

VCE Mathematical Methods Units 3 & 4

Written Examination 2

Suggested Solutions

SECTION 1

1	A	B	C	D	E
2	A	B	C	D	E
3	A	B	C	D	E
4	A	B	C	D	E
5	A	B	C	D	E
6	A	B	C	D	E
7	A	B	C	D	E
8	A	B	C	D	E
9	A	B	C	D	E
10	A	B	C	D	E
11	A	B	C	D	E

12	A	B	C	D	E
13	A	B	C	D	E
14	A	B	C	D	E
15	A	B	C	D	E
16	A	B	C	D	E
17	A	B	C	D	E
18	A	B	C	D	E
19	A	B	C	D	E
20	A	B	C	D	E
21	A	B	C	D	E
22	A	B	C	D	E

SECTION 1**Question 1**

$$\begin{aligned}
 y &= (2x + 4)^2 \\
 &= (2(x + 2))^2 \\
 &= 4(x + 2)^2
 \end{aligned}$$

$$y = a(x - h)^2 + k$$

$a = 4 \Rightarrow$ a dilation from the x -axis by a factor of 4.

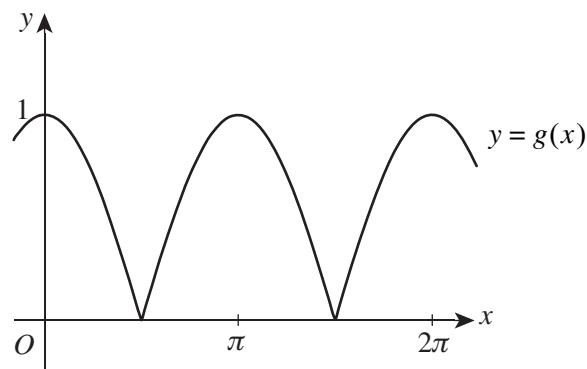
$h = -2 \Rightarrow$ a translation of 2 units to the left.

Answer D

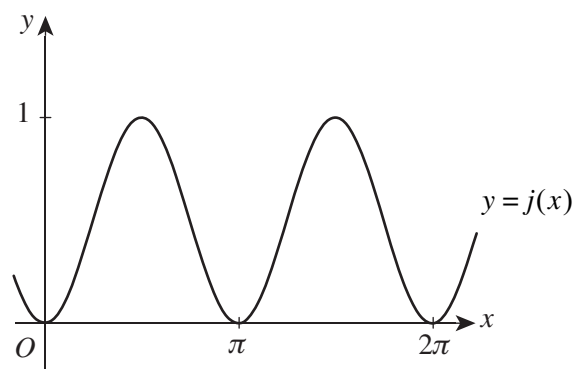
Question 2

$f(x) = \sin\left(\frac{x}{2} + \pi\right)$ has a period of 2π .

$g(x) = |\cos(x)|$ has a period of π , as may be seen from the graph below.



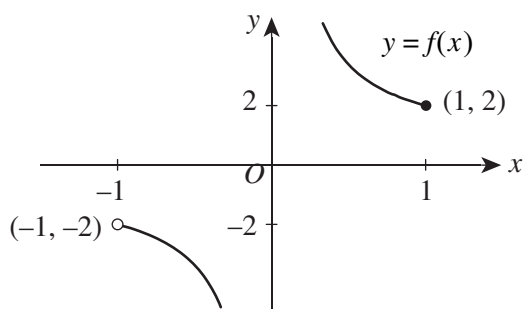
$j(x) = \sin^2(x)$ also has a period of π , as may be seen from the graph below.



$k(x) = 1 - \pi \cos\left(2\left(\frac{\pi}{8} - x\right)\right)$ has a period of $\frac{2\pi}{2} = \pi$.

Thus each of the three functions g , j and k has a period of π .

Answer E

Question 3

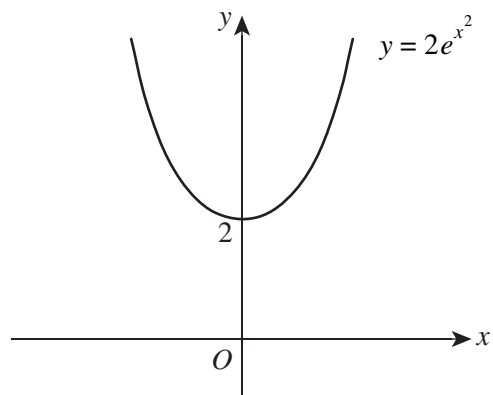
From the graph of $f(x)$ above it can be seen that its range is $\mathbb{R} \setminus [-2, 2)$.

Answer E

Question 4

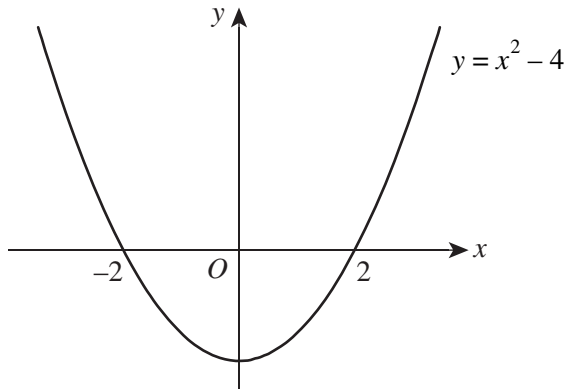
$$\begin{aligned}
 \log_e \left(\frac{1}{m^2 - n^2} \right) &= \log_e (m^2 - n^2)^{-1} \\
 &= -\log_e (m^2 - n^2) \\
 &= -\log_e ((m - n)(m + n)) \\
 &= -\log_e (m - n) - \log_e (m + n)
 \end{aligned}$$

Answer B

Question 5

For an inverse function to exist f must be a one-to-one function over the domain $(-\infty, a)$. Hence **A** is the only option that can be correct.

Answer A

Question 6

$$\begin{aligned}\sqrt{x^2 - 4} &\neq 0 \\ \Rightarrow x^2 - 4 &> 0 \\ \Rightarrow x < -2 \text{ or } x > 2\end{aligned}$$

Answer D**Question 7**

$y = \log_e |x|$ becomes $y = \log_e |3x|$ after a dilation of a factor of $\frac{1}{3}$ from the y-axis.

$y = \log_e |3x|$ becomes $y = \log_e |3(x + 3)|$ after a translation of -3 units parallel to the x-axis.

$y = \log_e |3(x + 3)|$ becomes $y = \log_e |3(-x + 3)| = \log_e |-3x + 9|$ after a reflection in the y-axis.

Answer B**Question 8**

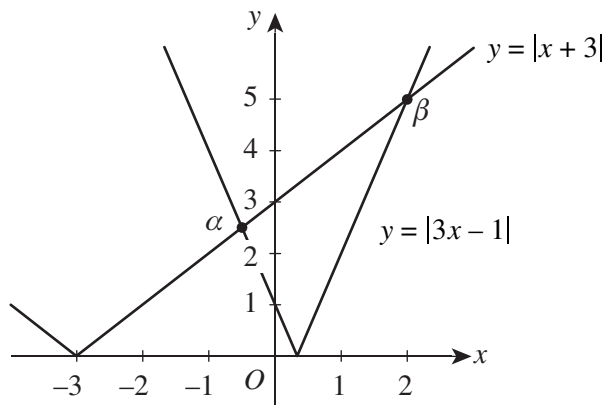
$$\begin{aligned}f(x) &= e^{\frac{1}{2}x} \\ f'(x) &= \frac{1}{2}e^{\frac{1}{2}x} \\ \therefore f'(2) &= \frac{1}{2}e^{\frac{1}{2} \times 2} \\ &= \frac{1}{2}e \\ \log_e(f'(2)) &= \log_e\left(\frac{1}{2}e\right) \\ &= \log_e\left(\frac{1}{2}\right) + \log_e(e) \\ &= -\log_e(2) + 1 \\ &= 1 - \log_e(2)\end{aligned}$$

Answer C**Question 9**

If $p(x)$ is divisible by $2x + 2$ (i.e. $2(x + 1)$), then $p(-1) = 0$.

$$\begin{aligned}\therefore -(-1)^9 - 2(-1)^7 + a(-1)^2 + 1 &= 0 \\ 1 + 2 + a + 1 &= 0 \\ a &= -4\end{aligned}$$

Answer C

Question 10

$$\alpha : x + 3 = -(3x - 1)$$

$$x + 3 = -3x + 1$$

$$4x = -2$$

$$x = -\frac{1}{2}$$

$$\beta : x + 3 = 3x - 1$$

$$-2x = -4$$

$$x = 2$$

So $|3x - 1| \geq |x + 3|$ when $x \leq -\frac{1}{2}$ or $x \geq 2$.

The points of intersection can also be found using a graphics calculator.

Answer A

Question 11

$$\Pr(\text{even}) = 0.5$$

X : number of even numbers recorded in 8 rolls

$$X \sim \text{Bi}(8, 0.5)$$

$$E(X) = 8 \times 0.5$$

$$= 4$$

$$\Pr(X \leq 4) = \binom{8}{0}(0.5)^4(0.5)^4 + \binom{8}{1}(0.5)^8 + \binom{8}{2}(0.5)^8 + \binom{8}{3}(0.5)^8 + \binom{8}{4}(0.5)^8$$

$$= 0.6367$$

Or, using a graphics calculator, the answer may be found from `bincdf(8, 0.5, 4)`.

Answer D

Question 12

$$\text{ran } f = R = \text{dom } f^{-1}$$

To find the rule for f^{-1} :

$$x = \log_e \sqrt{-y + 2}$$

$$e^x = \sqrt{-y + 2}$$

$$(e^x)^2 = -y + 2$$

$$y = -e^{2x} + 2$$

$$\therefore f^{-1} : R \rightarrow R, \text{ where } f^{-1}(x) = -e^{2x} + 2$$

Answer C

Question 13

The graphs in I could show $f(x) = \log_e(x)$ so that $f'(x) = \frac{1}{x}$, which is as shown.

The graphs in II could show $f(x) = |x|$ so that $f'(x) = \begin{cases} -1, & x < 0 \\ 1, & x > 0 \end{cases}$

The graph of f' shown looks like this except that it should not be defined at $x = 0$.

The graphs in III could be $f(x) = e^{-x} - c$ so that $f'(x) = -e^{-x}$, which is as shown.
Hence I and III are valid.

Answer D

Question 14

Option C would only necessarily be true if f were symmetrical about the y -axis. All the other options are true given the conditions stated in the question.

Answer C

Question 15

We require the area bounded by the curve $y = f(x)$ and the horizontal line $y = d$ between $x = a$ and $x = b$.

Using the rule that $\text{area} = \int (\text{upper function} - \text{lower function})dx$, the area is given by $\int_a^b (d - f(x))dx$.

$$\begin{aligned} \text{area} &= \int_a^b d dx - \int_a^b f(x) dx \\ &= [d \times x]_a^b - \int_a^b f(x) dx \\ &= d(b - a) - \int_a^b f(x) dx \end{aligned}$$

Answer B

Question 16

$$\begin{aligned} \int_2^3 \frac{3}{(x-1)(x+2)} dx &= \int_2^3 \left(\frac{1}{x-1} - \frac{1}{x+2} \right) dx \\ &= [\log_e |x-1| - \log_e |x+2|]_2^3 \\ &= \left[\log_e \left| \frac{x-1}{x+2} \right| \right]_2^3 \\ &= \log_e \left(\frac{2}{5} \right) - \log_e \left(\frac{1}{4} \right) \\ &= \log_e \left(\frac{2}{5} \times \frac{4}{1} \right) \\ &= \log_e \left(\frac{8}{5} \right) \end{aligned}$$

Answer D

Question 17

We require $\frac{d}{dx}\left(\frac{uv}{w}\right) = \frac{d}{dx}\left(\frac{u}{w} \times v\right)$.

Applying the product rule, we have $\frac{u}{w} \frac{d}{dx}(v) + v \frac{d}{dx}\left(\frac{u}{w}\right)$.

Applying the quotient rule to the second term, we have

$$\begin{aligned} \frac{u}{w} v' + v \left(\frac{wu' - uw'}{w^2} \right) &= \frac{uv'}{w} + \frac{vwu'}{w^2} - \frac{uvw'}{w^2} \\ &= \frac{vwu' + uwv' - uvw'}{w^2} \end{aligned}$$

Answer A**Question 18**

$$p + 2p + p + 0.2 = 1$$

$$4p = 0.8$$

$$p = 0.2$$

$$E(X) = 0 \times p + 1 \times 2p + 2 \times p + 4 \times 0.2$$

$$= 4p + 0.8$$

$$= 4 \times 0.2 + 0.8$$

$$= 1.6$$

Answer D**Question 19**

$$\int_0^6 A \sin\left(\frac{\pi}{6}x\right) dx = 1$$

$$A \left[-\frac{6}{\pi} \cos\left(\frac{\pi}{6}x\right) \right]_0^6 = 1$$

$$A \left[-\frac{6}{\pi} \cos(\pi) - \left(-\frac{6}{\pi} \cos(0) \right) \right] = 1$$

$$A \left(\frac{6}{\pi} + \frac{6}{\pi} \right) = 1$$

$$A \left(\frac{12}{\pi} \right) = 1$$

$$A = \frac{\pi}{12}$$

Answer B

Question 20

Let $f(x) = \frac{1}{\sqrt{x}} = x^{-\frac{1}{2}}$ so that $f'(x) = -\frac{1}{2}x^{-\frac{3}{2}}$.

Now $f(48.5) = f(49 - 0.5) \approx f(49) - 0.5f'(49)$

$$f(49) = \frac{1}{\sqrt{49}} = \frac{1}{7}$$

$$f'(49) = -\frac{1}{2(49)^{\frac{3}{2}}} = -\frac{1}{2(7)^3}$$

$$f(48.5) \approx \frac{1}{7} - \frac{1}{2} \left(-\frac{1}{2(7)^3} \right)$$

$$= \frac{1}{7} + \frac{1}{4} \times \frac{1}{7^3}$$

$$= \frac{1}{7} \left(1 + \frac{1}{4 \times 7^2} \right)$$

$$= \frac{1}{7} \left(1 + \frac{1}{4 \times 49} \right)$$

Answer E

Question 21

$X \sim N(\mu, 2)$

$\Pr(X \leq 510) = 0.95$

$$z = \frac{510 - \mu}{2} = \text{invNorm}(0.95)$$

$$510 - \mu = 2 \times 1.645$$

$$\mu = 506.7$$

$$= 507 \text{ to the nearest mL}$$

Answer D

Question 22

$$g(f(x)) = g(a^x)$$

$$\text{Let } y = g(a^x)$$

We require $\frac{dy}{dx}$, which represents $g'(f(x))$

$$\text{Now let } a^x = e^{kx}: \log_e(a^x) = \log_e(e^{kx}) = kx$$

$$x \log_e(a) = kx$$

$$k = \log_e(a)$$

$$\text{So } y = g(e^{x \log_e(a)})$$

$$\text{Now let } u = e^{x \log_e(a)}, \text{ so } y = g(u) \text{ and } \frac{dy}{du} = g'(u).$$

$$\frac{du}{dx} = \log_e(a) \times e^{x \log_e(a)} = \log_e(a) \times a^x$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= g'(u) \times \log_e(a) \times a^x$$

$$= a^x \log_e(a) g'(e^{x \log_e(a)})$$

$$= a^x \log_e(a) g'(a^x)$$

Answer E

SECTION 2

Question 1

- a. i. $f(2\pi) = 2\pi - \cos(4\pi)$
 $= 2\pi - 1$
 The coordinates are $(2\pi, 2\pi - 1)$. A1
- ii. $(0.515, 0)$, using graphics calculator. A1
- b. $f'(x) = 1 + 2\sin(2x)$ A1
 NB: $x \in [0, 2\pi]$
 $1 + 2\sin(2x) = 0$ for stationary points
 $\sin(2x) = -\frac{1}{2}$
 $2x = \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{19\pi}{6}, \frac{23\pi}{6}$ M1
 $x = \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{19\pi}{12}, \frac{23\pi}{12}$
 For a maximum, $x = \frac{7\pi}{12}$ or $\frac{19\pi}{12}$. A1
- c. i. $x - \cos(2x) = x + 1$
 $\cos(2x) = -1$
 $2x = \pi, 3\pi$
 $x = \frac{\pi}{2}$ or $\frac{3\pi}{2}$ A1
- ii. $g'(x) = 1$ A1
 $f'(x) = 1 + 2\sin(2x)$
 At $x = \frac{\pi}{2}$, $f'\left(\frac{\pi}{2}\right) = 1 + 2\sin(\pi)$ M1
 $= 1$
 $= g'(x)$
 At $x = \frac{3\pi}{2}$, $f'\left(\frac{3\pi}{2}\right) = 1 + 2\sin(3\pi)$
 $= 1$
 $= g'(x)$
 $g(x)$ touches $f(x)$ at $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$. A1
- iii. $c = -1$ A1

Question 2

a. i. $\Pr(X \geq 3) = 0.8413$
 $\text{cdfnorm}(3, 999, 4, 1)$
 84.1% A1

ii. $(0.8413)^5$ A1
 $= 0.422$ A1

b. Let T be the time spent completing security clearances.

$$T \sim N(\mu, 6)$$

$$\Pr(T \geq 30) = 0.1$$

$$\Pr(T < 30) = 0.9$$
 M1

$$\Pr\left(z < \frac{30 - \mu}{6}\right) = 0.9$$

$$\frac{30 - \mu}{6} = 1.2816$$
 M1

$$\mu = 22.31 \text{ minutes}$$

$$= 22 \text{ minutes and } 19 \text{ seconds}$$
 A1

c. i. Let X be the number of hits in 6 shots.

$$X \sim \text{Bi}(6, 0.85)$$
 M1

$$\Pr(X \geq 5) = \binom{6}{5} 0.85^5 \times 0.15 + \binom{6}{6} 0.85^6$$

$$\text{or } 1 - \text{bincdf}(6, 0.85, 4)$$

$$= 0.77648$$
 A1

$$= 0.776 \text{ (to three decimal places)}$$

ii. $E(X) = np$

$$= 6 \times 0.85$$

$$= 5.1$$
 A1

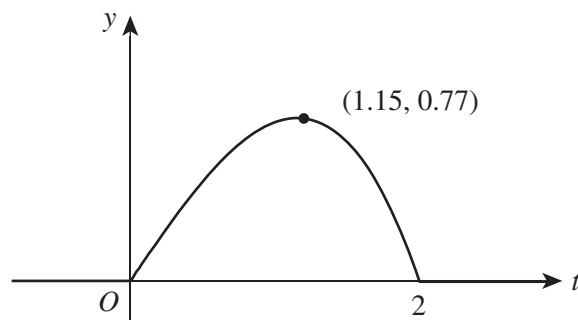
iii. Let Y be the number of satisfactory magazines.

$$Y \sim \text{Bi}(10, 0.77648)$$

$$E(Y) = 10 \times 0.77648$$

$$= 7.76$$
 A1

d. i.



Correct curve for $t \in [0, 2]$ A1

Correct curve ($y = 0$) for $t < 0$ and $t > 2$ A1

Correct coordinates of maximum point A1

ii. $1.15 \text{ hours} = 1 \text{ hour and } 9 \text{ minutes}$ A1

iii. $\Pr(0.75 < T < 1.5) = \int_{0.75}^{1.5} \frac{1}{4} t(4 - t^2) dt$ M1

$$= 0.547, \text{ using graphics calculator}$$
 A1

Question 3

- a. If the parabola is to touch the x -axis at $x = 2$, then $x^2 + px + q = (x - 2)^2$. M1

$$x^2 + px + q = x^2 - 4x + 4$$

$$\text{So } p = -4, q = 4.$$

A1

- b. As $y = x^2 - 4x + 4 + k$, $\frac{dy}{dx} = 2x - 4$.

The gradient of the line $y = 2x$ is 2, and so $2x - 4 = 2$. M1

Hence $x = 3$. A1

At $x = 3$, $y = 6$. Substituting these values into the original equation gives

$$3^2 - 4 \times 3 + 4 + k = 6$$

$$k = 5$$

A1

- c. The equation $x^2 + px + q = x$ must have two solutions.

This means that the discriminant of the equation $x^2 + (p - 1)x + q = 0$ must be positive. M1

Let $a = 1$, $b = p - 1$ and $c = q$.

Then $(p - 1)^2 - 4 \times 1 \times q > 0$.

$(p - 1)^2 > 4q$, as required. A1

- d. $x^2 + px + q = 2x$

$$x^2 + (p - 2)x + q = 0$$

M1

$$x = \frac{2 - p \pm \sqrt{(p - 2)^2 - 4q}}{2} = \frac{2 - p \pm \sqrt{p^2 - 4p + 4 - 4q}}{2}$$

A1

- e. i. $L_1 = \frac{1 - p - \sqrt{p^2 - 2p + 1 - 4q}}{2} - \frac{2 - p - \sqrt{p^2 - 4p + 4 - 4q}}{2}$ M1

$$= \frac{1}{2}(-1 + \sqrt{p^2 - 4p + 4 - 4q} - \sqrt{p^2 - 2p + 1 - 4q})$$

A1

$$L_2 = \frac{2 - p + \sqrt{p^2 - 4p + 4 - 4q}}{2} - \frac{1 - p - \sqrt{p^2 - 2p + 1 - 4q}}{2}$$

$$= \frac{1}{2}(1 + \sqrt{p^2 - 4p + 4 - 4q} - \sqrt{p^2 - 2p + 1 - 4q})$$

A1

- ii. $|L_1 - L_2| = \frac{1}{2} \times 2 = 1$

So $|L_1 - L_2|$ is constant. A1

- f. The pattern in the solutions for the parabola and each line is very similar to that in e. above.

For example, $ax^2 + px + q = 2x$

$$ax^2 + (p-2)x + q = 0$$

$$\begin{aligned} x &= \frac{2-p \pm \sqrt{(p-2)^2 - 4q}}{2a} \\ &= \frac{2-p \pm \sqrt{p^2 - 4p + 4 - 4q}}{2a} \end{aligned}$$

Similarly, for the parabola intersecting the line $y = x$, we get

$$x = \frac{1-p \pm \sqrt{p^2 - 2p + 1 - 4q}}{2a}$$

M1

Notice that the only change in the solutions is that the denominator is now $2a$.

$$\begin{aligned} \text{Hence } L_3 &= \frac{1-p - \sqrt{p^2 - 2p + 1 - 4q}}{2a} - \frac{2-p - \sqrt{p^2 - 4p + 4 - 4q}}{2a} \\ &= \frac{1}{2a}(-1 + \sqrt{p^2 - 4p + 4 - 4q} - \sqrt{p^2 - 2p + 1 - 4q}) \end{aligned}$$

$$\begin{aligned} L_4 &= \frac{2-p + \sqrt{p^2 - 4p + 4 - 4q}}{2a} - \frac{1-p - \sqrt{p^2 - 2p + 1 - 4q}}{2a} \\ &= \frac{1}{2a}(1 + \sqrt{p^2 - 4p + 4 - 4q} - \sqrt{p^2 - 2p + 1 - 4q}) \end{aligned}$$

M1

$$\begin{aligned} \text{Hence } |L_3 - L_4| &= 2 \times \frac{1}{2a}, \text{ as } a > 0 \\ &= \frac{1}{a} \end{aligned}$$

A1

Question 4

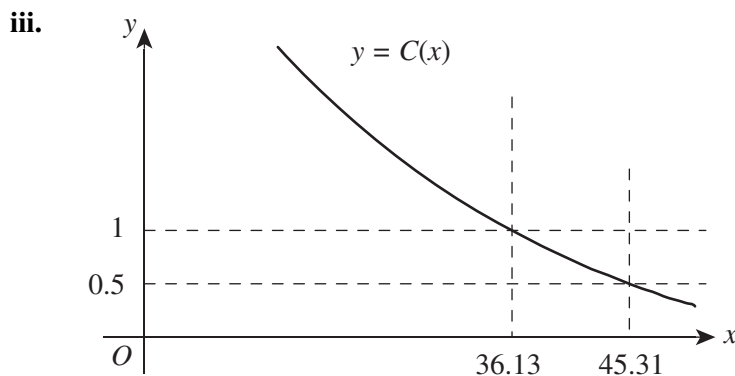
- a. i. 15.3 cm

A1

$$\begin{aligned} \text{ii. } C(x) &= We^{-bx} \\ 4.93 &= 15.3e^{-15b} \\ e^{-15b} &= \frac{4.93}{15.3} \end{aligned}$$

$$\begin{aligned} b &= -\frac{1}{15} \log_e \left(\frac{4.93}{15.3} \right) \\ &= 0.0755 \end{aligned}$$

A1



Using a graphics calculator, we find that the compacted layers will be between 0.5 and 1 cm in thickness at depths between 36.13 cm and 45.31 cm.

A1

A1

b. Let $x = \frac{\log_e(2)}{b}$.

$$C = We^{-b \frac{\log_e(2)}{b}}$$

$$= We^{-\log_e(2)}$$

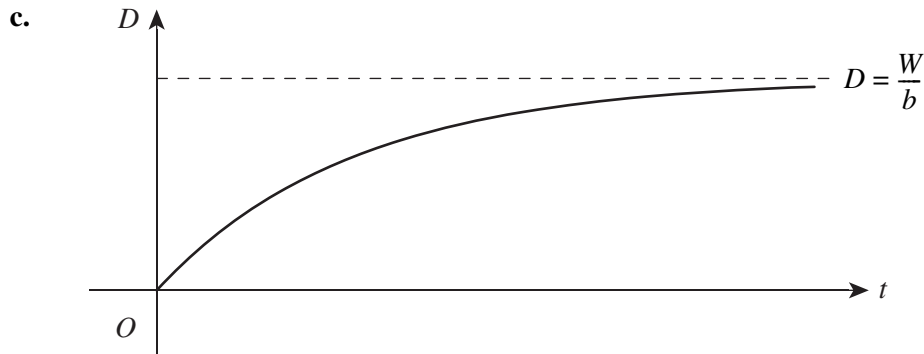
$$= We^{\log_e(2^{-1})}$$

M1

$$= W \times 2^{-1}$$

$$= \frac{1}{2}W$$

A1



Curve with asymptotic behaviour A1

Curve passes through the origin A1

Correct equation of asymptote A1

d. i. $36.2 = \frac{2.06}{b}(1 - e^{-20b})$

Using a graphics calculator,

$$y_1 = 36.2x$$

$$y_2 = 2.06(1 - e^{-20x})$$

M1

Solving gives $x = 0.01323$.

$$\therefore b = 0.01323$$

A1

ii. $D = \frac{2.06}{0.01323}(1 - e^{-0.01323t})$

As $t \rightarrow \infty$

M1

$$D \rightarrow \frac{2.06}{0.01323}$$

$$D \rightarrow 156 \text{ cm}$$

A1

iii. $\frac{dD}{dt} = 156(0.0132e^{-0.01323t})$

A1

$$= 2.06e^{-0.01323t}$$

$$1.3 = 2.06e^{-0.01323t}$$

M1

$$t = -\frac{1}{0.01323} \log_e\left(\frac{1.3}{2.06}\right)$$

$$= 34.87$$

As $t = 0$ at the end of 2005, then $t = 35$ at the end of 1970.So $t = 34.87$ at a point during 1971.

Hence the layer is from 1971.

A1