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MATHS METHODS 3 & 4 TRIAL EXAMINATION 1 SOLUTIONS 2007

Question 1

The function $f(x) = \log_e(x-2)$ is defined for

$$x - 2 > 0$$
$$x > 2$$

So $d_f = (2, \infty)$

(1 mark)

Question 2

a. $f(g(x))$ exists iff $r_g \subseteq d_f$.

Now, $r_g = R$ and $d_f = R \setminus \{0\}$.

Since $R \not\subset R \setminus \{0\}$,

$r_g \not\subset d_f$

so $f(g(x))$ does not exist.

(1 mark)

b. **i.**

$$g(f(x)) = g\left(\frac{1}{2x}\right) = \frac{1}{2x} + 1$$

(1 mark)

ii.

$$d_{g(f(x))} = d_f = R \setminus \{0\}$$

(1 mark)

Question 3

a. $f:(1,\infty)\to R,\ f(x)=\dfrac{3}{x-1}+2$

Let $y=\dfrac{3}{x-1}+2$

Swap x and y

$$x=\frac{3}{y-1}+2$$

(1 mark)

$$x-2=\frac{3}{y-1}$$

$$(x-2)(y-1)=3$$

$$y-1=\frac{3}{x-2}$$

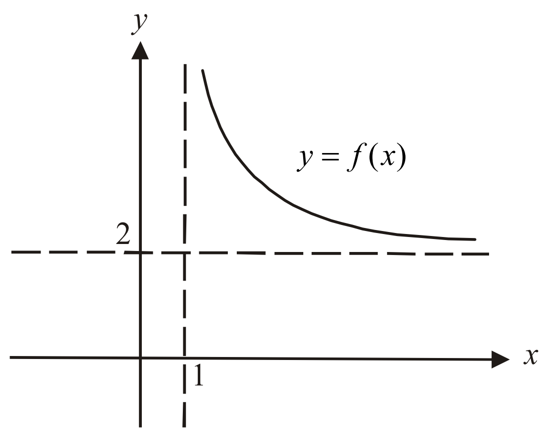
$$y=\frac{3}{x-2}+1$$

So $f^{-1}(x)=\dfrac{3}{x-2}+1$

(1 mark)

b. $d_{f^{-1}}=r_f$

Do a quick sketch of $y=\dfrac{3}{x-1}+2$



$r_f=(2,\infty)$

So $d_{f^{-1}}=(2,\infty)$

(1 mark)

Question 4

a. $f(x)=\sin\left(e^{2x}\right)$

Let $y=\sin\left(e^{2x}\right)$

$y=\sin(u)$ Let $u=e^{2x}$

$\frac{dy}{du}=\cos(u)$ $\frac{du}{dx}=2e^{2x}$

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx} \text{ (Chain rule)}$$
$$=\cos(u)\cdot 2e^{2x}$$
$$=2e^{2x}\cos\left(e^{2x}\right)$$

(1 mark) – for knowing to use the chain rule and attempting to use it
(1 mark) – correct answer

b.

$$y=\frac{\log_e(x)}{x^3+2x}$$
$$\frac{dy}{dx}=\frac{\left(x^3+2x\right)\times\frac{1}{x}-\left(3x^2+2\right)\log_e(x)}{\left(x^3+2x\right)^2}$$
$$=\frac{x^2+2-\left(3x^2+2\right)\log_e(x)}{\left(x^3+2x\right)^2}$$

(1 mark) – for knowing to use the quotient rule and attempting to use it
(1 mark) – correct answer

Question 5

For $y = 2\sin\left(2\left(x - \frac{\pi}{3}\right)\right)$ the period is $\frac{2\pi}{2} = \pi$ **(1 mark)** correct period

This means that for $x \in [-\pi, \pi]$ there will be two complete periods of the graph.

<u>x-intercepts</u>

<u>Method 1</u>

The graph is a sine graph with a period of π that has been translated $\frac{\pi}{3}$ units to the right so that the x-intercepts will occur at $\left(\frac{\pi}{3}, 0\right)$, at $\left(\frac{\pi}{2} + \frac{\pi}{3}, 0\right) = \left(\frac{5\pi}{6}, 0\right)$, at $\left(-\frac{\pi}{2} + \frac{\pi}{3}, 0\right) = \left(\frac{-\pi}{6}, 0\right)$ and at $\left(-\pi + \frac{\pi}{3}, 0\right) = \left(\frac{-2\pi}{3}, 0\right)$. **(1 mark)** for x- intercepts

<u>Method 2</u>

The x- intercepts occurs when $y = 0$.

$$y = 2\sin\left(2\left(x - \frac{\pi}{3}\right)\right)$$

becomes

$$0 = 2\sin\left(2\left(x - \frac{\pi}{3}\right)\right)$$

So, $\sin\left(2\left(x - \frac{\pi}{3}\right)\right) = 0$

$$2\left(x - \frac{\pi}{3}\right) = \ldots -3\pi, -2\pi, -\pi, 0, \pi, 2\pi, \ldots$$

$$x - \frac{\pi}{3} = \ldots \frac{-3\pi}{2}, -\pi, \frac{-\pi}{2}, 0, \frac{\pi}{2}, \pi, \ldots$$

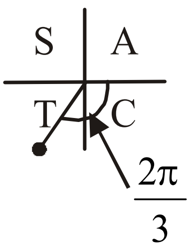
$$x = \ldots \frac{-7\pi}{6}, \frac{-2\pi}{3}, \frac{-\pi}{6}, \frac{\pi}{3}, \frac{5\pi}{6}, \frac{4\pi}{3}, \ldots$$

$$x = \frac{-2\pi}{3}, \frac{-\pi}{6}, \frac{\pi}{3}, \frac{5\pi}{6} \quad \text{since} \quad -\pi \le x \le \pi$$

(1 mark) for x- intercepts

<u>y - intercept</u> $(x = 0)$

$$\begin{aligned} y &= 2\sin\left(2\left(0 - \frac{\pi}{3}\right)\right) \\ &= 2\sin\left(-\frac{2\pi}{3}\right) \\ &= 2 \times -\sin\left(\frac{\pi}{3}\right) \\ &= 2 \times -\frac{\sqrt{3}}{2} \\ &= -\sqrt{3} \end{aligned}$$



Angle measured clockwise because it is negative.

y-intercept occurs at $(0, -\sqrt{3})$ **(1 mark)**

To find the endpoints:

Method 1
Because two complete periods occur for $x \in [-\pi, \pi]$ the right endpoint and left endpoint will have the same y-coordinate and it will be the same as the y-intercept.
So left endpoint is $(-\pi, -\sqrt{3})$ and the right endpoint is $(\pi, -\sqrt{3})$.

(1 mark)

Method 2
Left endpoint: $f(-\pi) = 2\sin\left(2\left(-\pi - \frac{\pi}{3}\right)\right)$

$$= 2\sin\left(2\left(\frac{-4\pi}{3}\right)\right)$$
$$= 2\sin\left(\frac{-8\pi}{3}\right)$$
$$= 2\sin\left(\frac{-2\pi}{3}\right)$$
$$= 2\times -\sin\left(\frac{\pi}{3}\right)$$
$$= 2\times -\frac{\sqrt{3}}{2}$$
$$= -\sqrt{3}$$

Left endpoint is $(-\pi, -\sqrt{3})$.

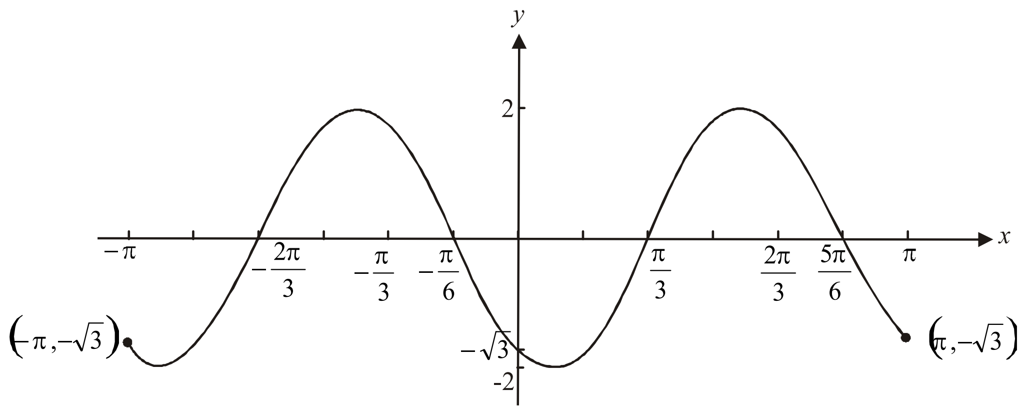
Right endpoint: $f(\pi) = 2\sin\left(2\left(\pi - \frac{\pi}{3}\right)\right)$

$$= 2\sin\left(2\left(\frac{2\pi}{3}\right)\right)$$
$$= 2\sin\left(\frac{4\pi}{3}\right)$$
$$= 2\times -\sin\left(\frac{\pi}{3}\right)$$
$$= 2\times -\frac{\sqrt{3}}{2}$$
$$= -\sqrt{3}$$

Right endpoint is $(\pi, -\sqrt{3})$.

(1 mark)

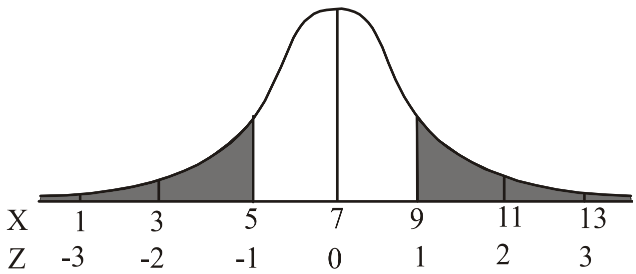
The graph of $y = 2\sin\left(2\left(x - \frac{\pi}{3}\right)\right)$ is shown below.



(1 mark) correct shape including amplitude

Question 6

a.



$$\begin{aligned}\Pr(X>9) &= \Pr(X<5) \text{ by symmetry}\\ &= \Pr(Z<-1)\\ &= 0\cdot 16 \ \text{(given)}\end{aligned}$$

(1 mark)

b. $\Pr(X<7|X<9)$ (conditional probability)

$=\dfrac{\Pr(X<7\cap X<9)}{\Pr(X<9)}$ **(1 mark)**

$=\dfrac{\Pr(X<7)}{1-\Pr(X>9)}$

$=\dfrac{0\cdot 5}{1-0\cdot 16}$ from part a. **(1 mark)**

$=\dfrac{0.5}{0.84}$

$=\dfrac{50}{84}$

$=\dfrac{25}{42}$

(1 mark)

Question 7

a. Since $f(x)$ represents a probability density function then

$$\int_1^3 (ax+1)\,dx = 1$$

(1 mark)

$$\left[\frac{ax^2}{2}+x\right]_1^3 = 1$$

$$\left\{\left(\frac{9a}{2}+3\right)-\left(\frac{a}{2}+1\right)\right\} = 1$$

$$\frac{8a}{2}+2 = 1$$

$$4a = -1$$

$$a = -\frac{1}{4} \quad \text{as required}$$

(1 mark)

b.

$$\Pr(X<2) = \int_1^2 \left(-\frac{1}{4}x+1\right)dx$$

(1 mark)

$$= \left[-\frac{1}{4}\times\frac{x^2}{2}+x\right]_1^2$$

$$= \left[-\frac{x^2}{8}+x\right]_1^2$$

$$= \left\{\left(-\frac{4}{8}+2\right)-\left(-\frac{1}{8}+1\right)\right\}$$

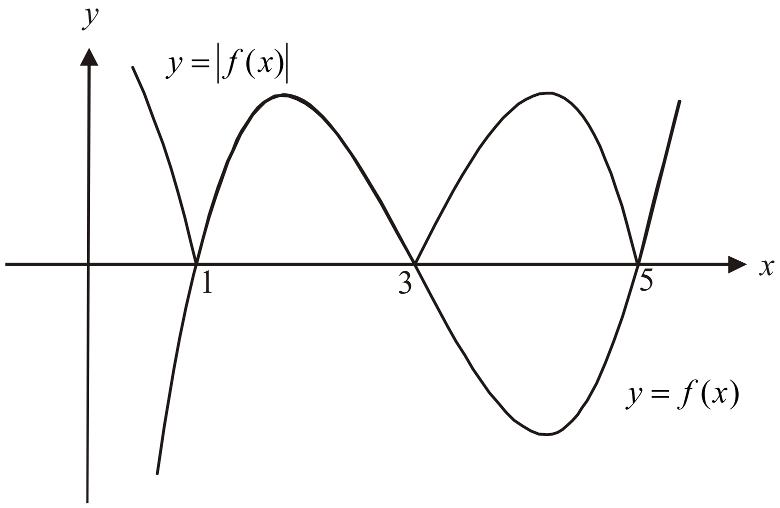
$$= -\frac{3}{8}+1$$

$$= \frac{5}{8}$$

(1 mark)

Question 8

a.



Note that at the points $(1,0), (3,0)$ and $(5,0)$ the graph of $y = |f(x)|$ has cusps; i.e. "pointy bits" not smooth curves since the graph of $y = f(x)$ is being reflected in the x-axis.

(1 mark)

b. $A = \int_1^3 f(x)dx - \int_3^5 f(x)dx$

(1 mark)

Question 9

a.

$$\begin{aligned}\text{area} &= \frac{1}{2} \times \text{base} \times \text{height}\\ &= \frac{1}{2} \times a \times b\\ &= \frac{1}{2} \times a \times f(a) \text{ where } f(x) = 1 - \frac{x}{5}\\ &= \frac{1}{2} \times a \times \left(1 - \frac{a}{5}\right)\\ &= \frac{a}{2} - \frac{a^2}{10} \text{ as required}\end{aligned}$$

(1 mark)

b. Let $A = \frac{a}{2} - \frac{a^2}{10}$

$$\frac{dA}{da} = \frac{1}{2} - \frac{2a}{10}$$

(1 mark)

Maximum occurs when $\frac{dA}{da} = 0$

$$\begin{aligned}\frac{1}{2} - \frac{a}{5} &= 0\\ \frac{a}{5} &= \frac{1}{2}\\ 2a &= 5\\ a &= \frac{5}{2}\end{aligned}$$

Note: we know we have a maximum because the graph of the function $A = \frac{a}{2} - \frac{a^2}{10}$ is an inverted parabola.

We have a maximum when $a = \frac{5}{2}$. **(1 mark)**

When $a = \frac{5}{2}$,
$$\begin{aligned}\text{area} &= \frac{a}{2} - \frac{a^2}{10}\\ &= \frac{5}{4} - \frac{25}{40}\\ &= \frac{50}{40} - \frac{25}{40}\\ &= \frac{25}{40}\\ &= \frac{5}{8} \text{ square units}\end{aligned}$$

or when $a = 2.5$,
$$\begin{aligned}\text{area} &= \frac{2.5}{2} - \frac{2.5^2}{10}\\ &= 1.25 - \frac{6.25}{10}\\ &= 1.25 - 0.625\\ &= 0.625 \text{ square units}\end{aligned}$$

(1 mark)

Question 10

$$y = 3x^2 + a$$
$$\frac{dy}{dx} = 6x$$

The gradient of a normal to $y = 3x^2 + a$ is $-\frac{1}{6x}$.

(1 mark)

Also the gradient of the normal $y = \frac{x}{3} + 1$ is $\frac{1}{3}$.

When
$$\begin{aligned} -\frac{1}{6x} &= \frac{1}{3} \\ -3 &= 6x \\ x &= -\frac{1}{2} \end{aligned}$$

(1 mark)

The x-coordinate of the point where the normal hits the curve is $-\frac{1}{2}$.

So
$$\begin{aligned} y &= -\frac{1}{2} \div 3 + 1 \\ &= -\frac{1}{2} \times \frac{1}{3} + 1 \\ &= -\frac{1}{6} + 1 \\ &= \frac{5}{6} \end{aligned}$$

The curve and the normal both pass through the point $\left(-\frac{1}{2}, \frac{5}{6}\right)$.

Substituting this point into

(1 mark)

$$y = 3x^2 + a$$

gives
$$\begin{aligned} \frac{5}{6} &= 3 \times \left(-\frac{1}{2}\right)^2 + a \\ \frac{5}{6} &= 3 \times \frac{1}{4} + a \end{aligned}$$

So
$$\begin{aligned} a &= \frac{5}{6} - \frac{3}{4} \\ &= \frac{10-9}{12} \\ &= \frac{1}{12} \end{aligned}$$

(1 mark)

Question 11

a. $y = e^{2x}$

$y = 6 - e^{x}$

At the point of intersection of the graphs,

$$e^{2x} = 6 - e^{x}$$

$$e^{2x} + e^{x} - 6 = 0$$

Let $e^{x} = m$

$$m^2 + m - 6 = 0$$ **(1 mark)**

$$(m+3)(m-2) = 0$$

$m = -3$ or $m = 2$

So $e^{x} = -3$ or $e^{x} = 2$

no solution $\quad x = \log_e(2)$

The x-coordinate of the point of intersection is $\log_e(2)$.

(1 mark)

b.

$$\text{Area} = \int_0^{\log_e(2)} \left\{ \left(6 - e^{x}\right) - \left(e^{2x}\right) \right\} dx$$ **(1 mark)**

$$= \int_0^{\log_e(2)} \left(6 - e^{x} - e^{2x}\right) dx$$

$$= \left[6x - e^{x} - \frac{e^{2x}}{2} \right]_0^{\log_e(2)}$$ **(1 mark)**

$$= \left\{ \left(6\log_e(2) - e^{\log_e(2)} - \frac{e^{2\log_e(2)}}{2} \right) - \left(0 - e^{0} - \frac{e^{0}}{2} \right) \right\}$$

$$= \log_e\left(2^6\right) - 2 - \frac{e^{\log_e\left(2^2\right)}}{2} + 1 + \frac{1}{2}$$ **(1 mark)**

$$= \log_e\left(2^6\right) - 2 - \frac{4}{2} + 1\frac{1}{2}$$

$$= \log_e(64) - \frac{5}{2} \text{ square units}$$

(1 mark)

Total 40 marks