

Mathematical Methods Exam 1: Solutions

Question 1

$$\begin{aligned} \text{a. } \frac{x+2}{x-3} &= 1 + \frac{5}{x-3} \\ \frac{x+2}{x-3} &= \frac{(x-3)+5}{x-3} \\ &= \frac{(x-3)}{x-3} + \frac{5}{x-3} \\ &= 1 + \frac{5}{x-3} \end{aligned}$$

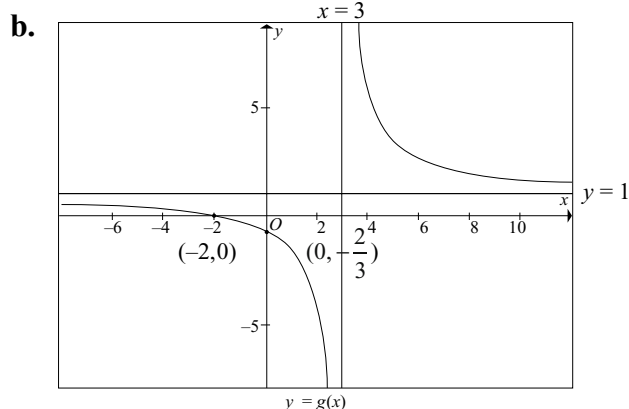
1M

Alternatively, use the long division algorithm.

$$\begin{array}{r} 1 \\ x-3 \overline{) x+2} \\ \underline{x-3} \\ 5 \end{array}$$

1M

$$\frac{x+2}{x-3} = 1 + \frac{5}{x-3}$$



Shape

1A

Asymptotes $y = 1$ and $x = 3$

1A

Intercepts $\left(0, -\frac{2}{3}\right)$ and $(-2, 0)$

1A

c. $(-\infty, -2] \cup (3, \infty)$

1A

Question 2

a. $a = \frac{1}{2}$

1A

b. Let $y = \log_e |2x - 1|$, where $x < \frac{1}{2}$

For the inverse swap x and y

$$x = \log_e |2y - 1|, \text{ where } y < \frac{1}{2}$$

1A

$$x = \begin{cases} \log_e (2y - 1), & y > \frac{1}{2} \\ \log_e (1 - 2y), & y < \frac{1}{2} \end{cases}$$

$$e^x = 1 - 2y$$

$$y = \frac{1 - e^x}{2}$$

$$f^{-1}(x) = \frac{1 - e^x}{2}$$

1A

Question 3

$$4^x - 5(2^x) = k$$

Let $a = 2^x, a > 0$

$$a^2 - 5a - k = 0$$

1M

$$a = \frac{5 \pm \sqrt{25 + 4k}}{2}$$

$$0 < \Delta < 25 \text{ as } a > 0$$

$$0 < 25 + 4k < 25$$

1M for discriminant

1A for restriction

$$-\frac{25}{4} < k < 0$$

1A

Question 4

a. $f(g(x)) = (1 - \log_e(2x))^{\frac{1}{3}}$ **1A**

b. By the chain rule,

$$f'(g(x)) \times g'(x) = \frac{1}{3}(1 - \log_e(2x))^{\frac{-2}{3}} \times \frac{-1}{x} \quad \mathbf{1M}$$

Substitute $x = \frac{1}{2}$ into the derivative to find m .

$$m = \frac{1}{3}(1 - \log_e(1))^{\frac{-2}{3}} \times -2$$

$$= \frac{-2}{3}$$

1M

$$f\left(g\left(\frac{1}{2}\right)\right) = (1 - \log_e(1))^{\frac{1}{3}} = 1$$

1M

The equation of the tangent is

$$y - 1 = \frac{-2}{3}\left(x - \frac{1}{2}\right)$$

$$y = -\frac{2}{3}x + \frac{4}{3}$$

1A**Question 5**

$$\text{Area} = -\int_{-1}^0 (x(x+1)^2) dx = \int_0^{-1} (x(x+1)^2) dx \quad \mathbf{1A}$$

$$= \int_0^{-1} (x^3 + 2x^2 + x) dx$$

$$= \left[\frac{x^4}{4} + \frac{2x^3}{3} + \frac{x^2}{2} \right]_0^{-1}$$

1M

$$= \left(\frac{1}{4} - \frac{2}{3} + \frac{1}{2} \right) - 0$$

$$= \frac{3 - 8 + 6}{12}$$

$$= \frac{1}{12} \text{ units}^2$$

1A**Question 6**

a. i. Range: $[-4 - 2, 4 - 2] = [-6, 2]$ **1A**

ii. Period $= \frac{2\pi}{\pi/6} = \frac{2\pi}{1} \times \frac{6}{\pi} = 12$ **1A**

b. Solve $4\sin\left(\frac{\pi}{6}t\right) - 2 = 0$.

$$\sin\left(\frac{\pi}{6}t\right) = \frac{1}{2} \quad \mathbf{1M}$$

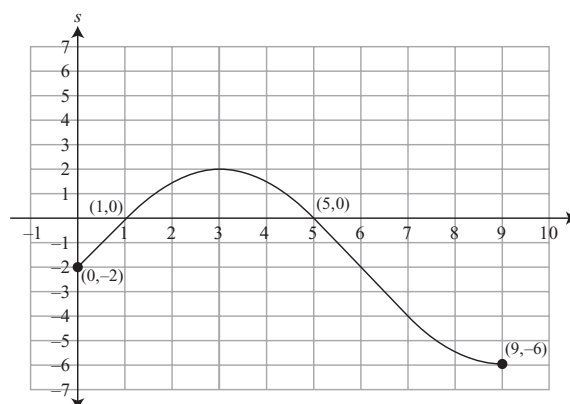
$$\frac{\pi}{6}t = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \dots$$

$$t = \frac{\pi}{6} \times \frac{6}{\pi}, \frac{5\pi}{6} \times \frac{6}{\pi}, \frac{13\pi}{6} \times \frac{6}{\pi}, \dots$$

Since $t \in [0, 9]$,

$$t = 1 \text{ or } t = 5 \quad \mathbf{1A}$$

c.



Correct shape: **1A**

Coordinates of x-axis intercepts labelled: **1A**

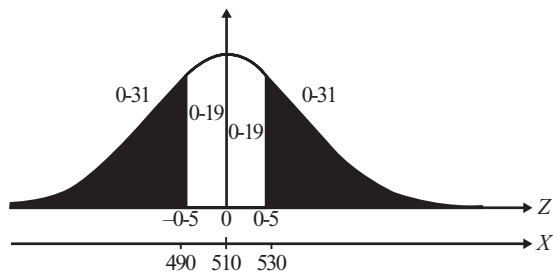
Endpoints labelled: **1A**

Question 7

- a. $\Pr(\mu - 2\sigma < X < \mu + 2\sigma) \approx 0.95$
Therefore $\Pr(430 < X < 590) \approx 0.95$
 $k = 430$

1A

p.



$$\begin{aligned}\Pr(X < 530) &= 1 - \Pr(Z < -0.5) & 1M \\ &= 1 - 0.31 \\ &= 0.69 & 1A\end{aligned}$$

$$\begin{aligned}\text{c. } \Pr(X < 530 | X > 510) &= \frac{\Pr(X > 510 \cap X < 530)}{\Pr(X > 510)} \\ &= \frac{\Pr(510 < X < 530)}{\Pr(X > 510)} & 1M \\ &= \frac{0.19}{0.5} = 0.19 \times 2 \\ &= 0.38 & 1A\end{aligned}$$

Question 8

- a. For f to be a probability density function,

$$\int_{-\infty}^{\infty} f(x) dx = 1. \text{ Therefore,}$$

$$0 + a \int_{-1}^3 (x+1) dx = 1$$

$$a \left[\frac{x^2}{2} + x \right]_{-1}^3 = 1$$

1M

$$a \left[\left(\frac{9}{2} + 3 \right) - \left(\frac{1}{2} - 1 \right) \right] = 1$$

$$8a = 1$$

$$a = \frac{1}{8}, \text{ as required}$$

1M

b.

$$\Pr(X < 0) = \frac{1}{8} \int_{-1}^0 (x+1) dx$$

$$= \frac{1}{8} \left[\frac{x^2}{2} + x \right]_{-1}^0$$

$$= \frac{1}{8} \left[0 - \left(\frac{1}{2} - 1 \right) \right]$$

$$= \frac{1}{16}$$

1A

c.

$$E(X) = \int_{-\infty}^{\infty} x f(x) dx$$

$$= 0 + \frac{1}{8} \int_{-1}^3 (x^2 + x) dx$$

$$= \frac{1}{8} \left[\frac{x^3}{3} + \frac{x^2}{2} \right]_{-1}^3$$

1M

$$= \frac{1}{8} \left[\left(9 + \frac{9}{2} \right) - \left(-\frac{1}{3} + \frac{1}{2} \right) \right]$$

$$= \frac{1}{8} \left[\frac{26}{2} + \frac{1}{3} \right]$$

$$= \frac{1}{8} \times \frac{40}{3}$$

$$= \frac{5}{3}$$

1A

d.

$$\frac{1}{8} \int_{-1}^m (x+1) dx = \frac{1}{2}$$

$$\int_{-1}^m (x+1) dx = 4$$

$$\left[\frac{x^2}{2} + x \right]_{-1}^m = 4 \quad \mathbf{1M}$$

$$\left[\left(\frac{m^2}{2} + m \right) - \left(\frac{1}{2} - 1 \right) \right] = 4$$

$$\frac{m^2}{2} + m + \frac{1}{2} = 4$$

$$m^2 + 2m - 7 = 0$$

Use quadratic formula or complete the square

$$m = \frac{-2 \pm \sqrt{4 + 28}}{2}$$

$$= \frac{-2 \pm \sqrt{32}}{2}$$

$$= \frac{-2 \pm 4\sqrt{2}}{2}$$

$$= -1 \pm 2\sqrt{2} \quad \mathbf{1M}$$

Note that $-1 - 2\sqrt{2}$ is outside the domain because $m > -1$.

$$\text{Median value is } -1 + 2\sqrt{2} \quad \mathbf{1A}$$