



THE SCHOOL FOR EXCELLENCE
UNIT 3 & 4 MATHEMATICAL METHODS 2007
COMPLIMENTARY WRITTEN EXAMINATION 1 - SOLUTIONS

QUESTION 1

a. $f(x) = (x^2 + 2x + 2)e^{-x}$
 $f'(x) = (2x + 2)e^{-x} - e^{-x}(x^2 + 2x + 2)$

M1

$$= e^{-x}(2x + 2 - x^2 - 2x - 2)$$

$$= -x^2 e^{-x}$$

$$= -\frac{x^2}{e^x}$$

A1

b. Stationary points occur when $f'(x) = 0$.

$$-\frac{x^2}{e^x} = 0$$

$$\therefore x = 0$$

$$f(0) = 2$$

$\therefore$ The coordinates of the stationary point are $(0, 2)$.

A1

c.

$x < 0$	$x = 0$	$x > 0$
eg. $f'(-1) = -\frac{1}{e^{-1}}$ $\therefore f'(x) < 0$	$f'(0) = 0$	eg. $f'(2) = -\frac{4}{e^2}$ $\therefore f'(x) < 0$

$\therefore$ The point $(0, 2)$ is a stationary point of inflexion.

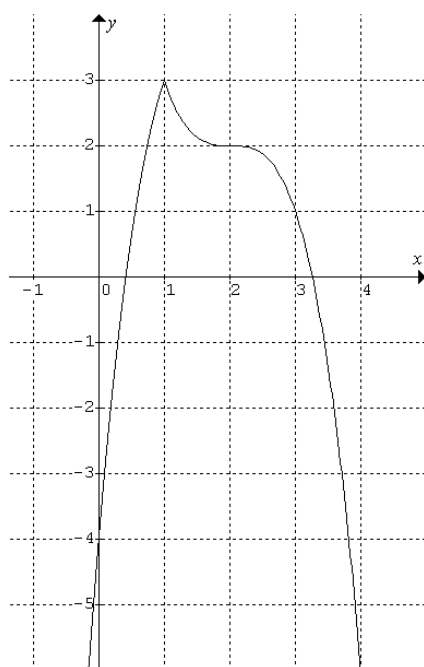
A1

QUESTION 2

a. $f(x) = 3 - |(x-2)^3 + 1|$

$$= \begin{cases} 4 + (x-2)^3, & x \leq 1 \\ 2 - (x-2)^3, & x > 1 \end{cases}$$

M1



A1

b. $f'(x) = \begin{cases} 3(x-2)^2, & x < 1 \\ -3(x-2)^2, & x > 1 \end{cases}$

M1

Note: The derivative does not exist at sharp corners i.e. at $x = 1$.

A1

c. (i) Since $x = \frac{1}{2} < 1$, use $f'(x) = 3(x-2)^2$:

$$f'\left(\frac{1}{2}\right) = 3\left(\frac{1}{2} - 2\right)^2 = \frac{27}{4}$$

A1

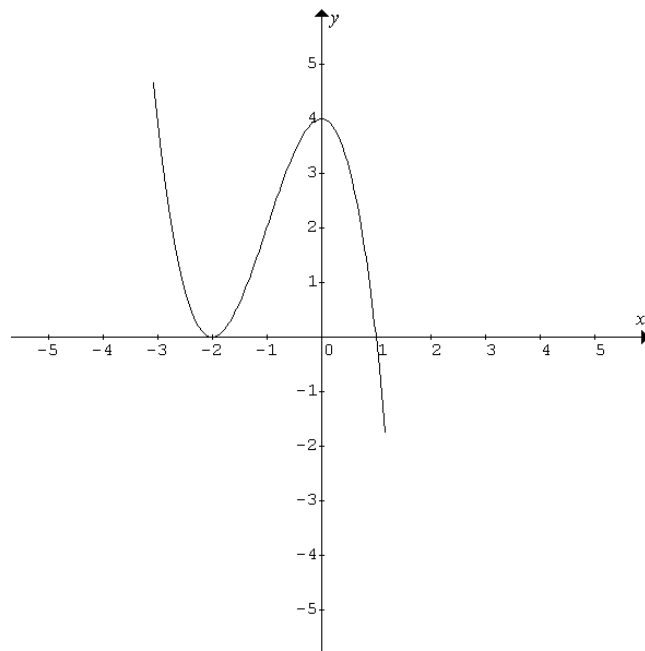
(ii) $f'(1)$ does not exist

A1

$f'(x)$ is undefined at $x = 1$ as the graph has a cusp at this point.

QUESTION 3

a.



Shape M1
Turning points M1
and axes intercepts M1

b. Model: $y = a(x + 2)^2(x - 1)$

The $(x + 2)^2$ comes from there being an x-intercept and also a turning point at $x = -2$.
The $(x - 1)$ comes from there being an x-intercept at $x = 1$.

As $f(0) = 4$:

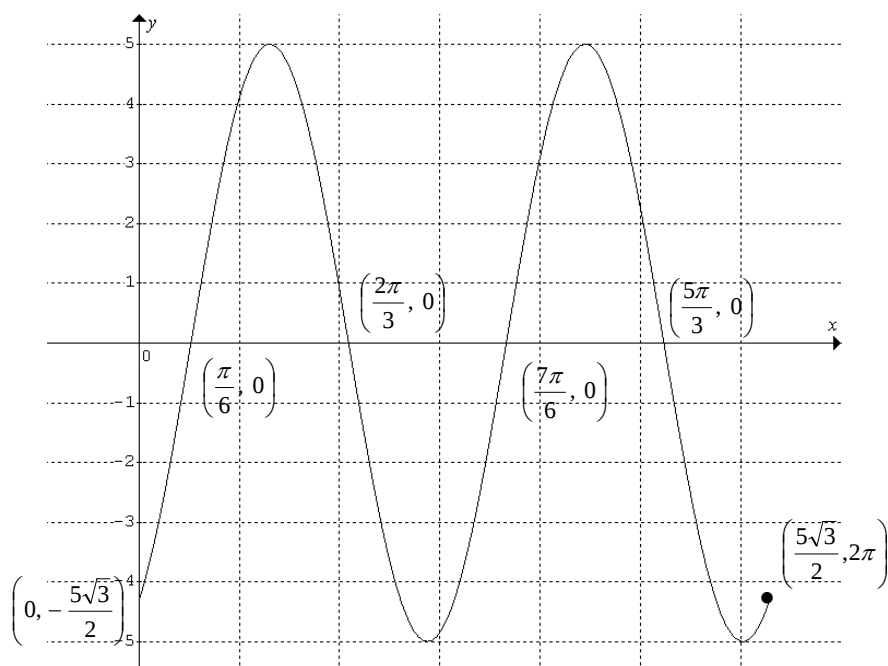
Substitute $(0, 4)$ into the model:

$$4 = a(0 + 2)^2(0 - 1)$$

$$\therefore a = -1$$

Therefore: $a = -1$, $b = 2$ and $c = -1$.

QUESTION 4



X intercepts, Let $y = 0$:

Shape
Axes intercepts
Endpoints

M1
M2
M1

$$y = 5 \sin 2 \left(x - \frac{\pi}{6} \right)$$

$$5 \sin 2 \left(x - \frac{\pi}{6} \right) = 0$$

$$\sin 2 \left(x - \frac{\pi}{6} \right) = 0$$

$$2 \left(x - \frac{\pi}{6} \right) = 0 \quad \text{or} \quad 2 \left(x - \frac{\pi}{6} \right) = \pi$$

$$x = \frac{\pi}{6}, \frac{2\pi}{3}$$

Add period to each solution: $x = \frac{\pi}{6}, \frac{2\pi}{3}, \frac{7\pi}{6}, \frac{5\pi}{3}$

Y intercepts, Let $x = 0$:

$$y = -\frac{5\sqrt{3}}{2}$$

QUESTION 5

a. $\int_{-1}^1 (x^4 - 2x^3 + 1 + a)dx = \frac{2}{5}$

$$\left[\frac{x^5}{5} - \frac{x^4}{2} + (1+a)x \right]_{-1}^1 = \frac{2}{5}$$

M1

$$\left(\frac{1}{5} - \frac{1}{2} + 1 + a \right) - \left(-\frac{1}{5} - \frac{1}{2} - 1 - a \right) = \frac{2}{5}$$

$$\frac{2}{5} + 2 + 2a = \frac{2}{5}$$

$$a = -1$$

A1

b. $\int_{-1}^0 (x^4 - 2x^3)dx - \int_0^1 (x^4 - 2x^3)dx$

M1

$$= \left[\frac{x^5}{5} - \frac{x^4}{2} \right]_{-1}^0 - \left[\frac{x^5}{5} - \frac{x^4}{2} \right]_0^1$$

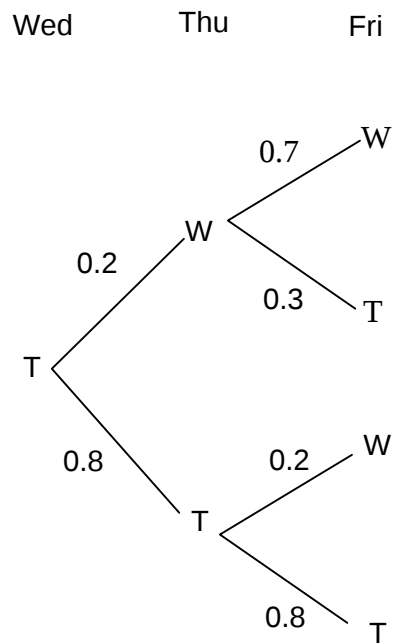
$$= 0 - \left(-\frac{1}{5} - \frac{1}{2} \right) - \left(\left(\frac{1}{5} - \frac{1}{2} \right) - 0 \right)$$

$$= 1 \text{ square units}$$

A1

QUESTION 6

- a. Let T = Tram
Let W = Walk



M2

- b. $\Pr(\text{Walks to school on Friday}) = (0.2)(0.7) + (0.8)(0.2) = 0.30$

A1

QUESTION 7

a. $\Pr(X > 36) = 1 - \Pr(X < 36)$

$$= 1 - \Pr\left(Z < \frac{36 - 22}{7}\right)$$

$$= 1 - \Pr(Z < 2)$$

$$= 1 - 0.98$$

$$= 0.02$$

A1

b. $\Pr(X < 8 | X < 22) = \frac{\Pr(X < 8 \text{ and } X < 22)}{\Pr(X < 22)}$

$$= \frac{\Pr(X < 8)}{\Pr(X < 22)}$$

$$= \frac{\Pr\left(Z < \frac{8 - 22}{7}\right)}{\Pr\left(Z < \frac{22 - 22}{7}\right)}$$

$$= \frac{\Pr(Z < -2)}{\Pr(Z < 0)}$$

M1

$$= \frac{1 - \Pr(Z < 2)}{\Pr(Z < 0)}$$

$$= \frac{0.02}{0.5} = 0.04$$

A1

QUESTION 8

$$\frac{dA}{dt} = 15 \text{ cm}^2 / \text{min}$$

Find $\frac{dr}{dt}$ when $r = 5 \text{ cm}$.

$$A = \pi r^2 \quad \therefore \frac{dA}{dr} = 2\pi r$$

$$\therefore \frac{dr}{dA} = \frac{1}{2\pi r}$$

M1

Chain rule: $\frac{dr}{dt} = \frac{dA}{dt} \times \frac{dr}{dA}$

M1

$$= 15 \times \frac{1}{2\pi r} = \frac{15}{2\pi r}$$

When $r = 5$, $\frac{dr}{dt} = \frac{15}{10\pi} = \frac{3}{2\pi} \text{ cm/min}$

A1

QUESTION 9

a. (i) Let $y = f^{-1}(x)$.

$$\text{Then: } x = \frac{1}{2} \log_e(y-1)$$

M1

$$\therefore 2x = \log_e(y-1)$$

$$\therefore y-1 = e^{2x}$$

$$\therefore y = e^{2x} + 1$$

$$\therefore f^{-1}(x) = e^{2x} + 1$$

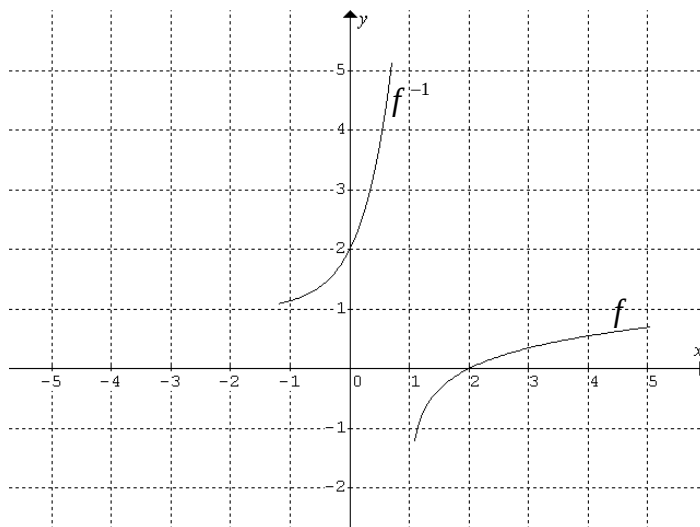
A1

(ii) $\text{dom } f^{-1} = \text{ran } f = R$

$$\text{ran } f^{-1} = \text{dom } f = (1, \infty)$$

A1

b.



A2

QUESTION 10

a. $k \int_{-a}^a (a - x) dx = 1$ M1

$$\therefore k \left[ax - \frac{x^2}{2} \right]_{-a}^a = 1$$

$$\therefore k \left(a^2 - \frac{a^2}{2} - \left(-a^2 - \frac{a^2}{2} \right) \right) = 1$$

$$\therefore k(2a^2) = 1$$

$$\therefore k = \frac{1}{2a^2} \quad \text{M1}$$

b. $E(X) = \frac{1}{2a^2} \int_{-a}^a x(a - x) dx$

$$\therefore -2 = \frac{1}{2a^2} \int_{-a}^a x(a - x) dx \quad \text{M1}$$

$$\therefore -2 = \frac{1}{2a^2} \left[\frac{ax^2}{2} - \frac{x^3}{3} \right]_{-a}^a$$

$$\therefore -2 = \frac{1}{2a^2} \left(\frac{a^3}{2} - \frac{a^3}{3} - \left(\frac{a^3}{2} + \frac{a^3}{3} \right) \right)$$

$$\therefore -2 = -\frac{1}{2a^2} \left(\frac{2a^3}{3} \right)$$

$$\therefore -2 = -\frac{a}{3}$$

$$\therefore a = 6 \quad \text{M1}$$