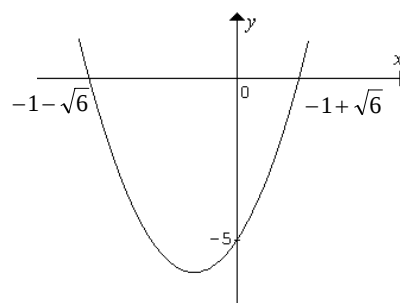
$$x = -1 \pm \sqrt{6}.$$

From the graph it can be seen that the solution to

$$\therefore x^2+2x-5 \leq 0 \text{ is therefore}$$

$$\{x: -1+\sqrt{6} \leq x \leq -1-\sqrt{6}\}.$$

Note that $x < \frac{3}{2}$ is satisfied.



Therefore the solution to $|2x-3| \geq x^2-2$ is $\{x: -1+\sqrt{6} < x < -1-\sqrt{6}\}$.

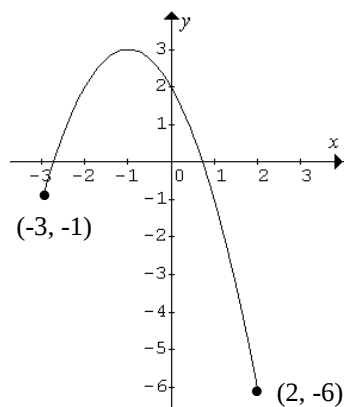
The answer is B.

QUESTION 4

From the graph of $f: [-3, 2] \rightarrow \mathbb{R}, f(x) = 3 - (x+1)^2$

the range is $[-6, 3]$.

The answer is E.



QUESTION 5

A function must be one-to-one in order to have an inverse function.

$f: [0, a] \rightarrow \mathbb{R}, f(x) = 3\cos(2x)$ is a one-to-one function for $a \leq \frac{\pi}{2}$.

The answer is A.

QUESTION 6

The graph of $y = a(x+b)^3(x+c)$ has:

- a stationary point of inflexion that is also an x -intercept at $x = -b$. This corresponds to $x = -1$ on the graph. Therefore $b = 1$.
- an x -intercept at $x = -c$. This corresponds to $x = 2$. Therefore $c = -2$.

Therefore $y = a(x+1)^3(x-2)$.

As $(0,1)$ is a point on the graph:

$$\therefore 1 = a(0+1)^3(0-2)$$

$$\therefore 1 = -2a$$

$$\therefore a = -\frac{1}{2}$$

$$\therefore a = -\frac{1}{2}, b = 1, c = -2$$

The answer is A.

QUESTION 7

Linear approximation: $f(x+h) \approx f(x) + hf'(x)$.

Let $x=1$ and $x+h=0.9$ $\therefore h=-0.1$.

$$f(x) = \frac{x-1}{\sqrt{x}} = \sqrt{x} - \frac{1}{\sqrt{x}} \text{ and } f(1) = 0.$$

$$f'(x) = \frac{1}{2} \frac{1}{\sqrt{x}} + \frac{1}{2} \frac{1}{x\sqrt{x}} \quad \therefore f'(1) = 1.$$

$$f(0.9) \approx f(1) - 0.1f'(1)$$

$$\approx 0 - 0.1(1)$$

$$\approx -0.1$$

$$\text{Change in } f = f(0.9) - f(1)$$

$$= -0.1 - 0$$

$$= -0.1$$

The answer is C.

QUESTION 8

Amplitude = 4

$$\text{Period} = \frac{1}{8} = \frac{2\pi}{n}$$

$$\therefore n = 16\pi$$

A. $f(t) = 2 \sin\left(\frac{\pi t}{4}\right)$ is incorrect as the period = 8

B. $f(t) = 4 \sin(16\pi t)$ is correct.

C. $f(t) = 4 \sin(8\pi t)$ is incorrect as period = $\frac{1}{4}$

D. $f(t) = 4 \sin\left(\frac{\pi t}{4}\right)$ is incorrect as period = 8

E. $f(t) = 2 \sin\left(\frac{\pi t}{8}\right)$ is incorrect as the period = 16

The answer is B.

QUESTION 9

$$f(x) = x + \sqrt{x+a}$$

$$\therefore f'(x) = 1 + \frac{1}{2\sqrt{x+a}}$$

For stationary points $f'(x) = 0$:

$$0 = 1 + \frac{1}{2\sqrt{x+a}}$$

$$\therefore \frac{1}{2\sqrt{x+a}} = -1$$

$$\therefore \sqrt{x+a} = -\frac{1}{2}.$$

This equation has no real solutions.

The answer is D.

QUESTION 10

$$\log_e x = \log_e (x-1) + b$$

$$\log_e x - \log_e (x-1) = b$$

$$\log_e \left(\frac{x}{x-1} \right) = b$$

$$\frac{x}{x-1} = e^b$$

$$x = e^b (x-1)$$

$$x(1 - e^b) = -e^b$$

$$x = \frac{e^b}{e^b - 1}$$

The answer is E.

QUESTION 11

$$f(x) = a \log_e (bx - c)$$

$$f(x) = a \log_e b \left(x - \frac{c}{b} \right)$$

The function has a vertical asymptote at $x = \frac{c}{b}$, therefore the domain is $\left(\frac{c}{b}, \infty \right)$.

The answer is D.

QUESTION 12

The answer is B.

QUESTION 13

The graph of f has a negative gradient throughout its domain. Therefore A and D are the only possibilities. As x increases the gradient approaches zero. Therefore A is most likely to be the correct graph.

The answer is A.

QUESTION 14

$$y = x^n e^{3x-n}$$

$$\frac{dy}{dx} = nx^{n-1}e^{3x-n} + 3x^n e^{3x-n}$$

$$= x^{n-1}e^{3x-n}(n+3)$$

$$\text{At the point, } (1, e^{3-n}), \frac{dy}{dx} = 1^{n-1}e^{3-n}(n+3)$$

$$= (n+3)e^{3-n}.$$

The answer is D.

QUESTION 15

$$\text{Using the chain rule: } \frac{d}{dx}(\log_e \sqrt{2x^2+1}) = \frac{2x}{2x^2+1}$$

The answer is D.

QUESTION 16

$$\begin{aligned} \int_a^b 1 - 3f(x) dx &= \int_a^b 1 dx - 3 \int_a^b f(x) dx \\ &= [x]_a^b - 3(1) \\ &= b - a - 3 \end{aligned}$$

The answer is A.

QUESTION 17

$$\text{Area} = - \int_{-a}^0 f(x) dx + \int_0^a f(x) dx - \int_a^c f(x) dx$$

$$= 2 \int_0^a f(x) dx - \int_a^c f(x) dx$$

Note that, based on the graph, an assumption of symmetry is made.

$$= 2 \int_0^a f(x) dx + \int_c^a f(x) dx$$

The answer is C.

QUESTION 18

$$\frac{d}{dx}(x \log_e(3x)) = 1 + \log_e(3x)$$

Integrate both sides:

$$x \log_e(3x) = x + c_1 + \int \log_e(3x) dx$$

$$\therefore \int \log_e(3x) dx = x \log_e(3x) - x - c_1$$

$$\begin{aligned} \therefore \int 2 \log_e(3x) dx &= 2(x \log_e(3x) - x - c_2) \\ &= 2x(\log_e(3x) - 1) - c \end{aligned}$$

The answer is D.

QUESTION 19

$$\Pr(X > k) = \frac{7}{8}$$

$$\therefore \int_k^1 \frac{3\sqrt{x}}{2} dx = \frac{7}{8} \quad \therefore \left[x^{\frac{3}{2}} \right]_k^1 = \frac{7}{8} \quad \therefore 1 - k^{\frac{3}{2}} = \frac{7}{8} \quad \therefore k^{\frac{3}{2}} = \frac{1}{8} \quad k = \frac{1}{4}.$$

The answer is C.

QUESTION 20

Let X be the number of defective MP3 players in a sample of n .

X is a binomial random variable where $p = 1 - 0.98 = 0.02$ and the sample size is n .

Note: $\Pr(\text{not defective}) = 0.98$

$\Pr(\text{defective}) = 0.02$

$$\begin{aligned} \Pr(X > 1) &= 1 - \Pr(X = 0) - \Pr(X = 1) \\ &= 1 - \binom{n}{0} (0.02)^0 (0.98)^n - \binom{n}{1} (0.02)^1 (0.98)^{n-1} \\ &= 1 - (0.98)^n - n(0.02)(0.98)^{n-1} \end{aligned}$$

The answer is B.

QUESTION 21

The standard deviation $\frac{\sigma}{2}$ will halve the spread so the graph will be narrow and taller. The mean will not affect the spread or height of the graph.

Option A is incorrect as the spread is too small.

Option B is incorrect as the spread is too large.

Option C is correct as its spread is half of the original graph and it is taller.

Option D is incorrect as the graph does not represent a normal distribution.

Option E is incorrect as the spread is too large.

The answer is C.

QUESTION 22

Let X = the number of goals in n trials.

X follows a binomial distribution with $p = 0.4$ and number of trials = n .

$$\Pr(X \geq 1) \geq 0.9$$

$$\therefore 1 - \Pr(X = 0) \geq 0.9$$

$$\therefore \Pr(X = 0) \leq 0.1$$

$$\therefore \binom{n}{0}(0.4)^0(0.6)^n \leq 0.1$$

$$\therefore (0.6)^n \leq 0.1$$

Using a graphics calculator, the solution to $(0.6)^n = 0.1$ is 4.508.

Therefore n must be 5 since $(0.6)^n \leq 0.1$ is required.

The answer is E.

SECTION 2 – EXTENDED ANSWER QUESTIONS

QUESTION 1

Note that $D_g = \mathbb{R} \setminus \{1\}$.

a. (i) $f(g(x)) = f(\log_e |x-1|)$
 $= e^{2\log_e |x-1|}$ A1
 $= e^{\log_e |x-1|^2}$
 $= (x-1)^2, \quad x \in \mathbb{R} \setminus \{1\} \text{ since } D_g = \mathbb{R} \setminus \{1\}.$

(ii) $(f \circ g)'(x) = 2(x-1), \quad x \in \mathbb{R} \setminus \{1\}$
 $\therefore (f \circ g)'(-1) = 2(-1-1) = -4.$ A1

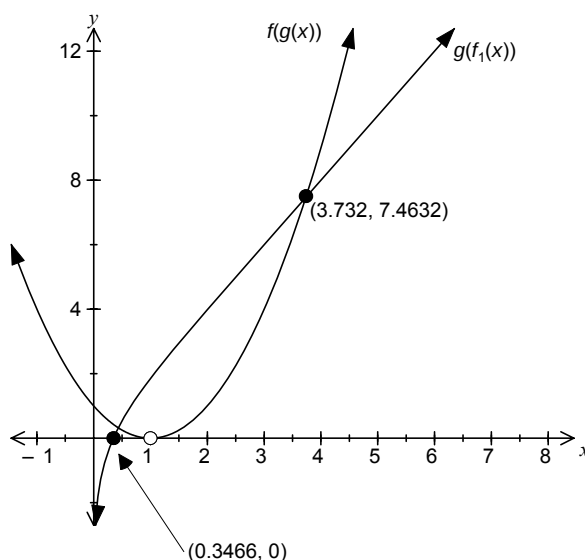
b. $\text{ran } f = (0, \infty)$ and $\text{dom } g = D_g = \mathbb{R} \setminus \{1\}$.
 $(0, \infty) \not\subset \mathbb{R} \setminus \{1\}$ Note: $\mathbb{R} \setminus \{1\} \equiv (-\infty, 1) \cup (1, +\infty)$
 $\therefore \text{ran } f \not\subset \text{dom } g$
 $\therefore f(g(x))$ does not exist. M1

c. (i) Require $\text{ran } f_1 = (1, \infty)$
 $\therefore \text{dom } f_1 = (0, \infty).$ A1
 $g(f_1(x)) = g(e^{2x}) = \log_e |e^{2x} - 1|, \quad 0 < x < \infty.$

(ii) Require $\text{ran } f_2 = (0, 1)$
 $\therefore \text{dom } f_2 = (-\infty, 0)$ A1
 $g(f_2(x)) = g(e^{2x}) = \log_e |e^{2x} - 1|, \quad -\infty < x < 0.$

d. (i) $f(g(x)) = g(f_1(x))$
 $\therefore (x-1)^2 = \log_e |e^{2x} - 1|, \quad x \neq 1.$
 $\therefore$ The coordinates of the point of intersection that lies to the right of the line $x = 1$ are (3.7, 7.5). A1

(ii)



$$\begin{aligned}\text{Area} &= \int_1^{3.732} (\log_e |e^{2x} - 1| - (x-1)^2) dx + \int_{3.732}^5 ((x-1)^2 - \log_e |e^{2x} - 1|) dx && \text{M1} \\ &\approx (12.858 - 6.797) + (14.536 - 11.072) \\ &= 6.061 + 3.464 \\ &= 9.525 \\ &= 9.5, \text{ correct to one decimal place.} && \text{A1}\end{aligned}$$

Note 1: $\int_1^{3.732} (\log_e |e^{2x} - 1| - (x-1)^2) dx$ is an improper integral since the lower integral terminal lies outside the domain of $f(g(x)) = (x-1)^2$ (see the above graph). However, the integral can be shown to exist via a limiting process, which is why the calculator gives a finite value.

Note 2: During a calculation, accuracy greater than that specified for the answer should be used so as to avoid the accumulation of rounding error.

QUESTION 2

- a. (i) $a = \text{amplitude} = \frac{1}{2} |h_{\max} - h_{\min}| = \frac{1}{2} ([18 + 2 + 100] - [18 + 2])$
 $= \frac{1}{2} (120 - 20) = 50 \text{ metres.}$ M1
- (ii) Minimum height = $18 + 2 = 20 \text{ metres} = (-\text{Amplitude} + \text{Vertical Translation})$
 $\therefore 20 = b - a$
 $\therefore 20 = b - 50$ M1
 $\therefore b = 70$

b. Period = 80 seconds = $\frac{4}{3}$ minutes

$$\therefore \frac{2\pi}{c} = \frac{4}{3}$$

$$\therefore c = 2\pi \times \frac{3}{4}$$

$$= \frac{3\pi}{2}$$

M1

c. $h = 70 + 50 \sin \frac{3\pi}{2}(t + d)$

When $t = 0$, $h = 20$

$$\therefore 20 = 70 + 50 \sin \left(\frac{3\pi d}{2} \right)$$

$$\therefore \sin \left(\frac{3\pi d}{2} \right) = -1$$

M1

$$\therefore \frac{3\pi d}{2} = \frac{3\pi}{2}$$

$\therefore d = 1$ is a possible value.

M1

d. Period = $\frac{4}{3}$ minutes.

Therefore the point P reaches the maximum height once every $\frac{4}{3}$ minutes:

$$36 \div \frac{4}{3} = 36 \times \frac{3}{4} = 27.$$

Therefore the point P reaches its maximum height 27 times during a 36 minute ride.

A1

e. $70 + 50 \sin \frac{3\pi}{2}(t + 1) \geq 95.$

First solve $70 + 50 \sin \frac{3\pi}{2}(t + 1) = 95$ where the smallest value of t satisfying

$0 \leq t \leq 36$ is required:

$$70 + 50 \sin \frac{3\pi}{2}(t + 1) = 95, \quad 0 \leq t \leq 36$$

$$\therefore \sin \frac{3\pi}{2}(t + 1) = \frac{1}{2}$$

$$\therefore \frac{3\pi}{2}(t + 1) = \frac{\pi}{6} + 2m\pi \quad \text{or} \quad \frac{5\pi}{6} + 2m\pi, \quad \text{where } m \in \mathbb{Z}.$$

M1

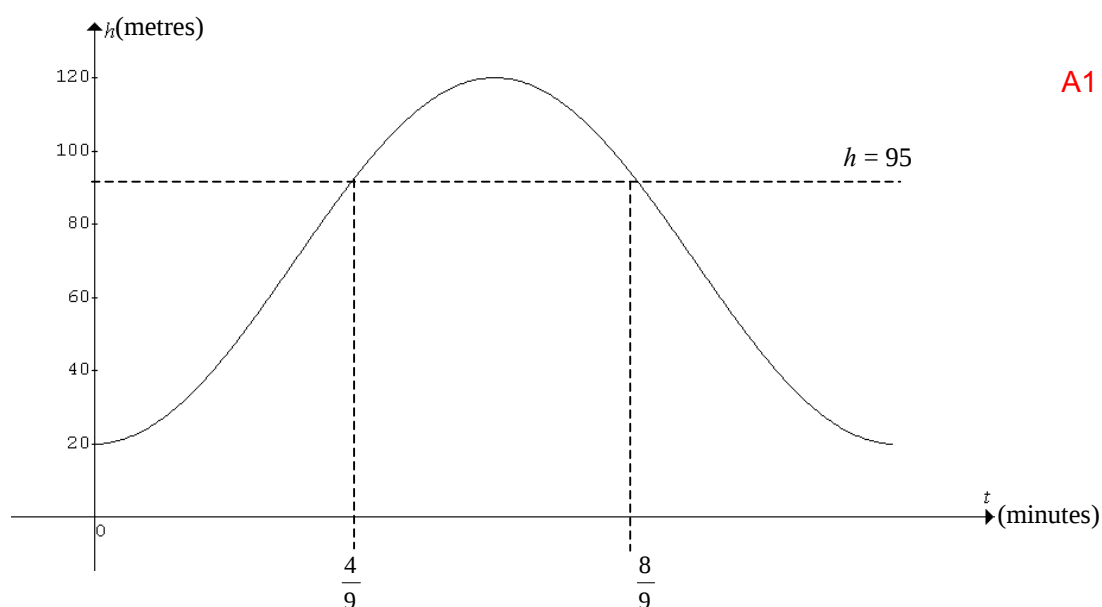
$$\therefore t+1 = \frac{1}{9} + \frac{4m}{3} \quad \text{or} \quad \frac{5}{9} + \frac{4m}{3}$$

$$\therefore t = -\frac{8}{9} + \frac{4m}{3} = \frac{12m-8}{9} \quad \text{or} \quad -\frac{4}{9} + \frac{4m}{3} = \frac{12m-4}{9}.$$

The smallest value of t satisfying $0 \leq t \leq 36$ is $t = \frac{4}{9}$ minutes. A1

Therefore the point P first reaches a height of at least 95 metres above ground level after $\frac{4}{9}$ minutes.

- f. During the first rotation P is at a height of 95 metres above the ground a second time when $t = \frac{8}{9}$ minutes (see **part e**).



From the graph it is clear that the number of minutes during one rotation that the point P is at least 95 metres above ground level is $\frac{8}{9} - \frac{4}{9} = \frac{4}{9}$ minutes:

$$\frac{4}{9} \text{ minutes} = \frac{4}{9} \times 60 \text{ seconds} = \frac{80}{3} \text{ seconds}$$

A1

g. $h = 70 + 50 \sin \frac{3\pi}{2}(t+1)$

$$\begin{aligned}\therefore \frac{dh}{dt} &= 50 \times \frac{3\pi}{2} \cos \frac{3\pi}{2}(t+1) \\ &= 75\pi \cos \frac{3\pi}{2}(t+1)\end{aligned}$$

M1

Let $\frac{dh}{dt} > 200$

$$\therefore 75\pi \cos \frac{3\pi}{2}(t+1) > 200$$

$$\therefore \cos \frac{3\pi}{2}(t+1) > \frac{8}{3\pi}$$

Enter $y = 75\pi \cos \frac{3\pi}{2}(t+1)$ and $y = \frac{8}{3\pi}$ in the **y = editor** of a graphics calculator.

Then use **2nd CALC 5: intersect** to find two consecutive solutions to

$$\cos \frac{3\pi}{2}(t+1) = \frac{8}{3\pi}.$$

$$\therefore t = 0.2151 \text{ minutes and } t = 0.4515 \text{ minutes.}$$

M1

$$0.452 - 0.215 = 0.2354 \text{ minutes} = 14.18 \text{ seconds.}$$

Therefore $\frac{dh}{dt} > 200$ m/s for 14.18 seconds, which is less than 20 seconds.

A1

The average person will not feel sick on the Southern Star Observation Wheel.

h. $h = 70 + 50 \sin c(t+1)$

$$\therefore \frac{dh}{dt} = 50c \cos(c(t+1))$$

- The smallest positive value of c such that $\frac{dh}{dt} > 200$ for no more than 10 seconds at a time and the wheel turns as quickly as possible is required.
- 10 seconds = $\frac{1}{6}$ minutes.
- Therefore the positive value of c is required such that if t_1 and $t_2 > t_1$ are two consecutive solutions to $50c \cos(c(t+1)) = 200$, then $t_1 - t_2 = \frac{1}{6}$.

M1

$$50c \cos(c(t+1)) = 200$$

$$\therefore \cos(c(t+1)) = \frac{4}{c}$$

$$\therefore c(t+1) = \pm \cos^{-1}\left(\frac{4}{c}\right)$$

$$\therefore t = \pm \frac{1}{c} \cos^{-1}\left(\frac{4}{c}\right) - 1$$

$$\therefore t_1 = -\frac{1}{c} \cos^{-1}\left(\frac{4}{c}\right) - 1 \quad \text{and} \quad t_2 = \frac{1}{c} \cos^{-1}\left(\frac{4}{c}\right) - 1$$

$$\therefore t_2 - t_1 = \frac{2}{c} \cos^{-1}\left(\frac{4}{c}\right)$$

M1

$$\therefore \frac{1}{6} = \frac{2}{c} \cos^{-1}\left(\frac{4}{c}\right)$$

$$\therefore \cos\left(\frac{c}{12}\right) = \frac{4}{c}$$

- From the graphics calculator:

$c = 4.267$, correct to three decimal places.

A1

QUESTION 3

- a. (i) When $t = 4$, $A = 9$:

$$\therefore 9 = \frac{a(4)}{3(4) + 4}$$

$$\therefore 9 = \frac{4a}{16}$$

$$\therefore 9 = \frac{a}{4}$$

$$\therefore a = 36$$

M1

(ii) $A = \frac{36t}{3t + 4}$

Using polynomial long division:

$$\begin{array}{r} 12 \\ 3t + 4 \overline{) 36t} \\ \underline{36t + 48} \\ -48 \end{array}$$

$$\therefore A = 12 - \frac{48}{3t + 4}$$

M1

$$= 12 \left(1 - \frac{4}{3t + 4} \right)$$

b. $D = A - B$

$$\therefore D = 12 \left(1 - \frac{4}{3t + 4} \right) - 12 \left(1 - \frac{3}{t + 3} \right)$$

$$\therefore D = 12 \left(\frac{3}{t + 3} - \frac{4}{3t + 4} \right)$$

M1

c. $D = 12 \left(\frac{3}{t + 3} - \frac{4}{3t + 4} \right)$

$$\therefore D = 12(3(t + 3)^{-1} - 4(3t + 4)^{-1})$$

$$\therefore \frac{dD}{dt} = 12 \left(-\frac{3}{(t + 3)^2} + \frac{12}{(3t + 4)^2} \right)$$

A2

- d. Maximum value of D occurs when $\frac{dD}{dt} = 0$:

$$0 = 12 \left(-\frac{3}{(t+3)^2} + \frac{12}{(3t+4)^2} \right), \quad 0 \leq t \leq 14.$$

M1

$$\therefore 0 = -\frac{3}{(t+3)^2} + \frac{12}{(3t+4)^2}$$

$$\therefore 0 = -\frac{1}{(t+3)^2} + \frac{4}{(3t+4)^2}$$

$$\therefore \frac{1}{(t+3)^2} = \frac{4}{(3t+4)^2}$$

$$\therefore (3t+4)^2 = 4(t+3)^2$$

$$\therefore 3t+4 = 2(t+3),$$

$$\therefore 3t+4 = 2t+6$$

$$\therefore t = 2$$

M1

$$\text{When } t = 2: D = 12 \left(\frac{3}{2+3} - \frac{4}{6+4} \right) = 2.4$$

Hence the maximum difference in new influenza cases between those previously unexposed and those exposed to the virus is $2.4 \times 100 = 240$.

A1

Note: D is measured in hundreds.

e. $C(t) = 4(t^2 + t)e^{-\frac{t}{k}}$

$$\therefore C'(t) = 4 \left((2t+1)e^{-\frac{t}{k}} - \frac{1}{k}e^{-\frac{t}{k}}(t^2 + t) \right)$$

$$= 4e^{-\frac{t}{k}} \left(2t+1 - \frac{1}{k}(t^2 + t) \right)$$

M1

For maximum concentration $C'(t) = 0$:

$$\therefore 2t+1 - \frac{1}{k}(t^2 + t) = 0$$

or

$$e^{-\frac{t}{k}} = 0$$

M1

$$\therefore 2tk + k - t^2 - t = 0$$

No solution

$$\text{When } t = 4, C'(t) = 0: 8k + k - 16 - 4 = 0$$

$$\therefore 9k = 20$$

$$\therefore k = \frac{20}{9}$$

A1

f. Let $A = C'(t)$

Substitute $k = \frac{20}{9}$:

$$\begin{aligned}\therefore A &= 4e^{-\frac{9t}{20}} \left(2t + 1 - \frac{9}{20}(t^2 + t) \right) \\ &= 4e^{-\frac{9t}{20}} \left(\frac{31t}{20} + 1 - \frac{9t^2}{20} \right) \\ &= \frac{1}{5}e^{-\frac{9t}{20}} (31t + 20 - 9t^2)\end{aligned}$$

$$\frac{dA}{dt} = \frac{1}{5} \left(-\frac{9}{20}e^{-\frac{9t}{20}} (31t + 20 - 9t^2) + e^{-\frac{9t}{20}} (31 - 18t) \right)$$

M1

For maximum absorption $\frac{dA}{dt} = 0$:

$$0 = \frac{1}{5} \left(-\frac{9}{20}e^{-\frac{9t}{20}} (31t + 20 - 9t^2) + e^{-\frac{9t}{20}} (31 - 18t) \right)$$

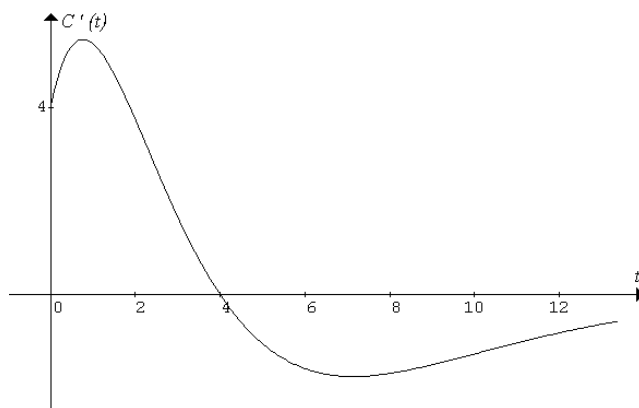
$$\therefore 0 = -\frac{9}{20} (31t + 20 - 9t^2) + 31 - 18t$$

$$\therefore t = 0.76 \text{ or } 7.13$$

M1

From the graph of $C'(t)$ below it can be seen that the maximum rate of absorption occurs after 0.76 hours.

A1



QUESTION 4

a. (i) $\Pr(5 < T < 15) = 0.98758 \approx 0.9876$. A1

(ii) $\Pr(T > 15) = 0.00621 \approx 0.0062$. A1

(iii) $\Pr(T > 5 | T < 15) = \frac{\Pr(T > 5 \cap T < 15)}{\Pr(T < 15)}$ M1

$$= \frac{\Pr(5 < T < 15)}{\Pr(T < 15)}$$
$$= \frac{0.98758}{1 - 0.00621}$$
$$= 0.9938$$
 A1

Note: During a calculation, accuracy greater than that specified for the answer should be used so as to avoid the accumulation of rounding error.

b. $\Pr(T > k) = 0.0095$

$$\therefore \Pr(T < k) = 1 - 0.0095 = 0.9905$$

$$\therefore k = \text{invnorm}(0.9915, 10, 2)$$
 M1

$$= 14.69$$

$$= 15 \text{ correct to the nearest minute.}$$
 A1

- c. (i) Let Y be the random variable “number of trains which arrive more than 15 minutes late”.

$$Y \sim \text{Binomial}(n = 10, p = \Pr(T > 15) = 0.00621).$$
 M1

$$\Pr(Y = 2) = \binom{10}{2} (0.00621)^2 (1 - 0.00621)^8 = 0.0017, \text{ correct to four decimal places.}$$

$$\text{Alternatively: } \Pr(Y = 2) = \text{binompdf}(10, 0.00621, 2) = 0.0017.$$
 A1

- (ii) $\Pr(\text{first train is more than 15 minutes late and the last 9 are not})$

$$= p(1 - p)^9$$

$$= 0.00621(1 - 0.00621)^9$$

$$= 0.0059, \text{ correct to four decimal places.}$$
 A1

d. $E(Y) = np = 200 \times 0.0062 = 1.24$

Therefore one train out of 200 will be more than 15 minutes late. A1

e. Let $X \sim \text{Normal}(\mu, \sigma^2)$

$$\Pr(X > 5) = 0.0095$$

$$\Pr(X < 4) = 0.205$$

$$\therefore \Pr(X < 5) = 0.9905$$

$$\therefore \Pr(Z < k_2) = 0.205$$

$$\therefore \Pr(Z < k_1) = 0.9905$$

$$\text{Where } k_2 = \text{invnorm}(0.205) = -0.82389$$

$$\text{Where } k_1 = \text{invnorm}(0.9905) = 2.34553$$

$$\therefore \frac{4 - \mu}{\sigma} = -0.82389 \quad \text{M1}$$

$$\therefore \frac{5 - \mu}{\sigma} = 2.34553 \quad \text{M1}$$

$$\therefore -0.82389\sigma = 4 - \mu \dots\dots\dots(2)$$

$$\therefore 2.34553\sigma = 5 - \mu \dots\dots\dots(1)$$

$$(1) - (2): 3.16942\sigma = 1$$

$$\therefore \sigma = 0.315515 \approx 0.3155. \quad \text{A1}$$

$$\text{From (1): } \mu = 4.2600, \text{ correct to four decimal places.} \quad \text{A1}$$

f. (i) $\Pr(X > 3) = 0.04 \int_3^5 x dx + 0.04 \int_5^{10} (10 - x) dx \quad \text{M1}$

$$= 0.04 \left[\frac{x^2}{2} \right]_3^5 + 0.04 \left[10x - \frac{x^2}{2} \right]_5^{10}$$

$$= 0.04 \left(\frac{25}{2} - \frac{9}{2} \right) + 0.04 \left(100 - 50 - \left(50 - \frac{25}{2} \right) \right)$$

$$= 0.8 \quad \text{A1}$$

(ii) $E(X) = 0.04 \int_0^5 x^2 dx + 0.04 \int_5^{10} (10x - x^2) dx \quad \text{M1}$

$$= 0.04 \left[\frac{x^3}{3} \right]_0^5 + 0.04 \left[5x^2 - \frac{x^3}{3} \right]_5^{10}$$

$$= 5 \text{ minutes.} \quad \text{A1}$$

$$(iii) \Pr(X > k) = 0.8$$

$$\therefore \Pr(X < k) = 0.2$$

$$\therefore 0.04 \int_0^5 x dx = 0.2$$

M1

$$\therefore \left[\frac{x^2}{2} \right]_0^k = 5$$

$$\therefore \frac{k^2}{2} = 5$$

$$\therefore k^2 = 10$$

$$\therefore k = \sqrt{10}$$

A1