

INSIGHT

Trial Exam Paper

2007

MATHEMATICAL METHODS

Written examination 1

Worked solutions

This book presents:

- correct answers
- worked solutions, giving you a series of points to show you how to work through the questions
- mark allocation details.

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Question 1

Let $f(x) = 2x - 5$ and $g(x) = \cos x$. Write down the rule of $f(g(x))$.

Solution

$$f(g(x)) = 2\cos x - 5$$

1 mark

Mark allocation

- 1 mark for the correct answer

Question 2

For the function $f : (2, \infty) \rightarrow \mathbb{R}$, $f(x) = 2\log_e(x - 1)$,

2a. find the rule for the inverse function f^{-1} .

Solution

Interchange x and y to give

$$x = 2\log_e(y - 1)$$

$$e^{\frac{x}{2}} = y - 1$$

$$y = 1 + e^{\frac{x}{2}}$$

2 marks

Mark allocation

- 1 mark for method
- 1 mark for the correct answer

2b. find the domain of the inverse function f^{-1} .

Solution

$$\begin{aligned} \text{dom } f^{-1} &= \text{ran } f \\ &= \mathbb{R}^+ \end{aligned}$$

1 mark

Mark allocation

- 1 mark for the correct answer

Question 3

For the function $f : [-\pi, \pi] \rightarrow \mathbb{R}$, $f(x) = -2\sin(3(x - \frac{\pi}{4}))$

3a. write down the amplitude and period of the function.

Solution

Amplitude is 2

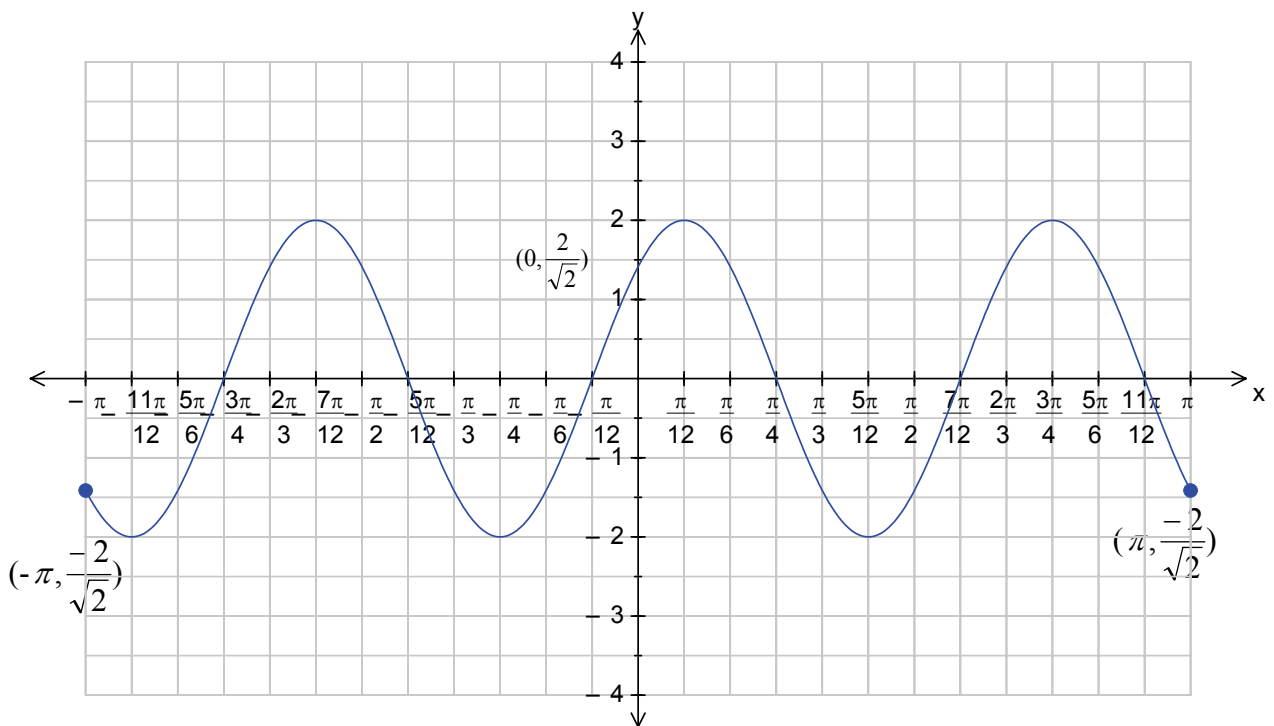
Period is $\frac{2\pi}{3}$

2 marks

Mark allocation

- 1 mark for each of amplitude and period

3b. on the set of axes below, sketch the graph of the function f . Label the axis intercepts with their coordinates. Label the end-points of the graph with their coordinates.

Solution

3 marks

Mark allocation

- 1 mark for shape – the graph must be a smooth, regular sine curve shape, must have 3 cycles and must have the correct amplitude
- 1 mark for correct intercepts – the intercepts must be correctly labelled with their coordinates; the x -intercepts are at $(\frac{\pi}{4}, 0)$, $(\frac{7\pi}{12}, 0)$, $(\frac{11\pi}{12}, 0)$, $(-\frac{\pi}{12}, 0)$, $(-\frac{5\pi}{12}, 0)$ and $(-\frac{3\pi}{4}, 0)$ and the y -intercept is at $(0, \frac{2}{\sqrt{2}})$
- 1 mark for correctly placing and labelling the graph end-points

Question 4

4a. Let $f(x) = \log_e(\sin(x))$. Find $f'(x)$.

Solution

Use the chain rule:

$$f'(x) = \frac{1}{\sin x} \cos x = \frac{\cos x}{\sin x}$$

1 mark

Mark allocation

- 1 mark for the correct answer

4b. Let $y = x^2 \cos(x)$. Evaluate $\frac{dy}{dx}$ when $x = \frac{\pi}{3}$.

Solution

Using the product rule gives $\frac{dy}{dx} = 2x \cos x - x^2 \sin x$

$$\begin{aligned} \text{At } x = \frac{\pi}{3}, \quad x = \frac{\pi}{3} \Rightarrow \frac{dy}{dx} &= \frac{2\pi}{3} \frac{1}{2} - \frac{\pi^2 \sqrt{3}}{9 \times 2} \\ &= \frac{\pi}{3} - \frac{\sqrt{3}\pi^2}{18} \end{aligned}$$

2 marks

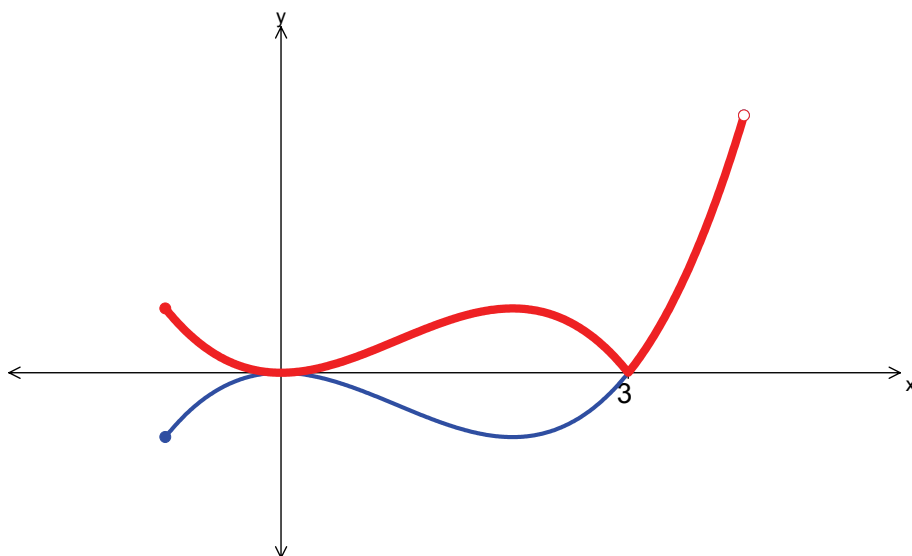
Mark allocation

- 1 mark for evidence of using the product rule
- 1 mark for the correct answer

Question 5

The graph of $f : [-1, 4] \rightarrow \mathbb{R}$ where $f(x) = x^3 - 3x^2$ is shown below.

5a. Let $g(x) = |f(x)|$. On the same set of axes, sketch the graph of g .

Solution

2 marks

Mark allocation

- 1 mark for graph drawn as shown: in the interval from $x = 3$ to $x = 4$, it must be obvious that the original graph has been drawn over
- 1 mark for correct end-points

5b. State the domain of the derivative function g' .

Solution

$(-1, 3) \cup (3, 4)$ –note that the graph is not differentiable at the end-points or at the cusp

1 mark

Mark allocation

- 1 mark for the correct answer

Question 6

Solve the equation $\sin(2x) - \sqrt{3} \cos(2x) = 0$ for $x \in [0, 2\pi]$, giving exact values in terms of π .

Solution

$$\sin 2x = \sqrt{3} \cos 2x$$

$$\tan 2x = \sqrt{3}$$

$$2x = \frac{\pi}{3}, \frac{4\pi}{3}, \frac{7\pi}{3}, \frac{10\pi}{3}$$

$$x = \frac{\pi}{6}, \frac{2\pi}{3}, \frac{7\pi}{6}, \frac{5\pi}{3}$$

3 marks

Mark allocation

- 1 mark for getting equation in terms of tan
- 1 mark for obtaining the first quadrant angle of $\frac{\pi}{3}$
- 1 mark for getting all answers correct

Question 7

The probability density function of a continuous random variable X is given by

$$f(x) = \begin{cases} \frac{x}{k} & 2 \leq x \leq 6 \\ 0 & \text{otherwise} \end{cases}$$

7a. Show that $k = 16$.

Solution

$$\int_2^6 \frac{x}{k} dx = 1$$

$$\frac{1}{k} \left[\frac{x^2}{2} \right]_2^6 = 1$$

$$\frac{32}{2k} = 1$$

$$\Rightarrow k = 16$$

2 marks

Mark allocation

- 1 mark for setting the integral equal to 1
- 1 mark for solving to get $k = 16$

7b. Find $\Pr(X > 4)$

Solution

$$\begin{aligned}\Pr(X > 4) &= \frac{1}{16} \int_4^6 x \, dx \\ &= \frac{1}{32} [x^2]_4^6 \\ &= \frac{1}{32} (36 - 16) \\ &= \frac{5}{8}\end{aligned}$$

2 marks

Mark allocation

- 1 mark for setting up an integral from 4 to 6
- 1 mark for the correct answer

7c. Find the median of X

Solution

$$\begin{aligned}\int_2^m \frac{x}{16} \, dx &= 0.5 \\ \frac{1}{32} [x^2]_2^m &= 0.5 \\ m^2 - 4 &= 16 \\ m^2 &= 20 \\ m &= 2\sqrt{5} \text{ (since } m > 0\text{)}\end{aligned}$$

2 marks

Mark allocation

- 1 mark for setting up the integral as equal to 0.5
- 1 mark for the correct answer

Question 8

The random variable X has the following probability distribution:

x	-1	0	1	2
$Pr(X=x)$	$a+b$	$2a-b$	$3a$	0.4

8a. Find the value of a .

Solution

The sum of the probabilities has to equal 1 so

$$a + b + 2a - b + 3a + 0.4 = 1$$

$$6a + 0.4 = 1$$

$$6a = 0.6$$

$$a = 0.1$$

1 mark

Mark allocation

- 1 mark for the correct answer

8b. If $E(X) = 0.95$, find the value of b .

Solution

$$E(X) = \sum x \Pr(X) = -1 \times (a + b) + 3a + 2 \times 0.4 = 0.95$$

$$2a - b = 0.15$$

$$b = 0.05$$

2 marks

Mark allocation

- 1 mark for equating the sum of the products to 0.95
- 1 mark for the correct answer

Question 9

The random variable X is normally distributed with mean 50 and standard deviation 5. The random variable Z is normally distributed with mean 0 and standard deviation 1.

9a. If $\Pr(X < 56) = \Pr(Z < a)$, find the value of a .

Solution

$$\Pr(X < 56) = \Pr\left(Z < \frac{56 - 50}{5}\right)$$

$$= \Pr\left(Z < \frac{6}{5}\right)$$

$$\Rightarrow a = \frac{6}{5}$$

2 marks

Mark allocation

- 1 mark for converting to a z-score
- 1 mark for the correct answer

9b. If $\Pr(50 < X < b) = 0.5 - \Pr(Z > 2)$, find the value of b .

Solution

Using symmetry, $0.5 - \Pr(Z > 2) \equiv \Pr(0 < Z < 2)$

Converting to standard normal distribution gives

$$\frac{b - 50}{5} = 2$$

$$b - 50 = 10$$

$$b = 60$$

2 marks

Mark allocation

- 1 mark for recognising that $0.5 - \Pr(Z > 2) \equiv \Pr(0 < Z < 2)$
- 1 mark for the correct answer

Question 10

A hemispherical bowl is being filled with water at a constant rate of $150\pi \text{ cm}^3/\text{min}$. When the depth of the water in the bowl is h cm, the volume, $V \text{ cm}^3$, of the water is given by

$V = \pi h^2(30 - \frac{2}{3}h)$. Find the rate at which the depth of the water is increasing when the depth is 5 cm.

Solution

$$\frac{dV}{dt} = 150\pi$$

$$V = \pi h^2(30 - \frac{2}{3}h)$$

$$\Rightarrow \frac{dV}{dh} = 60\pi h - 2\pi h^2$$

Using the chain rule:

$$\begin{aligned} \frac{dh}{dt} &= \frac{dV}{dt} \times \frac{dh}{dV} \\ &= 150\pi \times \frac{1}{60\pi h - 2\pi h^2} \\ &= \frac{150}{60h - 2h^2} \end{aligned}$$

$$\text{At } h = 5, \quad \frac{dh}{dt} = \frac{150}{250} = \frac{3}{5} \text{ cm/min}$$

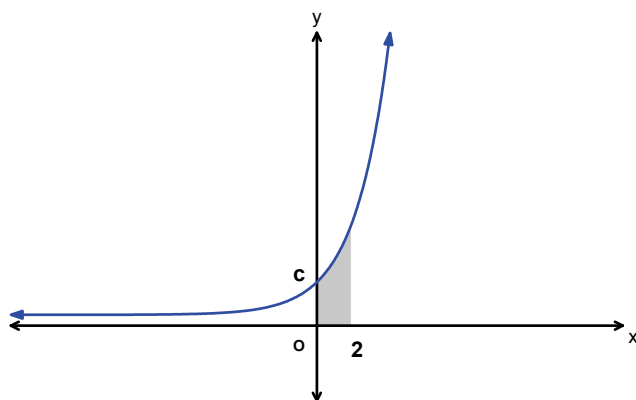
4 marks

Mark allocation

- 1 mark for recognising $\frac{dV}{dt}$
- 1 mark for determining $\frac{dV}{dh}$
- 1 mark for using the chain rule
- 1 mark for the correct answer

Question 11

Part of the graph of the function $f : \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = ae^{2x} + b$ is shown below. If the shaded area is $3e^4 + 1$ square units, find one set of possible values for a , b and c , where c is the y -intercept of the graph $y = f(x)$.

**Solution**

At $x = 0$, $y = a + b \Rightarrow a + b = c$

$$\int_0^2 ae^{2x} + b \, dx = 3e^4 + 1$$

$$\Rightarrow \left[\frac{a}{2} e^{2x} + bx \right]_0^2 = 3e^4 + 1$$

$$\Rightarrow \frac{a}{2} e^4 + 2b - \frac{a}{2} = 3e^4 + 1$$

equating coefficients on each side gives $\frac{a}{2} = 3$ so $a = 6$ and

$$2b - \frac{a}{2} = 1$$

$$2b = 4$$

$$b = 2$$

and

$$a + b = c \text{ so } c = 8$$

(Note other values for a , b , c are possible using the equation $b = -\frac{e^4 - 1}{4}a + \frac{3}{2}e^4 + \frac{1}{2}$.)

However, only one set of values was required, and the simplest set is that obtained by equating coefficients, so there is no need to pursue other solutions.)

5 marks

Mark allocation

- 1 mark for setting up the integral from 0 to 2 as equal to $3e^4 + 1$
- 1 mark for antidiifferentiating
- 1 mark for getting $a = 6$
- 1 mark getting $b = 2$
- 1 mark for getting $c = 8$

END OF SOLUTION BOOK