

Mathematical Methods Exam 2: SOLUTIONS

Solutions to Multiple Choice Questions

1. B 2. B 3. D 4. E 5. D 6. B 7. A 8. A 9. C
10. C 11. D 12. C 13. D 14. A 15. D 16. E
17. A 18. B 19. D 20. E 21. C 22. A

Question 1

$f(g(x)) = \frac{2}{x^2} + 1$. The range is $(1, \infty)$.

Question 2

$d_g = d_h \cap d_f$ which is $(1, 2)$.

$$g(x) = (x-1)^2(x+2)(1-x) \\ = -(x-1)^3(x+2)$$

Question 3

- a reflection in the x -axis: $f(x) \rightarrow -f(x)$
- dilation by a scale factor of 3 from the x -axis: $-f(x) \rightarrow -3f(x)$
- dilation by a scale factor of 2 from the y -axis $-3f(x) \rightarrow -3f\left(\frac{1}{2}x\right)$
- translation 5 units to the left $-3f\left(\frac{1}{2}x\right) \rightarrow -3f\left(\frac{1}{2}(x+5)\right)$

Question 4

The graph has the shape of the graph with rule $y = (ax)^{\frac{4}{3}} + 1$ or $y = (bx)^2 + 1$, where a and b are real constants. When $x = 2$, $y = 2$.

Thus, $y = x^2 + 1$ and $y = x^{\frac{4}{3}} + 1$ are

not possible. $y = \left(\frac{x}{2}\right)^{\frac{4}{3}} + 1$.

Question 5

At points of intersection,

$$\sin(2x) = \sqrt{3} \cos(2x)$$

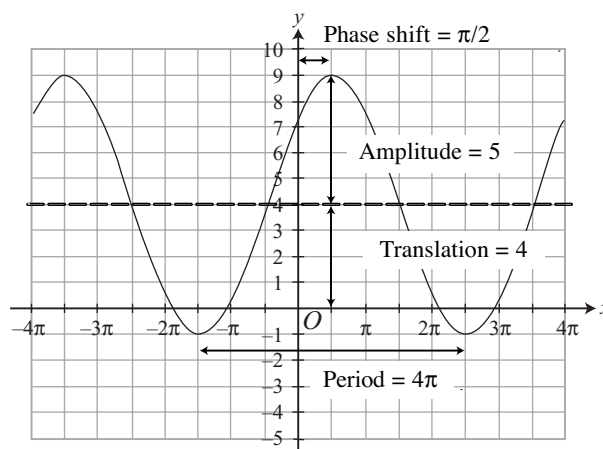
$$\frac{\sin(2x)}{\cos(2x)} = \sqrt{3}$$

$$\tan(2x) = \sqrt{3}$$

$$2x = -\frac{2\pi}{3}, \frac{\pi}{3}$$

$$x = -\frac{\pi}{3}, \frac{\pi}{6}$$

Question 6



Amplitude = 5

Period = 4π

Rule is of the form $y = a \cos(n(x - \epsilon)) + k$

$$a = 5, \epsilon = \frac{\pi}{2}, k = 4$$

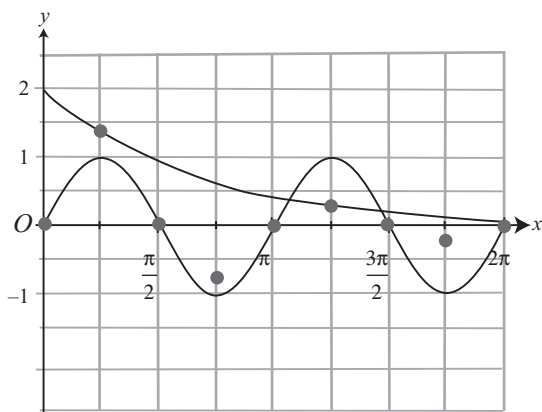
$$\text{Period} = \frac{2\pi}{n}, \text{ therefore } n = \frac{2\pi}{4\pi} = \frac{1}{2}$$

$$\text{Rule is } y = 5 \cos\left(\frac{1}{2}\left(x - \frac{\pi}{2}\right)\right) + 4$$

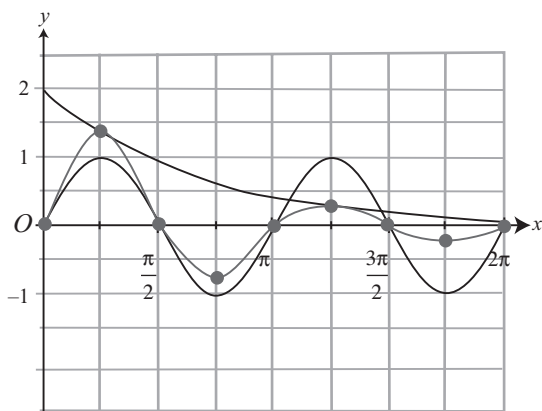
$$\text{or } y = 5 \cos\left(\frac{x}{2} - \frac{\pi}{4}\right) + 4$$

Question 7

When graphing product of functions, the key points occur when either function has the value of ± 1 or 0, as illustrated in the diagram below. Mark these key points.



Then “join the dots”.



Question 8

$$\begin{aligned} g(x) &= 3(e^{2(x-1)} + 3) \\ &= 3e^{2(x-1)} + 9 \end{aligned}$$

The graph of g has an asymptote at $y = 9$. Thus the graph of the inverse of g has an asymptote at $x = 9$.

A

Question 9

The graph of $y = e^x$ has been reflected in both x and y axes, and then translated up.

Reject option B because it does not show the translation up. None of the other options indicate the possibility of a translation parallel to the x -axis.

Ignoring the magnitude of any dilations (because there is no scale on the axes), the transformed graph will be of the form $y = -e^{-x} + k$. Since $b < 0$, require $y = -ae^{bx} + k$.

C

Question 10

$$\log_2(x+2) + \log_2(x-1) = 2$$

$$\log_2((x+2)(x-1)) = 2$$

$$(x+2)(x-1) = 4$$

$$x^2 + x - 6 = 0$$

$$(x+3)(x-2) = 2$$

$$x = 2 \text{ only as } x > 1$$

C

Question 11

The coordinates of the turning point of $y = 2(x-3)^6 + 4$ are $(3, 4)$. The tangent is $y = 4$.

D

Question 12

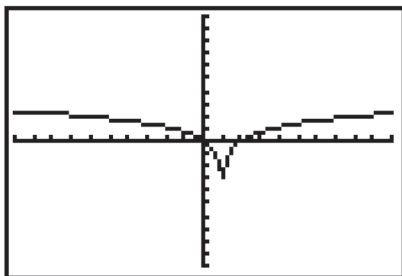
$$\begin{aligned} \text{The average rate of change} &= \frac{f(1) - f(-2)}{1 - (-2)} \\ &= \frac{\sqrt{1} - \sqrt{4}}{3} \\ &= -\frac{1}{3} \end{aligned}$$

C

A

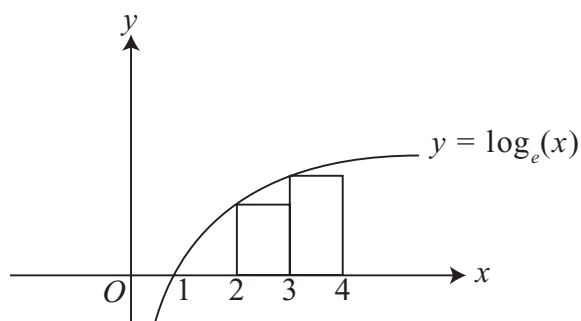
Question 13

The graph of $y = \log_e |x - 1|$ is shown below.



$$\frac{d}{dx} \log_e |x - 1| = \frac{1}{x - 1} \text{ for } x \in \mathbb{R} \setminus \{1\}$$

Question 14

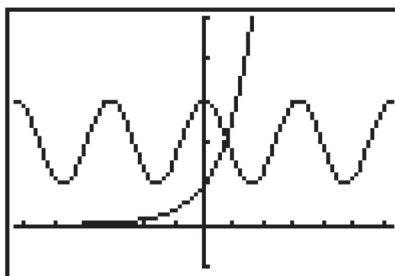


$$\begin{aligned} \text{The area of the rectangles} &= \log_e(2) + \log_e(3) \\ &= \log_e(6) \end{aligned}$$

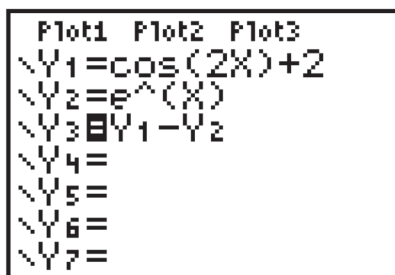
A

Question 15

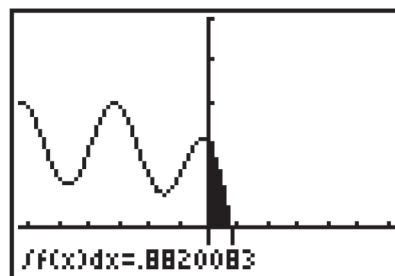
Sketch both curves and find the intersection.



D Graph $f(x) - g(x)$



$$\text{Find } \int_0^{0.7387} (f(x) - g(x)) dx$$



t

D

Question 16

$$\int_1^k (\log_e(x) + 1) dx = 1$$

$$[x \log_e(x) - x + x]_1^k = 1$$

$$[x \log_e(x)]_1^k = 1$$

$$k \log_e(k) - 1 \times 0 = 1$$

$$k \log_e(k) = 1$$

Question 17

f has a minimum turning point at $x = -1$ and a stationary point of inflection at $x = 2$.

Question 18

Let X denote the number of children with

blue eyes. $X \sim Bi\left(n=3, p=\frac{1}{4}\right)$

$$\Pr(X \geq 1) = 1 - \Pr(X = 0)$$

$$= 1 - \left(\frac{3}{4}\right)^3$$

$$= 1 - \left(\frac{27}{64}\right)$$

$$= \frac{37}{64}$$

Question 19

$$p(x) = \frac{x^2 + 1}{10}, \text{ for } x \in \{-1, 0, 1, 2\}$$

$$p(-1) = 0.2, p(0) = 0.1, p(1) = 0.2, p(2) = 0.5$$

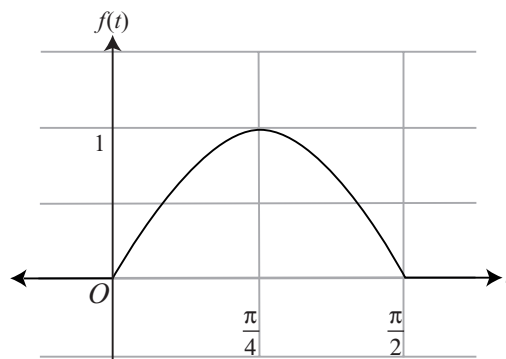
$$E(X) = \sum x p(x)$$

$$= -1 \times 0.2 + 0 \times 0.1 + 1 \times 0.2 + 2 \times 0.5$$

$$= 1$$

Question 20

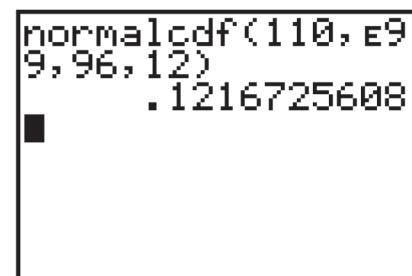
The mode is the value of t that gives the maximum value of $f(t)$.



By symmetry, this occurs at $\frac{\pi}{4}$.

Question 21

$$T \sim N(\mu = 96, \sigma^2 = 12^2)$$



$$\Pr(X > 110) = 0.1217$$

Approximately 12% exceeded the limit.

Question 22

$$\Pr(SSSM) = \frac{6}{12} \times \frac{5}{11} \times \frac{4}{10} \times \frac{6}{9} = \frac{2}{33}$$

Mathematical Methods Exam 2: SOLUTIONS

Section 2: Extended answers

Question 1

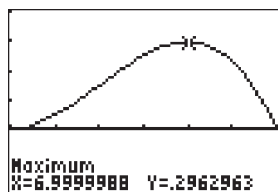
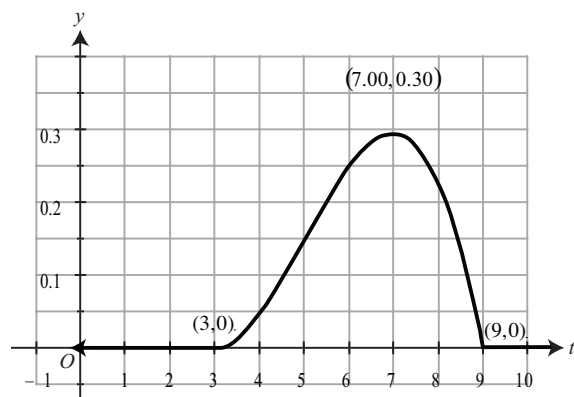
- a. For a probability density function,

$$\int_{-\infty}^{\infty} v(t) dt = 1. \text{ Therefore,}$$

$$\int_3^9 -\frac{1}{108}(t^3 - 15t^2 + 63t - 81) dt = 1$$

Showing integral with correct terminals **1M**
(dt must be shown to get the mark)

- b.



Correct shape and skew **1A**
Correct domain and axes intercepts **1A**
Correct stationary point **1A**

- c. From the graph, the mode is 7, as this is the value of t at which the maximum value occurs. **1 M**
(Consequential from part b)

- d.
- $\mu = \int_{-\infty}^{\infty} t v(t) dt$
- . Therefore,

$$\mu = -\frac{1}{108} \int_3^9 (t^3 - 15t^2 + 63t - 81) dt \quad \mathbf{1M}$$

$$= \frac{33}{5} \text{ or } 6.6$$

The mean incubation period is 6.6 days. **1A**

- e.
- $X \sim N(5, \sigma^2)$
- ,
- $Z \sim N(0, 1)$

$$\Pr(Z < z) = 0.9 \quad \mathbf{1M}$$

$$z = 1.28155$$

$$z = \frac{x - \mu}{\sigma}$$

$$1.28155 = \frac{6.25 - 5}{\sigma}$$

$$\sigma = 0.98$$

The standard deviation is 0.98 hours **1A**

- f. Let
- Y
- be the number of patients for whom the response time to the drug is more than 6 hours 15 minutes.

$$Y \sim Bi(10, 0.1) \quad \mathbf{1M}$$

$$\Pr(Y \geq 2) = \sum_{n=2}^{10} {}^{10}C_n (0.9)^n (0.1)^{10-n}$$

$$= 0.264 \quad \mathbf{1A}$$

g. $Y \sim Bi(k, 0.1)$

Consider the case where there is 95% chance.

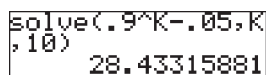
$$\Pr(Y \geq 1) = 0.95$$

$$1 - \Pr(Y = 0) = 0.95$$

$$1 - 0.9^k = 0.95$$

$$0.9^k = 0.05$$

$$k \approx 28.43 \quad (\text{using numerical solve/ solver})$$



Round up because $k \in \mathbb{Z}^+$ and the probability must be at least 95%.

The least value of k is 29 patients.

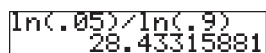
Alternative 1 to using numerical solve/ solver.

$$0.9^k = 0.05$$

$$\log_e(0.9^k) = \log_e(0.05)$$

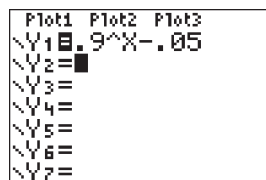
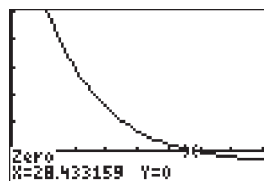
$$k \log_e(0.9) = \log_e(0.05)$$

$$k = \frac{\log_e(0.05)}{\log_e(0.9)} \approx 28.43$$



However, this involves more steps and still requires the use of a calculator.

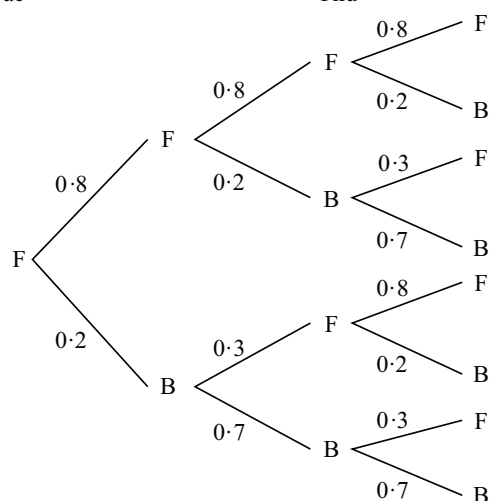
Alternative 2: graphical approach.

$$0.9^k - 0.05 = 0$$

$$k \approx 28.43$$

h. Tue Wed Thu Fri **1M**



$$\Pr(FFFB) = 0.8^2 \times 0.2 = 0.128$$

$$\Pr(FFBB) = 0.8 \times 0.2 \times 0.7 = 0.112$$

$$\Pr(FBFB) = 0.2 \times 0.3 \times 0.2 = 0.012$$

$$\Pr(FBBB) = 0.2 \times 0.7^2 = 0.098$$

$$0.128 + 0.112 + 0.012 + 0.098 = 0.35$$

The probability of Fluaway on Tuesday and Blugone on Friday is 0.35.

Question 2

a. $f(x) = A(x^{\frac{4}{3}} - 5)$

$$-2 = A(0 - 5)$$

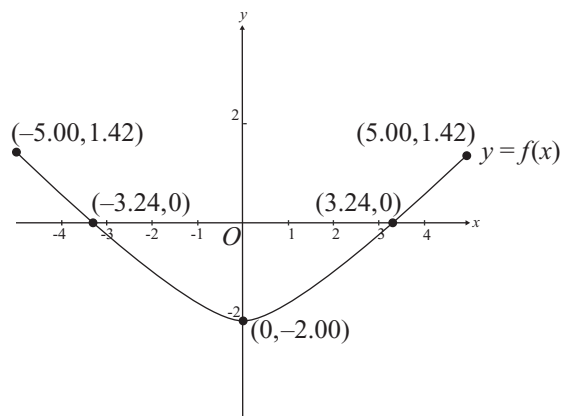
1M

$$A = \frac{2}{5}$$

b. endpoints $(-5.00, 1.42)$ and $(5.00, 1.42)$ 1A

turning point $(0, -2.00)$ and correct shape 1A

x -intercepts $(-3.24, 0)$ and $(3.24, 0)$ 1A



c. Max depth $= 2 + 1.42$

$$= 3.42 \text{ m}$$

1A

d. i. $A = -2 \times \frac{2}{5} \int_0^{5^{\frac{3}{4}}} (x^{\frac{4}{3}} - 5) dx$

1M

$$= -\frac{4}{5} \left[\frac{3}{7} x^{\frac{7}{3}} - 5x \right]_0^{5^{\frac{3}{4}}}$$

1M

$$= -\frac{4}{5} \left(\left(\frac{3}{7} (5^{\frac{3}{4}})^{\frac{7}{3}} - 5^{\frac{7}{4}} \right) - (0) \right)$$

1M

$$= 7.643 \text{ m}^2 \text{ correct to}$$

3 decimal places

1A

ii. $V = 1000A$

$$= 7643 \text{ m}^3$$

1A

e. i. $V = 1000A$

$$= 1000 \left(\frac{1}{2} \text{ base} \times \text{height} \right)$$

1M

$$= 2000 \times 5^{\frac{3}{4}}$$

$$= 6687.40 \text{ m}^3$$

1A

ii. $\frac{dV}{dt} = \frac{dh}{dt} \times \frac{dV}{dh}$

1A

$$\frac{dV}{dt} = -0.02 \times \frac{dV}{dh}$$

$$V = 1000 \frac{1}{2} bh$$

Using similar triangles:

$$\frac{\frac{1}{2}b}{h} = \frac{5^{\frac{3}{4}}}{2}$$

$$\frac{1}{2}b = \frac{5^{\frac{3}{4}}}{2}h$$

1M

$$V = 500 \times 5^{\frac{3}{4}} h^2$$

1A

$$\frac{dV}{dh} = 1000 \times 5^{\frac{3}{4}} h$$

$$\frac{dV}{dt} = -0.02 \times 1000 \times 5^{\frac{3}{4}} h$$

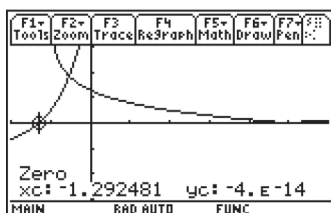
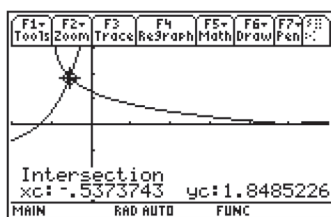
$$= -20 \times 5^{\frac{3}{4}} h \text{ m}^3/\text{h}$$

1A

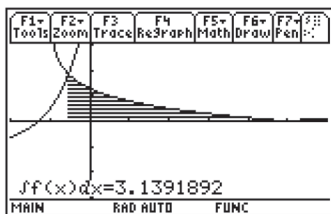
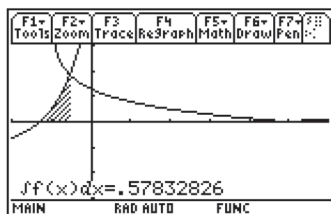
The rate at which water is decreasing

$$\text{is } 20 \times 5^{\frac{3}{4}} h \text{ m}^3/\text{h}$$

- iv. Sketch both graphs and find the intersection and x -intercepts.



$$A = \int_{-1.292481}^{-0.53737} g_3(x) dx + \int_{-0.53737}^5 |g_3^{-1}(x)| dx \quad 1M$$



$$A \approx 0.578328 + 3.139189 \\ = 3.7 \text{ units}^2 \text{ correct to one decimal place}$$

1A

Question 4

- a. By Pythagoras' theorem

$$x^2 + y^2 = 50^2$$

$$y = \sqrt{2500 - x^2} \text{ (reject negative solution)} \quad 1A$$

- b. Highest position when $x = 0$.

$$\text{Endpoint will be } (0, 50). y = 50 \text{ cm} \quad 1A$$

Lowest position when $x = \pm 30$. The corresponding y -axis value is

$$y = \sqrt{2500 - (30)^2} = 40 \text{ or}$$

$$y = \sqrt{2500 - (-30)^2} = 40$$

$$y = 40 \text{ cm}$$

1A

- c. Period along the x -axis $= \frac{2\pi}{\pi/8} = 16 \text{ sec} \quad 1A$

d. $30 \cos\left(\frac{\pi}{8}t\right) = 15$

$$\cos\left(\frac{\pi}{8}t\right) = \frac{1}{2}$$

$$\left(\frac{\pi}{8}t\right) = \frac{\pi}{3}, \frac{5\pi}{3}$$

$$t = \frac{8}{3}, \frac{40}{3}$$

1M

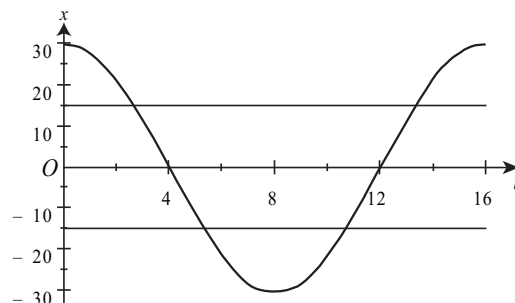
$$30 \cos\left(\frac{\pi}{8}t\right) = -15$$

$$\cos\left(\frac{\pi}{8}t\right) = -\frac{1}{2}$$

$$\left(\frac{\pi}{8}t\right) = \frac{2\pi}{3}, \frac{4\pi}{3}$$

$$t = \frac{16}{3}, \frac{32}{3}$$

1M



Time for which is given by

$$\left(\frac{16}{3} - \frac{8}{3}\right) + \left(\frac{40}{3} - \frac{32}{3}\right) = \frac{16}{3} \text{ seconds}$$

1A

e. $\frac{dx}{dt} = -\frac{15\pi}{4} \sin\left(\frac{\pi}{8}t\right)$ **1M**

When $x = 15$, $t = \frac{8}{3}, \frac{40}{3}, \dots$ (from part d.)

$$\begin{aligned} x'\left(\frac{8}{3}\right) &= -\frac{15\pi}{4} \sin\left(\frac{\pi}{8} \times \frac{8}{3}\right) \\ &= -\frac{15\pi}{4} \sin\left(\frac{\pi}{3}\right) \\ &= -\frac{15\pi}{4} \times \frac{\sqrt{3}}{2} \\ &= -\frac{15\sqrt{3}\pi}{8} \end{aligned}$$

1M

Speed is the magnitude of $x'(t)$

Speed = $\frac{15\sqrt{3}\pi}{8}$ cm/s **1A**

f. $x'(t) = -\frac{15\pi}{4} \sin\left(\frac{\pi}{8}t\right)$

The maximum speed occurs where

$$\frac{d}{dt}(x'(t)) = 0 \quad \textbf{1M}$$

$$\frac{d}{dt}\left(-\frac{15\pi}{4} \sin\left(\frac{\pi}{8}t\right)\right) = 0$$

$$-\frac{15\pi}{4} \times \frac{\pi}{8} \cos\left(\frac{\pi}{8}t\right) = 0$$

$$\cos\left(\frac{\pi}{8}t\right) = 0$$

$$\frac{\pi}{8}t = \frac{\pi}{2}, \frac{3\pi}{2}, \dots$$

$$t = \frac{\pi}{2} \times \frac{8}{\pi}, \frac{3\pi}{2} \times \frac{8}{\pi}, \dots \quad \textbf{1M}$$

$$t = 4, 12, \dots$$

(This is also evident from the graph of x v. t , which show the maximum gradient at $t = 4, 12, \dots$)

$$x'(4) = -\frac{15\pi}{4} \sin\left(\frac{\pi}{2}\right) = -\frac{15\pi}{4}$$

$$x'(12) = -\frac{15\pi}{4} \sin\left(\frac{3\pi}{2}\right) = \frac{15\pi}{4}$$

The speed is the magnitude of $x'(t)$.

Hence the maximum speed is $\frac{15\pi}{4}$ cm/s. **1A**