

VCE Mathematical Methods Units 3 & 4

Written Examination 2

Suggested Solutions

SECTION 1

1	A	B	C	D	E
2	A	B	C	D	E
3	A	B	C	D	E
4	A	B	C	D	E
5	A	B	C	D	E
6	A	B	C	D	E
7	A	B	C	D	E
8	A	B	C	D	E
9	A	B	C	D	E
10	A	B	C	D	E
11	A	B	C	D	E

12	A	B	C	D	E
13	A	B	C	D	E
14	A	B	C	D	E
15	A	B	C	D	E
16	A	B	C	D	E
17	A	B	C	D	E
18	A	B	C	D	E
19	A	B	C	D	E
20	A	B	C	D	E
21	A	B	C	D	E
22	A	B	C	D	E

Question 1

$$\begin{aligned}
 d &= 12 + 3 \cos\left(\frac{\pi}{12}t\right) \\
 &= 12 + 3 \cos\left(\frac{\pi}{12} \times 15\right) \quad \text{when } t = 15 \\
 &= 12 + 3 \cos\left(\frac{5\pi}{4}\right) \\
 &\approx 9.88
 \end{aligned}$$

So the depth is about 9.88 m.

Answer B

Question 2

$$\begin{aligned}
 \sin(3x) &= \frac{8}{16} & 0 \leq x \leq \pi \Rightarrow 0 \leq 3x \leq 3\pi \\
 \sin(3x) &= \frac{1}{2} \\
 3x &= \frac{\pi}{6}, \pi - \frac{\pi}{6}, 2\pi + \frac{\pi}{6}, 3\pi - \frac{\pi}{6} \\
 3x &= \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6} \\
 x &= \frac{\pi}{18}, \frac{5\pi}{18}, \frac{13\pi}{18}, \frac{17\pi}{18} \\
 \text{sum of answers} &= \frac{36\pi}{18} \\
 &= 2\pi
 \end{aligned}$$

Answer E

Question 3

$$\begin{aligned}
 y &= a \sin(b(x - c)) \\
 \text{amplitude} &= 2, \text{ period} = \frac{2\pi}{3} \\
 \therefore a &= \pm 2, b = 3
 \end{aligned}$$

The options with $a = 2$ are not correct as they would have a cycle beginning at $x = \frac{\pi}{4}$ or $x = \frac{3\pi}{4}$.

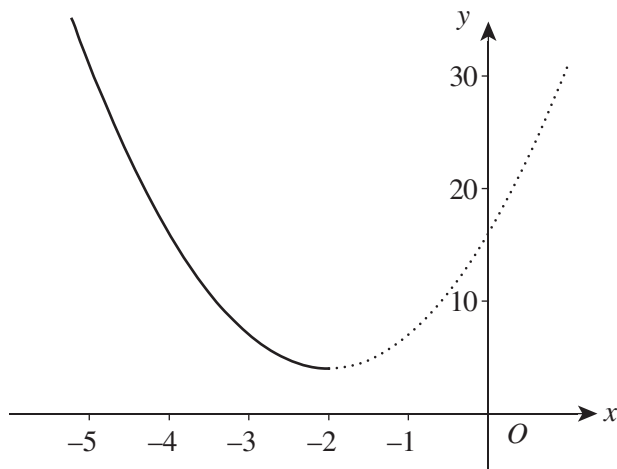
Answer **D** will be a vertically reflected sin graph with a cycle beginning at $x = \frac{\pi}{4}$, which is suitable.

Answer D

Question 4

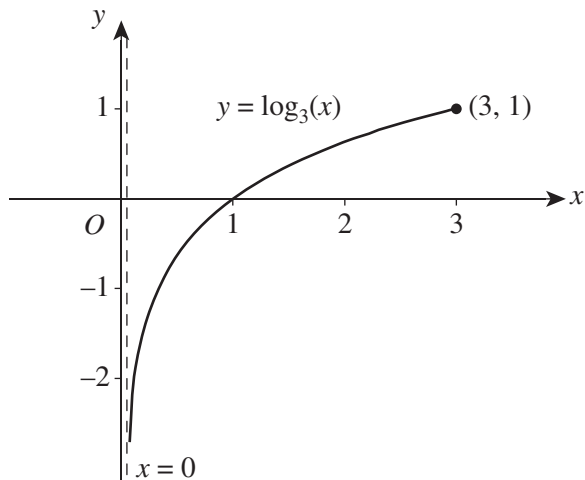
The function will have an inverse if it is one-to-one.

$$\therefore a \leq -2$$

**Answer A****Question 5**

$$f(x) = \log_3(x), x \in (0, 3]$$

$$f(3) = 1$$



The range of f is $(-\infty, 1]$.

Answer E**Question 6**

$$(x - a)^3(x^2 + b)(x^3 + c) = 0$$

$$\therefore x = a, \sqrt[3]{-c}$$

$x^2 + b = 0$ has no real solutions when $b > 0$.

$\therefore$ two real solutions

Answer C

Question 7

$$\begin{aligned} f(g(x)) &= \sqrt{9 - (\sqrt{x})^2} \\ &= \sqrt{9 - x} \end{aligned}$$

$\therefore$ require $\text{ran}_g \subseteq \text{dom}_f$

$$\text{ran}_g \subseteq [-3, 3]$$

$$\therefore -3 \leq g(x) \leq 3$$

$$0 \leq \sqrt{x} \leq 3$$

$$0 \leq x \leq 9 \text{ (which is a subset of } \text{dom}_g \text{)}$$

$$\therefore x \in [0, 9]$$

Answer B**Question 8**

$$y_1 = (x + 1)^2$$

$$y_2 = -(x + 1)^2 \quad \text{reflection in the } x\text{-axis}$$

$$y_3 = -(x + 1)^2 - 2 \quad \text{translation by 2 units in the negative } y\text{-direction}$$

$$y_4 = 3(-(x + 1)^2 - 2) \quad \text{dilation by a factor of 3 from the } x\text{-axis}$$

$$= -3(x + 1)^2 - 6$$

Answer E**Question 9**

$$16^x - 4^{x+1} = 32$$

$$4^{2x} - 4 \times 4^x = 32$$

$$(4^x)^2 - 4(4^x) - 32 = 0$$

$$(4^x + 4)(4^x - 8) = 0$$

$$4^x + 4 = 0 \quad 4^x - 8 = 0$$

$$4^x = -4 \quad 4^x = 8$$

$$\text{no solution} \quad 2^{2x} = 2^3$$

$$2x = 3$$

$$x = 1.5$$

Answer D

Question 10

$$4\log_2(x) + 4\log_2(\sqrt{x}) - \log_2(x^3) = -6$$

$$\log_2(x^4) + \log_2(x^2) - \log_2(x^3) = -6$$

$$\log_2\left(\frac{x^6}{x^3}\right) = -6$$

$$\log_2(x^3) = -6$$

$$x^3 = 2^{-6}$$

$$= \frac{1}{64}$$

$$x = \frac{1}{4} \text{ (which is a suitable solution)}$$

Answer C**Question 11**

$$f(x) = e^{-2x} \sin(x)$$

$$f'(x) = -2e^{-2x} \sin(x) + e^{-2x} \cos(x)$$

Answer E**Question 12**

$$y = \log_e\left(\frac{1}{\tan(x)}\right)$$

$$\frac{dy}{dx} = \frac{\frac{d}{dx}\left(\frac{1}{\tan(x)}\right)}{\left(\frac{1}{\tan(x)}\right)}$$

$$\text{Now } \frac{1}{\tan(x)} = (\tan(x))^{-1}$$

$$\frac{d}{dx}\left(\frac{1}{\tan(x)}\right) = -(\tan(x))^{-2} \sec^2(x)$$

$$= -\frac{\sec^2(x)}{\tan^2(x)}$$

$$= -\frac{1}{\sin^2(x)}$$

$$\frac{dy}{dx} = \frac{-\frac{1}{\sin^2(x)}}{\frac{\cos(x)}{\sin(x)}}$$

$$= -\frac{1}{\sin^2(x)} \times \frac{\sin(x)}{\cos(x)}$$

$$= \frac{-1}{\sin(x) \cos(x)}$$

Answer D

Question 13

$$\begin{aligned}
 y &= \frac{20p}{(1-2p)^4} \\
 \frac{dy}{dp} &= \frac{20(1-2p)^4 - 20p \times 4(1-2p)^3(-2)}{(1-2p)^8} \\
 &= \frac{20(1-2p)^3[(1-2p) + 8p]}{(1-2p)^8} \\
 &= \frac{20(1+6p)}{(1-2p)^5}
 \end{aligned}$$

Answer B**Question 14**

$$\begin{aligned}
 y &= 3x^3 - 6x^2 \\
 \frac{dy}{dx} &= 9x^2 - 12x
 \end{aligned}$$

when $x = 1$, $y = -3$ and $m_T = -3$

$$\therefore m_N = \frac{1}{3}$$

$$y - (-3) = \frac{1}{3}(x - 1)$$

$$3y + 9 = x - 1$$

$$3y = x - 10$$

Answer A**Question 15**

$$\begin{aligned}
 f(0) &= (0 + 1)e^0 \\
 &= 1
 \end{aligned}$$

$$\begin{aligned}
 f(2) &= (2 + 1)e^4 \\
 &= 3e^4
 \end{aligned}$$

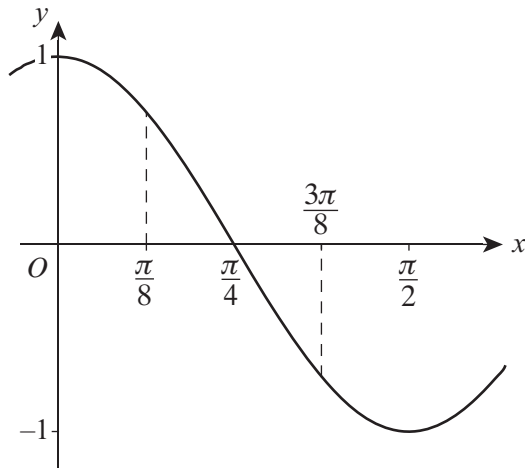
$$\begin{aligned}
 \text{average rate of change} &= \frac{f(2) - f(0)}{2 - 0} \\
 &= \frac{3e^4 - 1}{2}
 \end{aligned}$$

Answer D

Question 16

$$\int_{\frac{\pi}{8}}^a \cos(2x) dx = 0 \Rightarrow \text{that the area bounded by the curve } y = \cos(2x), \text{ the line } x = \frac{\pi}{8} \text{ and the } x\text{-axis will be}$$

equal to the area bounded by the curve, the line $x = a$ and the x -axis.



$$a = \frac{3\pi}{8} \text{ by symmetry.}$$

Answer C**Question 17**

$$\begin{aligned} \text{area} &\approx w(f(2) + f(3) + f(4)) \\ &= 1(\sqrt{2} + 1 + \sqrt{3} + 1 + \sqrt{4} + 1) \\ &= 5 + \sqrt{2} + \sqrt{3} \end{aligned}$$

Answer D**Question 18**

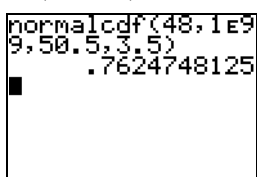
$$\begin{aligned} \text{It is possible that } f'(x) &= k(x+2)(x-1) \\ &= k(x^2 + x - 2) \end{aligned}$$

$$\therefore \text{it is possible that } f(x) = k\left(\frac{x^3}{3} + \frac{x^2}{2} - 2x + c\right)$$

Answer E**Question 19**

$$X \sim N(50.5, 3.5^2)$$

$$\Pr(X > 48) \approx 0.762$$

**Answer D**

Question 20

$$X \sim \text{Bi}(5, 0.3)$$

$$\Pr(X \geq 2) = 1 - \Pr(X \leq 1)$$

$$\approx 0.472$$

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1-binomcdf(5,.3,
1)
.47178
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Answer C**Question 21**

$$X \sim N(997, \sigma^2), Z \sim N(0, 1)$$

$$\Pr(Z < z) = 0.98$$

$$z \approx 2.0537$$

$$z = \frac{x - \mu}{\sigma}$$

$$\sigma = \frac{x - \mu}{z}$$

$$= \frac{1005 - 997}{2.0537}$$

$$= 3.9$$

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invNorm(.98)
2.053748911
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Answer C**Question 22**

$$\text{require } \int_1^a \frac{2}{x} = 1$$

$$2[\log_e(x)]_1^a = 1, x > 0$$

$$2[\log_e(a) - \log_e(1)] = 1$$

$$2\log_e(a) = 1$$

$$\log_e(a) = \frac{1}{2}$$

$$a = e^{\frac{1}{2}}$$

$$= \sqrt{e}$$

Answer A

SECTION 2

Question 1

a. i. $c = -1$ (from horizontal asymptote) A1

ii.
$$y = \frac{a}{(x+b)^2} - 1$$

$$f(0) = 3 \therefore 3 = \frac{a}{b^2} - 1$$

$$\frac{a}{b^2} = 4$$

$$a = 4b^2$$

A1

$$f(2) = 0 \therefore 0 = \frac{a}{(2+b)^2} - 1$$

$$\frac{a}{(2+b)^2} = 1$$

$$a = (2+b)^2$$

A1

$$\therefore 4b^2 = (2+b)^2$$

$$4b^2 = 4 + 4b + b^2$$

$$3b^2 - 4b - 4 = 0$$

$$(3b+2)(b-2) = 0$$

M1

$$b = 2 \text{ as } b > 0$$

$$\text{and } a = 4b^2$$

$$= 4(2)^2$$

$$= 16$$

(for a and b) A1

b. $f: y = \frac{16}{(x+2)^2} - 1$ $\text{dom}_f: [0, \infty), \text{ran}_f: (-1, 3]$

$$f^{-1}: x = \frac{16}{(y+2)^2} - 1$$

M1

$$\frac{16}{(y+2)^2} = x + 1$$

$$(y+2)^2 = \frac{16}{x+1}$$

$$y+2 = \pm \sqrt{\frac{16}{x+1}}$$

M1

$$y = -2 \pm \sqrt{\frac{16}{x+1}}$$

$$\text{but } y > 0 \therefore y = -2 + \sqrt{\frac{16}{x+1}}$$

$$= -2 + \frac{4}{\sqrt{x+1}}$$

$$f^{-1}(x) = -2 + \frac{4}{\sqrt{x+1}}$$

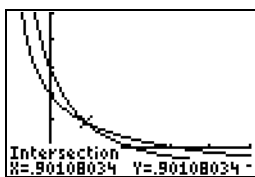
A1

$$\text{dom}_{f^{-1}}: (-1, 3], \text{ran}_{f^{-1}}: [0, \infty)$$

A1

c. $x = 0.9011$

A1



d.
$$\begin{aligned} \text{area} &= \int_0^{0.9011} \left(-2 + \frac{4}{\sqrt{x+1}} \right) dx + \int_{0.9011}^2 \left(\frac{16}{(x+2)^2} - 1 \right) dx \\ &= \int_0^{0.9011} \left(-2 + 4(x+1)^{-\frac{1}{2}} \right) dx + \int_{0.9011}^2 (16(x+2)^{-2} - 1) dx \\ &= \left[-2x + 2 \times 4(x+1)^{\frac{1}{2}} \right]_0^{0.9011} + \left[-16(x+2)^{-1} - x \right]_{0.9011}^2 \\ &= \left(-2(0.9011) + 8(1.9011)^{\frac{1}{2}} \right) - \left(-2(0) + 8(1)^{\frac{1}{2}} \right) \\ &\quad + \left(-16(4)^{-1} - 2 \right) - \left(-16(2.9011)^{-1} - 0.9011 \right) \\ &\approx 1.64 \text{ cm}^2 \end{aligned}$$

M1, A1

A1

A1

Question 2

a. $A = 20$

A1

$B = 25$

A1

$T = 100 = \frac{2\pi}{n}$

$\therefore n = \frac{\pi}{50}$

A1

b. $f(x) = 20 \cos\left(\frac{\pi x}{50}\right) + 25$

$f'(x) = -\frac{\pi}{50} \times 20 \sin\left(\frac{\pi x}{50}\right)$

$= -\frac{2\pi}{5} \sin\left(\frac{\pi x}{50}\right)$

A1

c. i. $r = f(64)$

$= 20 \cos\left(\frac{\pi(64)}{50}\right) + 25$

≈ 12.2515

A1

ii. $f'(64) = -\frac{2\pi}{5} \sin\left(\frac{\pi(64)}{50}\right)$

≈ 0.9683

A1

- d.** $f'(x) = C \frac{D}{D(x-50)}$ $64 < x < p$
- $$= \frac{C}{x-50}$$
- M1
- $$\frac{C}{64-50} = 0.9683 \quad (\text{smooth join when } x = 64)$$
- $$C = 13.5556$$
- A1
- also $r = C \log_e(D(64-50))$ (smooth join when $x = 64$)
- $$\log_e(14D) = \frac{r}{C}$$
- A1
- $$14D = e^{\frac{r}{C}}$$
- $$D = \frac{1}{14} e^{\frac{r}{C}}$$
- $$\approx 0.1764$$
- A1
- e.** Gradient of logarithmic curve at p is 0.4.
- $$\therefore \frac{C}{p-50} = 0.4$$
- M1
- $$p-50 = \frac{C}{0.4}$$
- $$p = \frac{C}{0.4} + 50$$
- $$\approx 83.8889$$
- $$\approx 84$$
- A1
- $$q = C \log_e(D(p-50))$$
- $$\approx 24.2351$$
- $$\approx 24$$
- A1
- f.** When $y = 0$, $0 = p + q - x$
- $$\therefore x = p + q$$
- $$\approx 108$$
- A1

Question 3

a. $V_{\text{cone}} = \frac{1}{3}\pi r^2 h$

$$4\pi = \frac{1}{3}\pi r^2 h$$

$$r^2 h = 12$$

$$h = \frac{12}{r^2}$$

A1

b. $A = \pi r \sqrt{r^2 + h^2}$

$$= \pi r \sqrt{r^2 + \left(\frac{12}{r^2}\right)^2}$$

A1

$$= \pi r \sqrt{r^2 + \frac{144}{r^4}}$$

$$= \pi r \sqrt{\frac{r^6 + 144}{r^4}}$$

M1

$$= \pi r \times \frac{\sqrt{r^6 + 144}}{r^2}$$

$$= \frac{\pi \sqrt{r^6 + 144}}{r}$$

A1

c. $A = \frac{\pi(r^6 + 144)^{\frac{1}{2}}}{r}$

$$\therefore \frac{dA}{dr} = \frac{\pi \left[\frac{1}{2}(r^6 + 144)^{-\frac{1}{2}} \times 6r^5 \times r - (r^6 + 144)^{\frac{1}{2}} \right]}{r^2}$$

M1

$$= \frac{\pi \left[\frac{3r^6}{\sqrt{r^6 + 144}} - \sqrt{r^6 + 144} \right]}{r^2}$$

$$= \frac{\pi \left[\frac{3r^6 - (r^6 + 144)}{\sqrt{r^6 + 144}} \right]}{r^2}$$

M1

$$= \frac{\pi(2r^6 - 144)}{r^2 \sqrt{r^6 + 144}}$$

A1

$$= \frac{2\pi(r^6 - 72)}{r^2 \sqrt{r^6 + 144}}$$

d. $\frac{dA}{dr} = 0$

$$r^6 - 72 = 0$$

M1

$$r^6 = 72$$

$$r = \sqrt[6]{72}$$

$$\approx 2.04$$

A1

$$h = \frac{12}{(\sqrt[6]{72})^2}$$

$$\approx 2.88$$

So the dimensions are height 2.88 m and radius 2.04 m.

A1

e. i. $V = \frac{1}{3}\pi R^2 H$ and $R = \frac{2}{3}H$ by similar triangles

$$\therefore V = \frac{1}{3}\pi \left(\frac{2H}{3}\right)^2 H$$

$$= \frac{4}{27}\pi H^3$$

A1

ii. $\frac{dV}{dt} = -0.5\sqrt{H}$
 $\frac{dH}{dt} = \frac{dV}{dt} \times \frac{dH}{dV}$ (so $\frac{dH}{dV}$ is required)

$$\frac{dV}{dH} = \frac{4}{9}\pi H^2$$

$$\therefore \frac{dH}{dV} = \frac{9}{4\pi H^2}$$

$$\therefore \frac{dH}{dt} = -0.5\sqrt{H} \times \frac{9}{4\pi H^2}$$

M1, A1

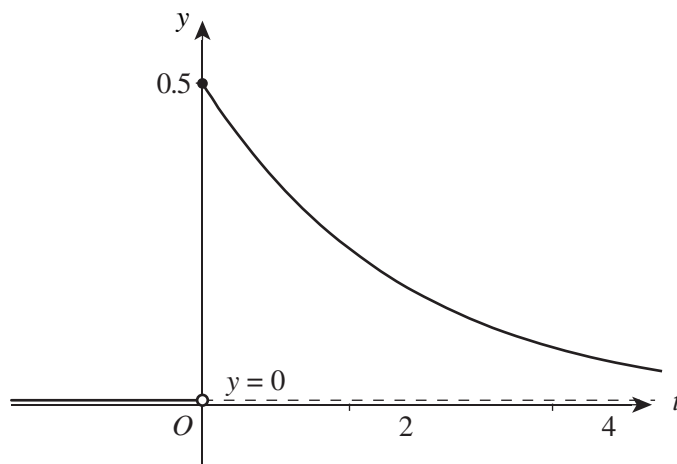
when $H = 1.5$, $\frac{dH}{dt} = -0.5\sqrt{1.5} \times \frac{9}{4\pi(1.5)^2}$
 ≈ -0.19

So the height is changing at -0.19 m/min (or decreasing at 0.19 m/min).

A1

Question 4

a.



Correct intercept and shape A1

Correct asymptote A1

Zero elsewhere A1

b. require m such that $\int_0^m \frac{1}{2} e^{-\frac{t}{2}} dt = \frac{1}{2}$ M1

$$\left[-e^{-\frac{t}{2}} \right]_0^m = \frac{1}{2}$$

$$-e^{-\frac{m}{2}} + e^0 = \frac{1}{2}$$

$$-e^{-\frac{m}{2}} + 1 = \frac{1}{2}$$

$$-e^{-\frac{m}{2}} = -\frac{1}{2}$$

$$e^{-\frac{m}{2}} = \frac{1}{2} \quad \text{A1}$$

$$-\frac{m}{2} = \log_e\left(\frac{1}{2}\right)$$

$$m = -2\log_e\left(\frac{1}{2}\right)$$

$$= 2\log_e(2)$$

$$= \log_e(4)$$

(or equivalent) A1

c. $\Pr(T \geq 2) = 1 - \Pr(T \leq 2)$

$$= 1 - \int_0^2 \frac{1}{2} e^{-\frac{t}{2}} dt \quad \text{M1}$$

$$= 1 - \left[-e^{-\frac{t}{2}} \right]_0^2 \quad \text{A1}$$

$$= 1 + \left[e^{-\frac{t}{2}} \right]_0^2$$

$$= 1 + (e^{-1} - e^0)$$

$$= 1 + \frac{1}{e} - 1$$

$$= \frac{1}{e} \quad \text{A1}$$

d. $\Pr(T < 2) = 1 - \Pr(T \geq 2)$

$$= 1 - \frac{1}{e} \quad \text{A1}$$

Let Y equal the number of time intervals that are less than 2 minutes.

$$Y \sim \text{Bi}\left(5, 1 - \frac{1}{e}\right)$$

$$\Pr(Y = 3) = \binom{5}{3} \left(1 - \frac{1}{e}\right)^3 \left(\frac{1}{e}\right)^2 \quad \text{M1}$$

$$\approx 0.3418 \quad \text{A1}$$

$$\mathbf{e.} \quad \Pr(T \geq 3 \mid T \geq 2) = \frac{\Pr(T \geq 3 \cap T \geq 2)}{\Pr(T \geq 2)} \quad \mathbf{M1}$$

$$= \frac{\Pr(T \geq 3)}{\Pr(T \geq 2)} \quad \mathbf{A1}$$

$$\Pr(T \geq 3) = 1 + \left[e^{-\frac{t}{2}} \right]_0^3 \text{ (similar to part c.)}$$

$$= 1 + e^{-\frac{3}{2}} \quad \mathbf{A1}$$

$$\therefore \Pr(T \geq 3 \mid T \geq 2) = \frac{e^{-\frac{3}{2}}}{e^{-1}}$$

$$= e^{-\frac{1}{2}}$$

$$= \frac{1}{\sqrt{e}} \quad \mathbf{A1}$$