# The Mathematical Association of Victoria
# MATHEMATICAL METHODS

## Trial written examination 2

**2007**

**Reading time: 15 minutes**
**Writing time: 2 hours**

Student's Name: ..............................................................

## QUESTION AND ANSWER BOOK

**Structure of book**

| *Section* | *Number of questions* | *Number of questions to be answered* | *Number of marks* |
|---|---|---|---|
| 1 | 22 | 22 | 22 |
| 2 | 4 | 4 | 58 |
| | | | Total 80 |

**Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.**

*These questions have been written and published to assist students in their preparations for the 2007 Mathematical Methods Examination 2. The questions and associated answers and solutions do not necessarily reflect the views of the Victorian Curriculum and Assessment Authority. The Association gratefully acknowledges the permission of the Authority to reproduce the formula sheet.*



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## Working space

## MULTIPLE CHOICE ANSWER SHEET

Student Name: ..............................................................................

Circle the letter that corresponds to each correct answer

| | | | | | |
|---|---|---|---|---|---|
| 1 | A | B | C | D | E |
| 2 | A | B | C | D | E |
| 3 | A | B | C | D | E |
| 4 | A | B | C | D | E |
| 5 | A | B | C | D | E |
| 6 | A | B | C | D | E |
| 7 | A | B | C | D | E |
| 8 | A | B | C | D | E |
| 9 | A | B | C | D | E |
| 10 | A | B | C | D | E |
| 11 | A | B | C | D | E |
| 12 | A | B | C | D | E |
| 13 | A | B | C | D | E |
| 14 | A | B | C | D | E |
| 15 | A | B | C | D | E |
| 16 | A | B | C | D | E |
| 17 | A | B | C | D | E |
| 18 | A | B | C | D | E |
| 19 | A | B | C | D | E |
| 20 | A | B | C | D | E |
| 21 | A | B | C | D | E |
| 22 | A | B | C | D | E |

## Working space

# SECTION 1

**Instructions for Section 1**

Answer **all** questions in pencil on the answer sheet provided for multiple-choice questions.
Choose the response that is **correct** for the question.
A correct answer scores 1, an incorrect answer scores 0.
Marks will **not** be deducted for incorrect answers.
No marks will be given if more than one answer is completed for any question.

**Question 1**

If $R \to R$, where $f(x) = 2x + 1$ and $g: R\backslash\{0\} \to R$, where $g(x) = \dfrac{1}{x^2}$ then the range of $f(g(x))$ is

**A.** $R\backslash\{1\}$

**B.** $(1, \infty)$

**C.** $(0, \infty)$

**D.** $R\backslash\{-\frac{1}{2}\}$

**E.** $R\backslash\{0\}$

**Question 2**

If $h: (1,2] \to R$, where $h(x) = (x-1)^2(x+2)$ and $f: [-1,2) \to$ R, where $f(x) = 1 - x$ then $g = hf$ is defined by

**A.** $g: R \to R$, where $h(x) = -(x-1)^3(x+2)$

**B.** $g: (1,2) \to R$, where $h(x) = -(x-1)^3(x+2)$

**C.** $g: (1,2] \to R$, where $h(x) = -(x-1)^3(x+2)$

**D.** $g: (1,2) \to R$, where $h(x) = -(x-1)^2(x+2)(x+1)$

**E.** $g: [-1,2] \to R$, where $h(x) = -(x-1)^3(x+2)$

**Question 3**

The graph of $y = f(x)$ undergoes the following transformations:

- a reflection in the $x$-axis, then
- a dilation by a scale factor of 3 from the $x$-axis, then
- a dilation by a scale factor of 2 from the $y$-axis, then
- a translation 5 units to the left.

The equation of the transformed graph is

**A.** $y = 3f(-2(x-5))$

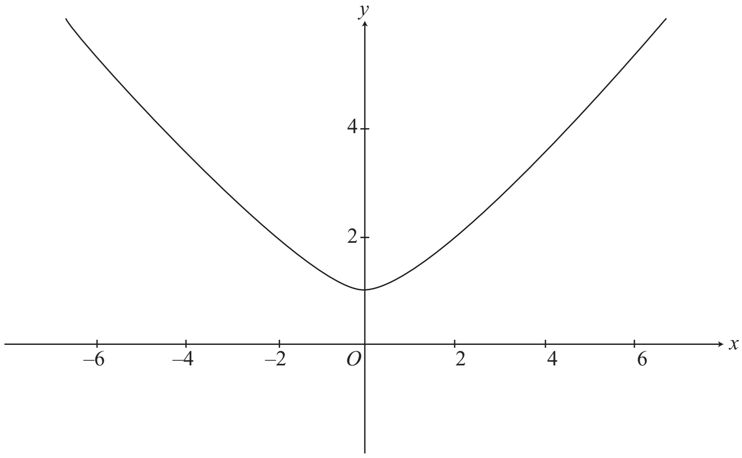
**B.** $y = -3f(2(x+5))$

**C.** $y = -3f\left(\frac{1}{2}(x-5)\right)$

**D.** $y = -3f\left(\frac{1}{2}(x+5)\right)$

**E.** $y = -2f(3(x-5))$

**Question 4**




The graph shown above passes through the point (2, 2). A possible equation for this graph could be

**A.** $y = x^{\frac{4}{3}} + 1$

**B.** $y = x^{\frac{2}{3}} + 1$

**C.** $y = x^{\frac{5}{3}} + 1$

**D.** $y = x^2 + 1$

**E.** $y = \left(\frac{x}{2}\right)^{\frac{4}{3}} + 1$

**Question 5**

The $x$-coordinate(s) of the point(s) of intersection of the graphs of $y = \sin(2x)$ and $y = \sqrt{3}\cos(2x)$, for $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, are

**A.** $\frac{\pi}{3}$ only

**B.** $\frac{\pi}{6}$ only

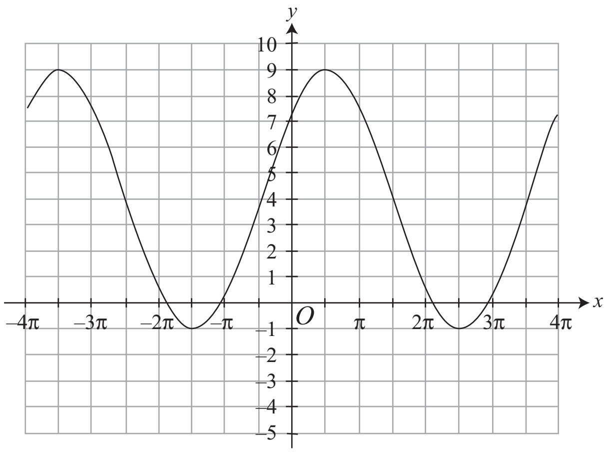
**C.** $-\frac{\pi}{3}, \frac{\pi}{3}$

**D.** $-\frac{\pi}{3}, \frac{\pi}{6}$

**E.** $-\frac{\pi}{6}$ and $\frac{\pi}{3}$

**Question 6**

The diagram below shows the graph of a circular function.




The **amplitude**, **period** and **rule**, respectively, of this graph are

**A.** 10, 8π, $y = 10\cos\left(\frac{x}{2} - \frac{\pi}{2}\right) + 4$

**B.** 5, 4π, $y = 5\cos\left(\frac{x}{2} - \frac{\pi}{4}\right) + 4$

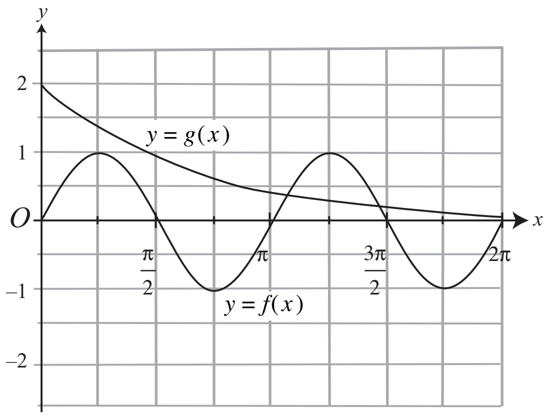
**C.** 5, 4π, $y = 5\cos\left(2x - \frac{\pi}{2}\right) + 4$

**D.** 10, 4π, $y = 10\cos\left(2x + \frac{\pi}{2}\right) + 4$

**E.** 5, 8π, $y = 5\cos\left(\frac{x}{4} - \frac{\pi}{2}\right) + 4$

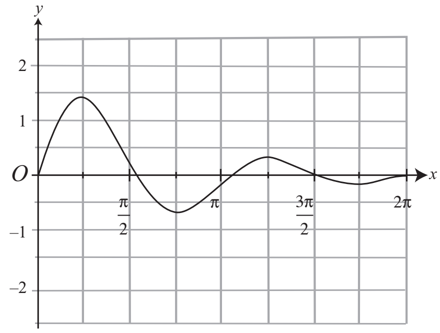
**Question 7**

The diagram below shows the graphs of two functions, $f$ and $g$.

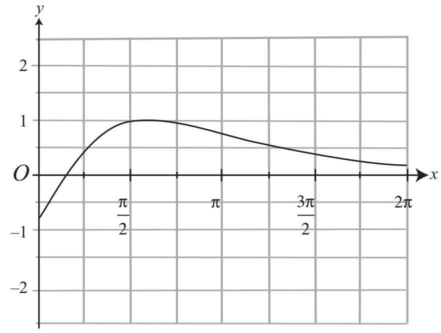



Which one of the following could be the graph of the product function $y = (fg)(x)$?

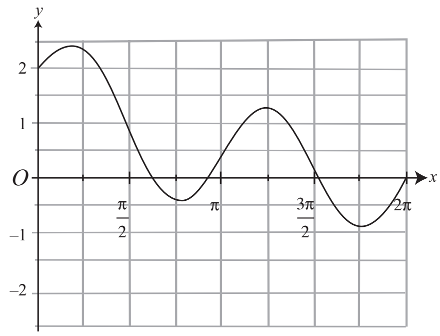
**A.**



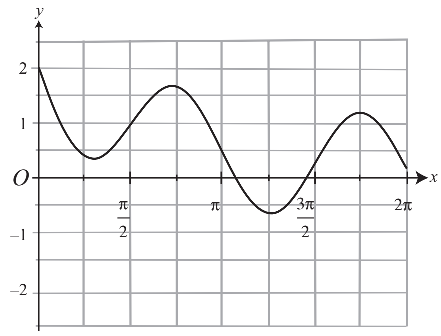
**B.**



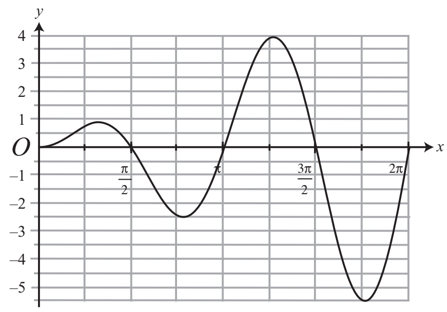
**C.**



**D.**



**E.**



**Question 8**

Let $R \rightarrow R$, where $f(x) = e^{2(x-1)} + 3$. If the graph of $g$ is the graph of $f$ dilated by a factor of 3 from the $x$-axis then the graph of the inverse function $g^{-1}$ has

**A.** a vertical asymptote with equation $x = 9$.

**B.** a horizontal asymptote with equation $y = 3$.

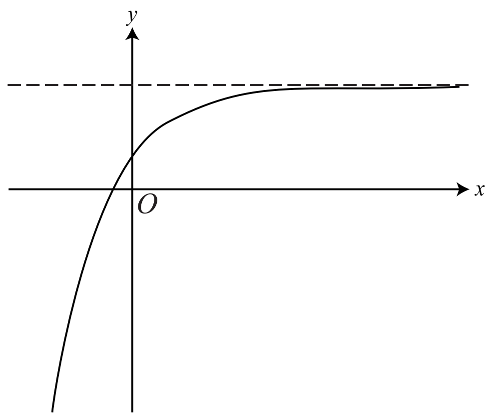
**C.** a vertical asymptote with equation $x = 1$.

**D.** a horizontal asymptote with equation $y = 1$.

**E.** a vertical asymptote with equation $x = 3$.

**Question 9**

Part of the graph of an exponential function with rule $y = f(x)$ is shown below.




If $a > 0$, $b < 0$ and $k > 0$, the rule for $f$ could be

**A.** $f(x) = ae^{bx} + k$

**B.** $f(x) = ae^{b(x-k)}$

**C.** $f(x) = -ae^{bx} + k$

**D.** $f(x) = ae^{-bx} + k$

**E.** $f(x) = ae^{bx} - k$

**Question 10**

If $\log_2(x + 2) + \log_2(x - 1) = 2$ then $x$ equals

**A.** –3 or 2

**B.** –3 only

**C.** 2 only

**D.** –2 or 3

**E.** 3 only

**Question 11**

The equation of the tangent to the graph with equation $y = 2(x - 3)^6 + 4$ at its turning point is

**A.** $x = 3$

**B.** $y = 3$

**C.** $x = -3$

**D.** $y = 4$

**E.** $x = 4$

**Question 12**

The average rate of change of $f(x)=\sqrt{2-x}$ from $x = -2$ to $x = 1$ is

**A.** $-1$

**B.** $-\dfrac{1}{2}$

**C.** $-\dfrac{1}{3}$

**D.** $-\dfrac{1}{4}$

**E.** $\dfrac{1}{3}$

**Question 13**

If $f(x)=\log_e|x-1|$, then the derivative of $f$ is

**A.** $\dfrac{1}{x-1}$, for $x > 1$

**B.** $\dfrac{1}{x-1}$, for $x < 1$

**C.** $\dfrac{1}{x-1}$, for $x > 0$

**D.** $\dfrac{1}{x-1}$, for $x \in R\backslash\{1\}$

**E.** $\dfrac{1}{x-1}$, for $x \in R\backslash\{-1\}$

**Question 14**

The area bounded by the curve with equation $y = \log_e(x)$, the $x$-axis and the line $x = 4$ is approximated using left-end rectangles of width 1 unit. This area, in units$^2$, is

**A.** $\log_e 6$

**B.** $\log_e 24$

**C.** $8\log_e(2)-4$

**D.** 1.55

**E.** 1.79

**Question 15**

The area, correct to three decimal places, bounded by the $y$-axis and the curves with equations $f(x) = \cos(2x) + 2$ and $g(x) = e^x$ is

**A.** 3.357

**B.** 0.879

**C.** 0.878

**D.** 0.882

**E.** $-0.882$

**Question 16**

If $\frac{d}{dx}(x\log_e(x)-x)=\log_e(x)$ and $\int_1^k \left(\log_e(x)+1\right)dx=1$, where $k$ is a real constant, then $k$ can be found by solving

**A.** $k^2-3k+1=0$

**B.** $k\log_e(k)-k=0$

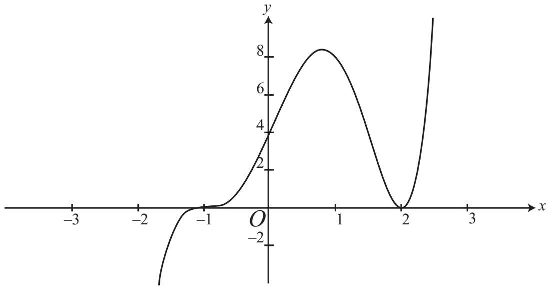
**C.** $k\log_e(k)=2$

**D.** $k\log_e(k)+k=2$

**E.** $k\log_e(k)=1$

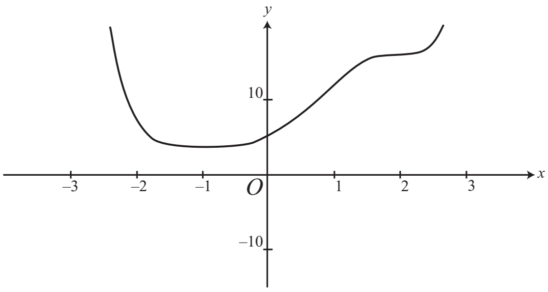
**Question 17**

Part of the graph of the derivative of $f$ is shown below.

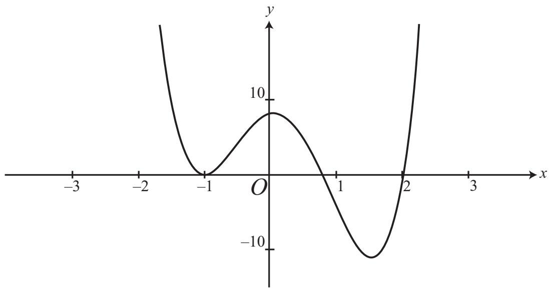


Part of the graph of $f$ could be

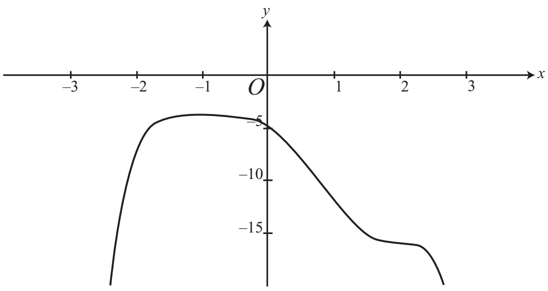
**A.**



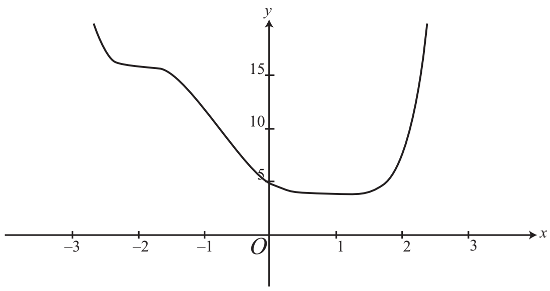
**B.**



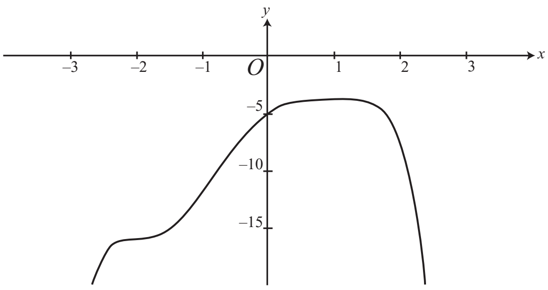
**C.**



**D.**



**E.**



**Question 18**

Miriam and her husband have brown eyes, but both carry the recessive gene for blue eyes. It is known that couples in these circumstances will have a child with blue eyes in 1 out of every 4 births. If they have three children altogether, the probability that at least one will have blue eyes is

**A.** $\frac{27}{64}$

**B.** $\frac{37}{64}$

**C.** $\frac{1}{4}$

**D.** $\frac{3}{4}$

**E.** $\frac{9}{64}$

**Question 19**

The probability distribution for a discrete random variable, $X$, is defined by the probability function

$p(x) = \frac{x^2 + 1}{10}$, for $x \in \{-1, 0, 1, 2\}$.

The expectation of $X$, $E(X)$, is

**A.** 1.5

**B.** 1.4

**C.** 1.1

**D.** 1

**E.** 0.8

**Question 20**

The probability distribution for a continuous random variable, $T$, is defined by the probability density function

$$f(t) = \begin{cases} \sin(2t) & 0 \le t \le \frac{\pi}{2} \\ 0 & \text{otherwise} \end{cases}$$

The mode of the distribution is

**A.** 0

**B.** 1

**C.** $\pi$

**D.** $\frac{\pi}{2}$

**E.** $\frac{\pi}{4}$

**Question 21**

The speed of vehicles travelling along a particular section of a rural freeway is modelled as a normally distributed random variable with a mean of 96 km/h and a standard deviation of 12 km/h.
According to this model, the percentage of vehicles that are exceeding the speed limit of 110 km/h is closest to

**A.** 5%

**B.** 10%

**C.** 12%

**D.** 14%

**E.** 20%

**Question 22**

Graham randomly selected and ate four sandwiches from a tray containing 6 salad and 6 meat sandwiches.
The probability that the first three sandwiches that he ate were salad and the fourth one was meat is

**A.** $\frac{2}{33}$

**B.** $\frac{1}{16}$

**C.** $\frac{1}{4}$

**D.** $\frac{31}{33}$

**E.** $\frac{3}{16}$

**END OF SECTION 1**

# SECTION 2

**Instructions for Section 2**

Answer **all** questions in the spaces provided.

A decimal approximation will not be accepted if an **exact** answer is required to a question.

In questions where more than one mark is available, appropriate working **must** be shown.

Where an instruction to **use calculus** is stated for a question, you must show an appropriate derivative or antiderivative.

Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

**Question 1**

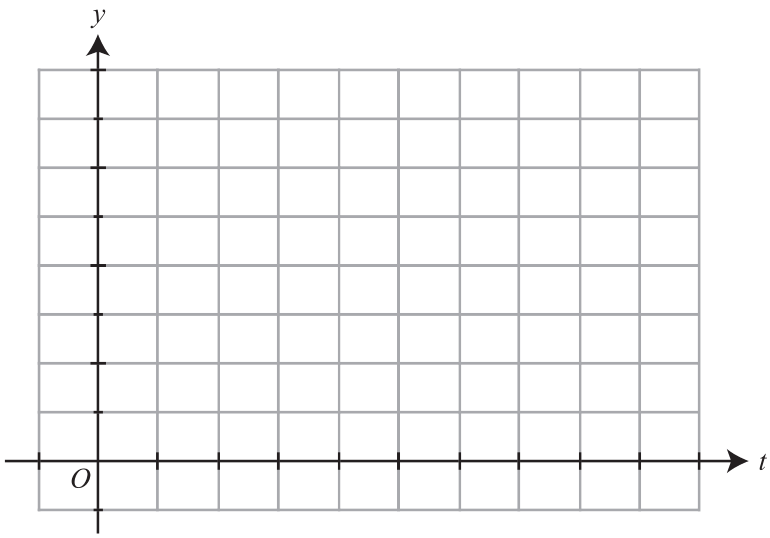
The incubation period for a new strain of influenza virus, commonly known as blue flu, $T$ days after coming in contact with the virus, is modelled by

$$v(t) = \begin{cases} -\dfrac{1}{108}\left(t^3 - 15t^2 + 63t - 81\right) & 3 \le t \le 9 \\ 0 & \text{otherwise} \end{cases}$$

**a.** Show that $v(t)$ is a probability density function.

1 mark

**b.** Sketch the graph of $y = v(t)$ on the axes below. Label axes intercepts. Label any stationary points with their coordinates, correct to two decimal places.




3 marks

**c.** What is the mode of the distribution?

1 mark

**d.** Find the mean incubation period for this virus.

2 marks

A new drug, Fluaway, has been developed to treat the symptoms of blue flu. The time taken for a standard dose of Fluaway to take effect is modelled by a normally distributed random variable with a mean of 5 hours.

For 90% of patients, a standard dose of Fluaway takes effect within 6 hours and 15 minutes (that is, 6.25 hours) of being administered.

**e.** Find the standard deviation, in hours, correct to two decimal places.

2 marks

Vivienne is a medical practitioner at an infectious disease clinic.

**f.** Vivienne randomly selects ten patients with blue flu and administers a standard dose of Fluaway to each of them. Find the probability that, in at least two of the patients, the drug will take more than 6 hours and 15 minutes to take effect. Give the answer correct to three decimal places.

2 marks

**g.** Vivienne randomly selected $k$ patients with blue flu and administered a standard dose of Fluaway to each of them. What is the minimum value of $k$ required to ensure that there is at least a 95% chance that at least one patient will take more than 6 hours and 15 minutes to respond to the drug?

2 marks

An alternative drug, Blugone, has been released. At the clinic, Vivienne's decision as to which of the two drugs to prescribe to patients on a particular day depends only on which drug was prescribed the previous day. If Fluaway was prescribed yesterday, there is a probability of 0.2 that Blugone will be prescribed today. However, if Blugone was prescribed yesterday, the probability that it will be prescribed again today is 0.7.

**h.** On Tuesday of this week Vivienne prescribed Fluaway to her blue flu patients. What is the probability that she will prescribe Blugone on Friday of the same week?

3 marks

Total 16 marks

**Question 2**

The shape of the vertical cross-section of an irrigation channel can be modelled by the following function:

$$f:[-5,\ 5] \to R, \text{ where } f(x) = A\left(x^{\frac{4}{3}} - 5\right) \text{ and } A \text{ is a real constant.}$$

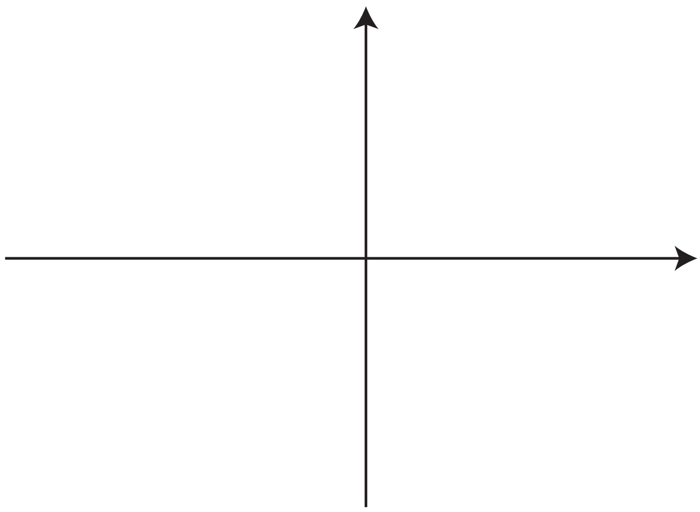
The $x$-axis represents normal ground level. $f$ is the vertical distance, in metres, of the edge of the channel above ground level. $x$ is the horizontal distance, in metres, of the edge of the channel from the centre of the channel.

**a.** The lowest point of the channel is represented by the turning point of the graph of $f$ which has coordinates (0, –2).

Show that the value of $A$ is $\frac{2}{5}$.

1 mark

**b.** Sketch the graph of $f$ on the axes below, clearly labelling the endpoints and axial intercepts with their coordinates, correct to two decimal places.



3 marks

**c.** What is the maximum possible depth, in metres correct to two decimal places, of the water in the channel?

1 mark

On a particular day, the maximum depth of the water in the channel is 2 m.

**d.** **i.** Use calculus to find the area, in $m^2$ correct to three decimal place, bounded by the $x$-axis and the graph of $f$.

**ii.** **Hence**, if the length of the channel is 1 km, find the volume of water in the channel to the nearest $m^3$.

4 + 1 = 5 marks

Due to weeds at the bottom of the channel, a more accurate estimate of the vertical cross-sectional area of water in the channel, on the day given in **part d**, can be found by using a triangle with vertices $(0, -2)$, $(-5^{\frac{3}{4}}, 0)$ and $(5^{\frac{3}{4}}, 0)$. Assume the length of the channel is 1 km.

**e.** **i.** What is the estimated volume, in $m^3$ correct to two decimal places, of water in the channel using this method?

**ii.** If the depth of water in the channel starts to decrease by 2 cm/h because of evaporation, at what rate, in $m^3/h$, is the volume of water decreasing in the channel?

Give your answer in terms of the maximum depth of water in the channel, $h$ m. Assume no water is entering the channel.

2 + 4 = 6 marks

Total 16 marks

**Question 3**

Let $g : R \to R$, where $g(x) = A \times B^x$, and $A$ and $B \in Z^+$.

**a.** Find values for $A$ and $B$ if the graph of $g$ passes through the points (2, 12) and (4, 48).

2 marks

The graph of $g_2$ is the graph of $g$ dilated by a factor of $\frac{1}{3}$ from the $y$-axis, followed by a translation of $\frac{4}{3}$ units parallel to the $x$-axis and 1 unit parallel to the $y$-axis.

**b.** Write down the rule for $g_2$ and state the range.

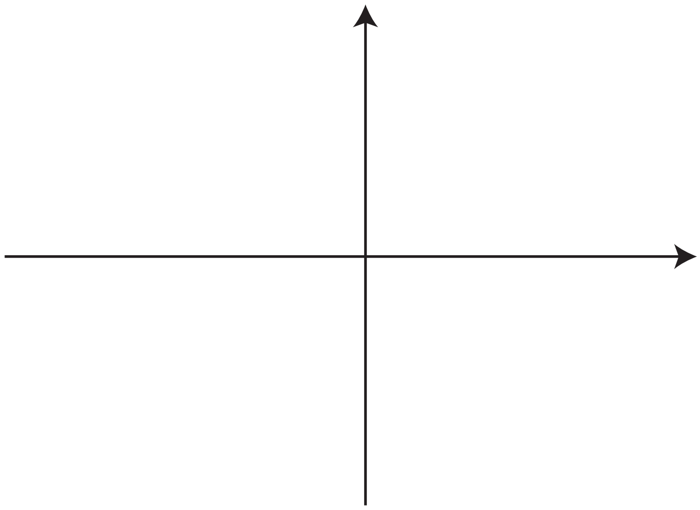
3 marks

Let $g_3 : R \to R$, where $g_3(x) = 3 \times 2^{(2x+1)} - 1$.

**c.** **i.** Show that $g_3^{-1}(x) = \frac{1}{2}\log_2(\frac{x+1}{6})$.

ii. State the transformations that have occurred to the graph of $y = \log_2(x)$ to get the graph of $g_3^{-1}$.

iii. Sketch the graph of $g_3^{-1}$ on the set of axes below, labelling any asymptotes with their equations and intercepts with their coordinates.



iv. Find the area, in units$^2$ correct to one decimal place, bounded by the $x$-axis and the graphs of $|g_3^{-1}|$ and $g_3$.

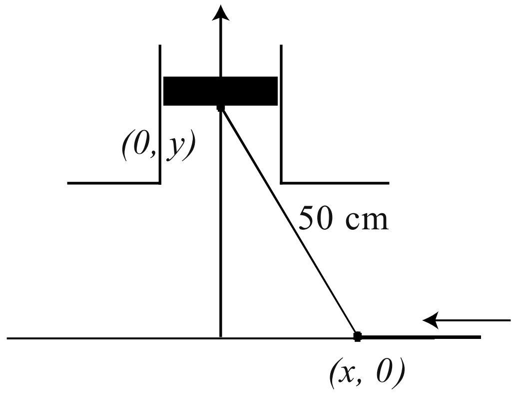
2 + 2 + 2 + 2 = 8 marks

Total 13 marks

**Question 4**

A machine uses a 50 cm long moveable rod to move a piston on a vertical shaft. The coordinates of the ends of the rod are $(x, 0)$ and $(0, y)$, as shown as in the diagram.

The position, $x$ cm, of the end of the rod on the $x$-axis, at time $t$ seconds, is given by $x(t) = 30\cos\left(\frac{\pi}{8}t\right)$, $t \geq 0$.




**a.** Show that the distance $y$ cm in terms of $x$ is given by $y = \sqrt{2500 - x^2}$.

1 mark

**b.** The piston is at its highest position on the $y$-axis when $x = 0$. The piston is at its lowest position on the $y$-axis when $x = 30$ or $x = -30$. Find the distance $y$ cm, from the origin to the highest and lowest positions of the end of the rod on the $y$-axis.

2 marks

**c.** What is the period of oscillation of the lower end of the rod along the ***x*-axis**?

1 mark

**d.** For $t \in [0,\ 16]$, solve the equations $x(t) = 15$ and $x(t) = -15$. Hence find, in the first 16 seconds of the motion of the rod, the total length of time for which $-15 \le x \le 15$.

3 marks

**e.** Show that the speed of the end of the rod on the ***x*-axis,** when the $x$-axis endpoint is (15, 0), is $\frac{15\sqrt{3}\pi}{8}$ cm/s.

3 marks

**f.** Find the **exact** maximum speed of the end of the rod on the ***x*-axis**.

3 marks

Total 13 marks

# Mathematical Methods Exam 2: SOLUTIONS

## Solutions to Multiple Choice Questions

1. B 2. B 3. D 4. E 5. D 6. B 7. A 8. A 9. C
10. C 11. D 12. C 13. D 14. A 15. D 16. E
17. A 18. B 19. D 20. E 21. C 22. A

## Question 1

$f(g(x)) = \frac{2}{x^2} + 1$. The range is $(1, \infty)$. **B**

## Question 2

$d_g = d_h \cap d_f$ which is (1, 2).

$g(x) = (x-1)^2(x+2)(1-x)$

$\quad = -(x-1)^3(x+2)$ **B**

## Question 3

- a reflection in the $x$-axis: $f(x) \to -f(x)$
- dilation by a scale factor of 3 from the $x$-axis: $-f(x) \to -3f(x)$
- dilation by a scale factor of 2 from the $y$-axis $-3f(x) \to -3f\left(\frac{1}{2}x\right)$
- translation 5 units to the left $-3f\left(\frac{1}{2}x\right) \to -3f\left(\frac{1}{2}(x+5)\right)$ **D**

## Question 4

The graph has the shape of the graph with rule $y = (ax)^{\frac{4}{3}} + 1$ or $y = (bx)^2 + 1$, where $a$ and $b$ are real constants. When $x = 2$, $y = 2$. Thus, $y = x^2 + 1$ and $y = x^{\frac{4}{3}} + 1$ are not possible. $y = \left(\frac{x}{2}\right)^{\frac{4}{3}} + 1$. **E**

## Question 5

At points of intersection,

$\sin(2x) = \sqrt{3}\cos(2x)$

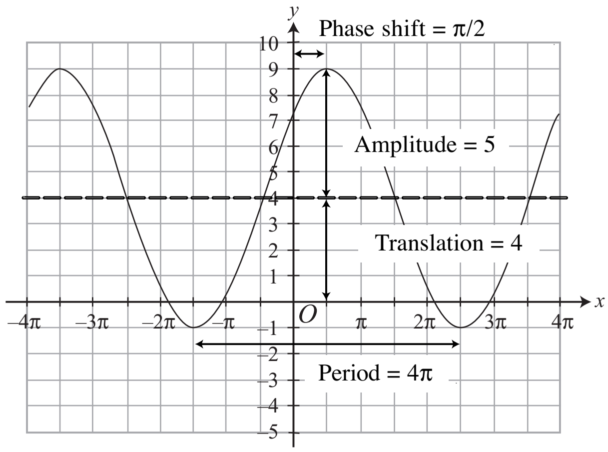
$\frac{\sin(2x)}{\cos(2x)} = \sqrt{3}$

$\tan(2x) = \sqrt{3}$

$2x = -\frac{2\pi}{3}, \frac{\pi}{3}$

$x = -\frac{\pi}{3}, \frac{\pi}{6}$ **D**

## Question 6




Amplitude = 5

Period = $4\pi$

Rule is of the form $y = a\cos(n(x - \varepsilon)) + k$

$a = 5,\ \varepsilon = \frac{\pi}{2},\ k = 4$

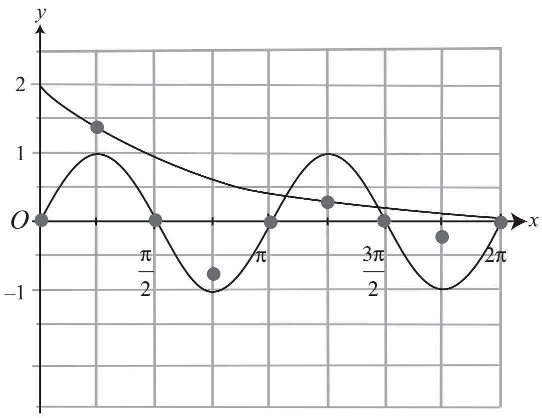
Period $= \frac{2\pi}{n}$, therefore $n = \frac{2\pi}{4\pi} = \frac{1}{2}$

Rule is $y = 5\cos\left(\frac{1}{2}\left(x - \frac{\pi}{2}\right)\right) + 4$

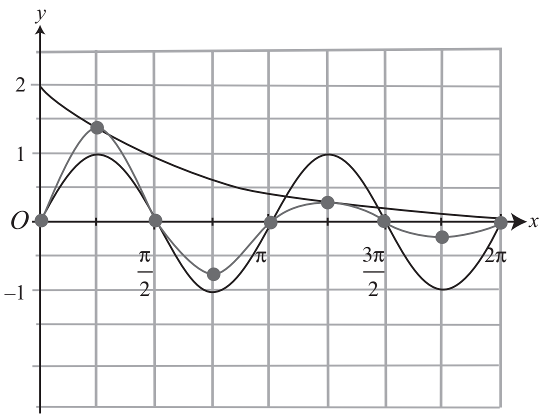
or $y = 5\cos\left(\frac{x}{2} - \frac{\pi}{4}\right) + 4$ **B**

## Question 7

When graphing product of functions, the key points occur when either function has the value of ±1 or 0, as illustrated in the diagram below. Mark these key points.




Then "join the dots".




**A**

## Question 8

$g(x) = 3(e^{2(x-1)} + 3)$

$= 3e^{2(x-1)} + 9$

The graph of $g$ has an asymptote at $y = 9$. Thus the graph of the inverse of $g$ has an asymptote at $x = 9$. **A**

## Question 9

The graph of $y = e^x$ has been reflected in both $x$ and $y$ axes, and then translated up.

Reject option B because it does not show the translation up. None of the other options indicate the possibility of a translation parallel to the $x$-axis.

Ignoring the magnitude of any dilations (because there is no scale on the axes), the transformed graph will be of the form $y = -e^{-x} + k$. Since $b < 0$, require $y = -ae^{bx} + k$. **C**

## Question 10

$\log_2(x + 2) + \log_2(x - 1) = 2$

$\log_2((x + 2)(x - 1)) = 2$

$(x + 2)(x - 1) = 4$

$x^2 + x - 6 = 0$

$(x + 3)(x - 2) = 2$

$x = 2$ only as $x > 1$ **C**

## Question 11

The coordinates of the turning point of $y = 2(x - 3)^6 + 4$ are $(3, 4)$. The tangent is $y = 4$. **D**

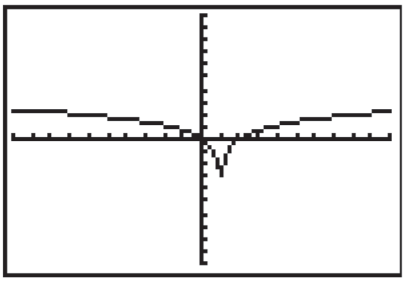
## Question 12

$$\text{The average rate of change} = \frac{f(1) - f(-2)}{1 - -2} = \frac{\sqrt{1} - \sqrt{4}}{3} = -\frac{1}{3}$$

**C**

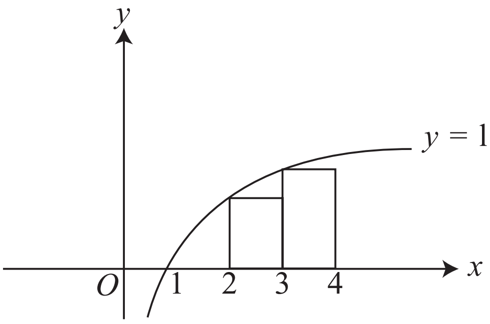
## Question 13

The graph of $y = \log_e|x-1|$ is shown below.



$\frac{d}{dx}\log_e|x-1| = \frac{1}{x-1}$ for $x \in R\backslash\{1\}$ **D**

## Question 14

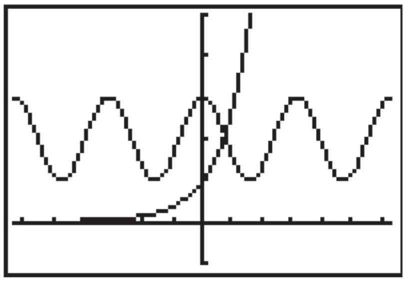



The area of the rectangles $= \log_e(2) + \log_e(3)$

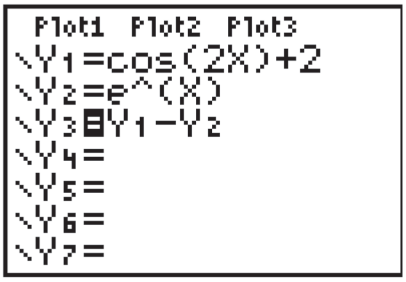
$= \log_e(6)$ **A**

## Question 15

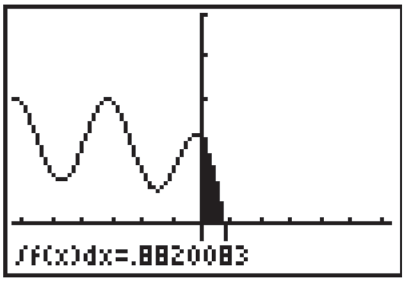
Sketch both curves and find the intersection.



Graph $f(x) - g(x)$




Find $\int_0^{0.7387} (f(x) - g(x))dx$




*t*

**D**

## Question 16

$$\int_1^k \left(\log_e(x)+1\right)dx = 1$$

$$\left[x\log_e(x) - x + x\right]_1^k = 1$$

$$\left[x\log_e(x)\right]_1^k = 1$$

$k\log_e(k) - 1 \times 0 = 1$

$k\log_e(k) = 1$ **E**

## Question 17

$f$ has a minimum turning point at $x = -1$ and a stationary point of inflection at $x = 2$. **A**

## Question 18

Let $X$ denote the number of children with blue eyes. $X \sim Bi\left(n = 3, p = \frac{1}{4}\right)$

$$\Pr(X \geq 1) = 1 - \Pr(X = 0)$$

$$= 1 - \left(\frac{3}{4}\right)^3$$

$$= 1 - \left(\frac{27}{64}\right)$$

$$= \frac{37}{64}$$ **B**

## Question 19

$p(x) = \dfrac{x^2+1}{10}$, for $x \in \{-1, 0, 1, 2\}$

$p(-1) = 0.2$, $p(0) = 0.1$, $p(1) = 0.2$, $p(2) = 0.5$

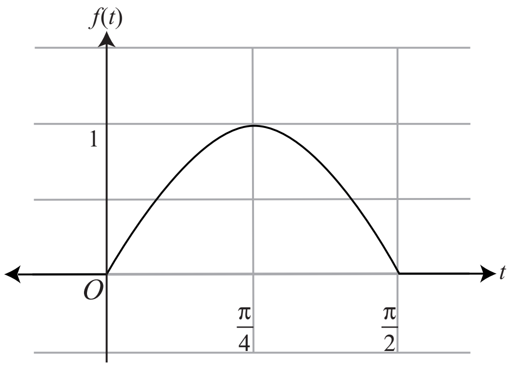
$$E(X) = \sum x\,p(x)$$

$$= -1\times 0.2 + 0\times 0.1 + 1\times 0.2 + 2\times 0.5$$

$$= 1$$ **D**

## Question 20

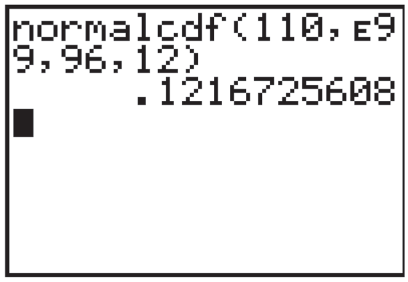
The mode is the value of $t$ that gives the maximum value of $f(t)$.




By symmetry, this occurs at $\frac{\pi}{4}$. **E**

## Question 21

$T \sim N(\mu = 96, \sigma^2 = 12^2)$




$\Pr(X > 110) = 0.1217$

Approximately 12% exceeded the limit. **C**

## Question 22

$$\Pr(SSSM) = \frac{6}{12}\times\frac{5}{11}\times\frac{4}{10}\times\frac{6}{9} = \frac{2}{33}$$ **A**

# Mathematical Methods Exam 2: SOLUTIONS

## Section 2: Extended answers

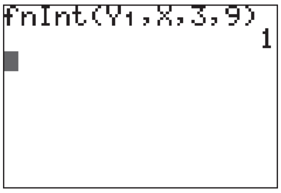
## Question 1

**a.** For a probability density function,

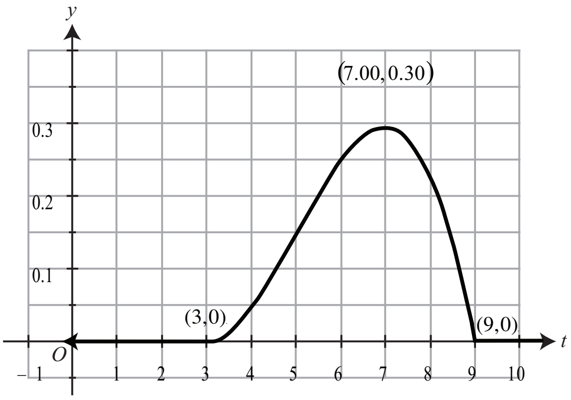
$\int_{-\infty}^{\infty} v(t)\,dt = 1$. Therefore,

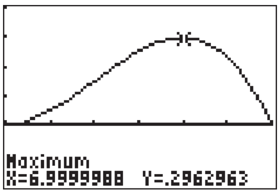
$$\int_3^9 -\frac{1}{108}\left(t^3 - 15t^2 + 63t - 81\right) dt = 1$$

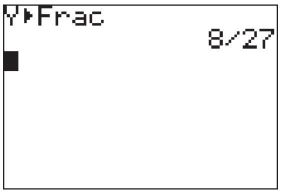
Showing integral with correct terminals **1M**
(*dt* must be shown to get the mark)




**b.**




Correct shape and skew **1A**
Correct domain and axes intercepts **1A**
Correct stationary point **1A**

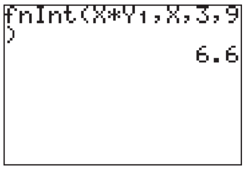
**c.** From the graph, the mode is 7, as this is the value of *t* at which the maximum value occurs. **1 M**
(Consequential from part b)

**d.** $\mu = \int_{-\infty}^{\infty} t\, v(t)\,dt$. Therefore,

$$\mu = -\frac{1}{108}\int_3^9 \left(t\left(t^3 - 15t^2 + 63t - 81\right)\right) dt \quad \textbf{1M}$$

$$= \frac{33}{5} \text{ or } 6.6$$

The mean incubation period is 6.6 days. **1A**




**e.** $X \sim N(5, \sigma^2)$, $Z \sim N(0,1)$

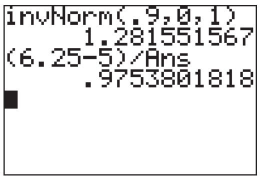
$\Pr(Z < z) = 0.9$ **1M**

$z = 1.28155$

$$z = \frac{x-\mu}{\sigma}$$

$$1.28155 = \frac{6.25-5}{\sigma}$$

$\sigma = 0.98$



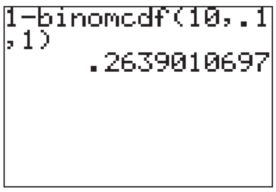

The standard deviation is 0.98 hours **1A**

**f.** Let *Y* be the number of patients for whom the response time to the drug is more than 6 hours 15 minutes.

$Y \sim Bi(10, 0.1)$ **1M**

$$\Pr\left(Y \geq 2\right) = \sum_{n=2}^{10} {}^{10}C_n \left(0.9\right)^n \left(0.1\right)^{10-n}$$

$= 0.264$ **1A**



**g.** $Y \sim Bi(k, 0.1)$

Consider the case where there is 95% chance.

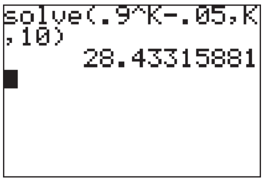
$\Pr(Y \geq 1) = 0.95$

$1 - \Pr(Y = 0) = 0.95$ **1M**

$1 - 0.9^k = 0.95$

$0.9^k = 0.05$

$k \approx 28.43$ (using numerical solve/ solver)




Round up because $k \in Z^+$ and the probability must be at least 95%.

The least value of $k$ is 29 patients. **1A**

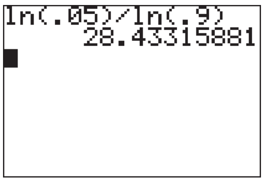
Alternative 1 to using numerical solve/ solver.

$0.9^k = 0.05$

$\log_e\left(0.9^k\right) = \log_e\left(0.05\right)$

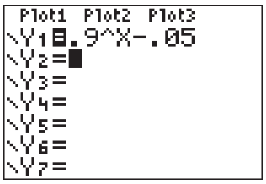
$k \log_e\left(0.9\right) = \log_e\left(0.05\right)$

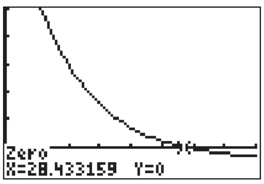
$k = \dfrac{\log_e\left(0.05\right)}{\log_e\left(0.9\right)} \approx 28.43$




However, this involves more steps and still requires the use of a calculator.

Alternative 2: graphical approach.

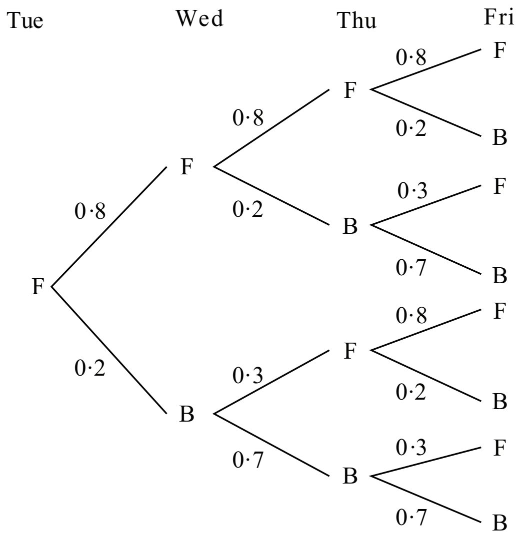



$0.9k - 0.05 = 0$

$k \approx 28.43$

**h.** **1M**




$\Pr(FFFB) = 0.8^2 \times 0.2 = 0.128$

$\Pr(FFBB) = 0.8 \times 0.2 \times 0.7 = 0.112$

$\Pr(FBFB) = 0.2 \times 0.3 \times 0.2 = 0.012$

$\Pr(FBBB) = 0.2 \times 0.7^2 = 0.098$ **1M**

$0.128 + 0.112 + 0.012 + 0.098 = 0.35$

The probability of Fluaway on Tuesday and Blugone on Friday is 0.35. **1A**

## Question 2

**a.** $f(x) = A(x^{\frac{4}{3}} - 5)$

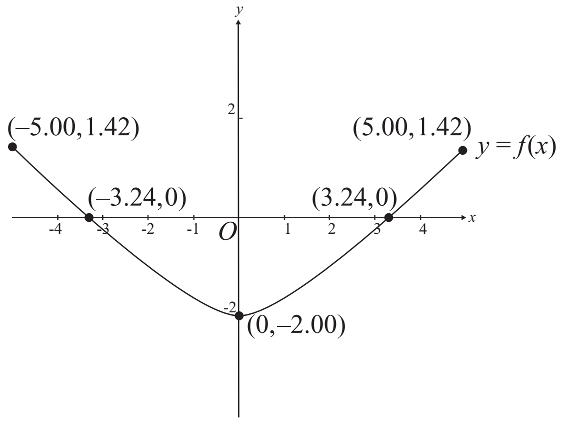
$-2 = A(0-5)$ **1M**

$A = \dfrac{2}{5}$

**b.** endpoints $(-5.00, 1.42)$ and $(5.00, 1.42)$ **1A**

turning point $(0, -2.00)$ and correct shape **1A**

$x$-intercepts $(-3.24, 0)$ and $(3.24, 0)$ **1A**




**c.** Max depth $= 2 + 1.42$

$= 3.42$ m **1A**

**d. i.** $A = -2 \times \dfrac{2}{5}\displaystyle\int_0^{5^{\frac{3}{4}}} (x^{\frac{4}{3}} - 5)\,dx$ **1M**

$= -\dfrac{4}{5}\left[\dfrac{3}{7}x^{\frac{7}{3}} - 5x\right]_0^{5^{\frac{3}{4}}}$ **1M**

$= -\dfrac{4}{5}\left(\left(\dfrac{3}{7}(5^{\frac{3}{4}})^{\frac{7}{3}} - 5^{\frac{7}{4}}\right) - (0)\right)$ **1M**

$= 7.643$ m$^2$ correct to 3 decimal places **1A**

**ii.** $V = 1000A$

$= 7643$ m$^3$ **1A**

**e. i.** $V = 1000A$

$= 1000(\frac{1}{2}\text{base} \times \text{height})$ **1M**

$= 2000 \times 5^{\frac{3}{4}}$

$= 6687.40$ m$^3$ **1A**

**ii.** $\dfrac{dV}{dt} = \dfrac{dh}{dt} \times \dfrac{dV}{dh}$ **1A**

$\dfrac{dV}{dt} = -0.02 \times \dfrac{dV}{dh}$

$V = 1000\dfrac{1}{2}bh$

Using similar triangles:

$\dfrac{\frac{1}{2}b}{h} = \dfrac{5^{\frac{3}{4}}}{2}$

$\dfrac{1}{2}b = \dfrac{5^{\frac{3}{4}}}{2}h$ **1M**

$V = 500 \times 5^{\frac{3}{4}}h^2$ **1A**

$\dfrac{dV}{dh} = 1000 \times 5^{\frac{3}{4}}h$

$\dfrac{dV}{dt} = -0.02 \times 1000 \times 5^{\frac{3}{4}}$

$= -20 \times 5^{\frac{3}{4}}h$ m$^3$/h **1A**

The rate at which water is decreasing is $20 \times 5^{\frac{3}{4}}h$ m$^3$/h

## Question 3

**a.** $12 = A \times B^2 \quad \ldots(1)$

$48 = A \times B^4 \quad \ldots(2)$

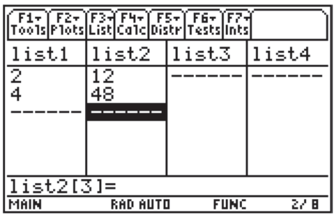
Divide (2) by (1)

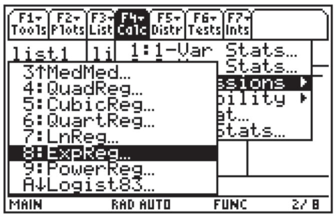
$4 = B^2$

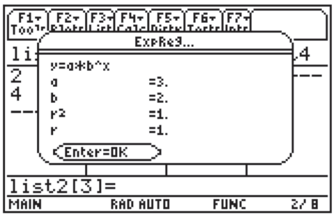
$B = 2$, as $B > 0$ **1A**

$A = \frac{12}{4} = 3$ **1A**

OR use ExpReg on the calculator.




**b.** Dilation by a factor of $\frac{1}{3}$ from the $y$-axis:

$y_1 = 3 \times 2^{3x}$ **1A**

Translation of $\frac{4}{3}$ units parallel to the $x$-axis:

$y_2 = 3 \times 2^{(3x-4)} = 3 \times 2^{\left(3(x-\frac{4}{3})\right)}$

Translation of 1 unit parallel to the $y$-axis:

$g_2(x) = 3 \times 2^{(3x-4)} + 1 = 3 \times 2^{\left(3(x-\frac{4}{3})\right)} + 1$ **1A**

Range $(1, \infty)$ **1A**

**c. i.** $g_3(x) = 3 \times 2^{(2x+1)} - 1$

Let $y = 3 \times 2^{(2x+1)} - 1$

Inverse: Swap $x$ and $y$

$x = 3 \times 2^{(2y+1)} - 1$ **1M**

$2^{2y+1} = \frac{x+1}{3}$

$2y + 1 = \log_2\left(\frac{x+1}{3}\right)$

$g_3^{-1}(x) = \frac{1}{2}\log_2\left(\frac{x+1}{3}\right) - \frac{1}{2}$ **1M**

$= \frac{1}{2}\log_2\left(\frac{x+1}{6}\right)$

**ii.** Dilation by a factor of $\frac{1}{2}$ from the $x$-axis

Dilation by a factor of 6 from the $y$-axis. **1A**

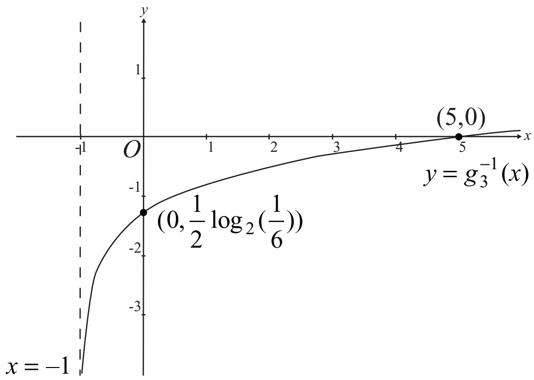
Translation of 1 unit to the left **1A**

**iii.** Asymptote: $x = -1$ and shape **1A**

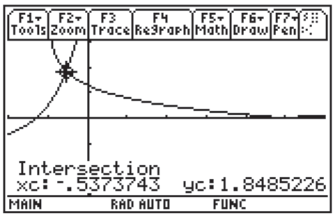
Coordinates of the $x$-intercept: $(5, 0)$

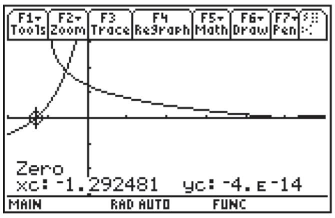
Coordinates of the $y$-intercept:

$\left(0, \frac{1}{2}\log_2\left(\frac{1}{6}\right)\right)$ **1A**

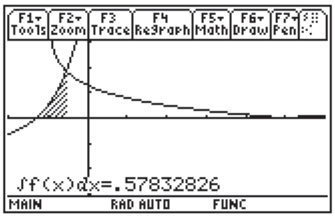


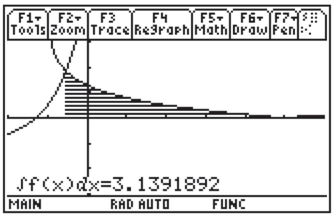
**iv.** Sketch both graphs and find the intersection and $x$-intercepts.




$$A = \int_{-1.292481}^{-0.53737} g_3(x)dx + \int_{-0.53737}^{5} \left|g_3^{-1}(x)\right| dx$$ **1M**




$A \approx 0.578328 + 3.139189$

$= 3.7$ units$^2$ correct to one decimal place **1A**

## Question 4

**a.** By Pythagoras' theorem

$x^2 + y^2 = 50^2$

$y = \sqrt{2500 - x^2}$ (reject negative solution) **1A**

**b.** Highest position when $x = 0$.
Endpoint will be (0, 50). $y = 50$ cm **1A**

Lowest position when $x = \pm 30$. The corresponding $y$-axis value is

$y = \sqrt{2500 - (30)^2} = 40$ or

$y = \sqrt{2500 - (-30)^2} = 40$

$y = 40$ cm **1A**

**c.** Period along the $x$-axis $= \dfrac{2\pi}{\pi/8} = 16$ sec **1A**

**d.** $30\cos\left(\dfrac{\pi}{8}t\right) = 15$

$\cos\left(\dfrac{\pi}{8}t\right) = \dfrac{1}{2}$

$\left(\dfrac{\pi}{8}t\right) = \dfrac{\pi}{3}, \dfrac{5\pi}{3}$

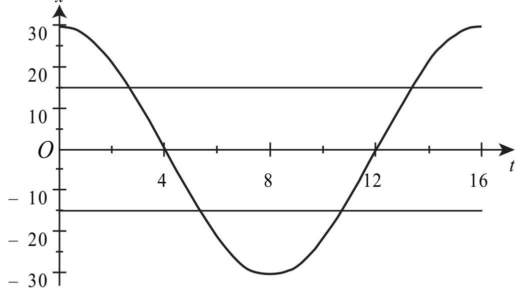
$t = \dfrac{8}{3}, \dfrac{40}{3}$ **1M**

$30\cos\left(\dfrac{\pi}{8}t\right) = -15$

$\cos\left(\dfrac{\pi}{8}t\right) = -\dfrac{1}{2}$

$\left(\dfrac{\pi}{8}t\right) = \dfrac{2\pi}{3}, \dfrac{4\pi}{3}$

$t = \dfrac{16}{3}, \dfrac{32}{3}$ **1M**




Time for which is given by

$\left(\dfrac{16}{3} - \dfrac{8}{3}\right) + \left(\dfrac{40}{3} - \dfrac{32}{3}\right) = \dfrac{16}{3}$ seconds **1A**

**e.** $\frac{dx}{dt} = -\frac{15\pi}{4}\sin\left(\frac{\pi}{8}t\right)$ **1M**

When $x = 15$, $t = \frac{8}{3}, \frac{40}{3}, \ldots$ (from part **d.**)

$$x'\left(\frac{8}{3}\right) = -\frac{15\pi}{4}\sin\left(\frac{\pi}{8}\times\frac{8}{3}\right)$$
$$= -\frac{15\pi}{4}\sin\left(\frac{\pi}{3}\right)$$
$$= -\frac{15\pi}{4}\times\frac{\sqrt{3}}{2}$$
$$= -\frac{15\sqrt{3}\pi}{8}$$ **1M**

Speed is the magnitude of $x'(t)$

Speed = $\frac{15\sqrt{3}\pi}{8}$ cm/s **1A**

**f.** $x'(t) = -\frac{15\pi}{4}\sin\left(\frac{\pi}{8}t\right)$

The maximum speed occurs where

$$\frac{d}{dt}\left(x'(t)\right) = 0$$ **1M**

$$\frac{d}{dt}\left(-\frac{15\pi}{4}\sin\left(\frac{\pi}{8}t\right)\right) = 0$$

$$-\frac{15\pi}{4}\times\frac{\pi}{8}\cos\left(\frac{\pi}{8}t\right) = 0$$

$$\cos\left(\frac{\pi}{8}t\right) = 0$$

$$\frac{\pi}{8}t = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$$

$$t = \frac{\pi}{2}\times\frac{8}{\pi}, \frac{3\pi}{2}\times\frac{8}{\pi}, \ldots$$ **1M**

$t = 4, 12, \ldots$

(This is also evident from the graph of $x$ v. $t$, which show the maximum gradient at $t = 4, 12, \ldots$)

$$x'(4) = -\frac{15\pi}{4}\sin\left(\frac{\pi}{2}\right) = -\frac{15\pi}{4}$$

$$x'(12) = -\frac{15\pi}{4}\sin\left(\frac{3\pi}{2}\right) = \frac{15\pi}{4}$$

The speed is the magnitude of $x'(t)$.

Hence the maximum speed is $\frac{15\pi}{4}$ cm/s. **1A**

# MATHEMATICAL METHODS AND MATHEMATICAL METHODS (CAS)

## Written examinations 1 and 2

### FORMULA SHEET

**Directions to students**

| |
|---|
| Detach this formula sheet during reading time.<br>This formula sheet is provided for your reference. |



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# Mathematical Methods and Mathematical Methods CAS Formulas

## Mensuration

| | | | |
|---|---|---|---|
| area of a trapezium: | $\frac{1}{2}(a+b)h$ | volume of a pyramid: | $\frac{1}{3}Ah$ |
| curved surface area of a cylinder: | $2\pi rh$ | volume of a sphere: | $\frac{4}{3}\pi r^3$ |
| volume of a cylinder: | $\pi r^2 h$ | area of a triangle: | $\frac{1}{2}bc\sin A$ |
| volume of a cone: | $\frac{1}{3}\pi r^2 h$ | | |

## Calculus

$$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$$

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + c, n \neq -1$$

$$\frac{d}{dx}\left(e^{ax}\right) = ae^{ax}$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + c$$

$$\frac{d}{dx}\left(\log_e(x)\right) = \frac{1}{x}$$

$$\int \frac{1}{x}\,dx = \log_e|x| + c$$

$$\frac{d}{dx}\left(\sin(ax)\right) = a\cos(ax)$$

$$\int \sin(ax)\,dx = -\frac{1}{a}\cos(ax) + c$$

$$\frac{d}{dx}\left(\cos(ax)\right) = -a\sin(ax)$$

$$\int \cos(ax)\,dx = \frac{1}{a}\sin(ax) + c$$

$$\frac{d}{dx}\left(\tan(ax)\right) = \frac{a}{\cos^2(ax)} = a\sec^2(ax)$$

product rule: $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

quotient rule: $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

chain rule: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

approximation: $f(x+h) \approx f(x) + hf'(x)$

## Probability

$\Pr(A) = 1 - \Pr(A')$

$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$

$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$

mean: $\mu = \mathrm{E}(X)$

variance: $\mathrm{var}(X) = \sigma^2 = \mathrm{E}((X-\mu)^2) = \mathrm{E}(X^2) - \mu^2$

| probability distribution | | mean | variance |
|---|---|---|---|
| discrete | $\Pr(X = x) = p(x)$ | $\mu = \sum x\,p(x)$ | $\sigma^2 = \sum (x-\mu)^2\,p(x)$ |
| continuous | $\Pr(a < X < b) = \int_a^b f(x)\,dx$ | $\mu = \int_{-\infty}^{\infty} x\,f(x)\,dx$ | $\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx$ |

**END OF FORMULA SHEET**

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