

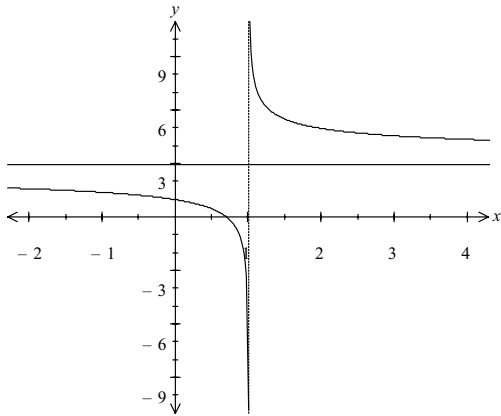
Mathematical Methods Exam 2: SOLUTIONS

Section 1: Multiple Choice

Question 1

Answer B

$$y = \frac{2}{\sqrt[3]{(x-1)}} + 3$$



$$x - 1 \neq 0$$

$x = 1$ is a vertical asymptote

$y = 3$ is a horizontal asymptote

Question 2

Answer C

$$(x - a)^3(x + b)^2(x^2 - c) = 0$$

Using the Null Factor Law

$$x - a = 0$$

$$x = a$$

or

$$x + b = 0$$

$$x = -b$$

or

$$x^2 - c = 0$$

$$x^2 = c$$

no real solution as $c < 0$

There are two distinct solutions.

Question 3

Answer E

The graph could be of the form

$y = A(x - 1)^9 + 3$ where A is a positive real constant.

The y -intercept is negative

$$-A + 3 < 0$$

$$A > 3$$

$$\text{Hence } y = 4(x - 1)^9 + 3$$

Alternatively,

The y -intercept is negative. Substituting $x = 0$ into each of the equations gives:

A. $-1 + 3 = 2$ No!

B. $+1 + 3 = 4$ No!

C. $-(-1) + 3 = 4$ No!

D. $-4 + 3$ Possibility

E. $-4 + 3$ Possibility

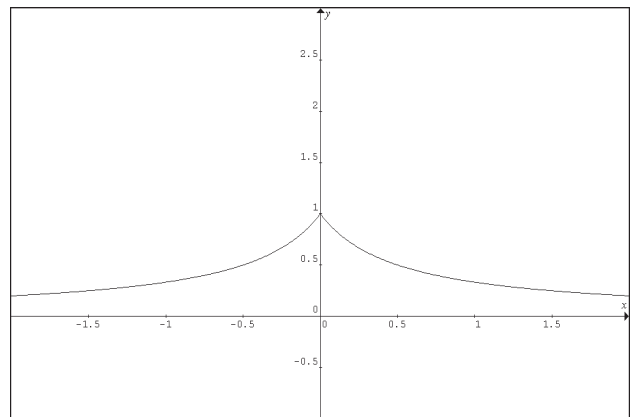
Try the graphs of these last two on the graphical calculator.

$$\text{Hence } y = 4(x - 1)^9 + 3$$

Question 4

Answer A

$$h(k(x)) = \frac{1}{|2x| + 1} \text{ with domain } R.$$



The range is $(0, 1]$.

Question 5

Answer C

$$f(g(x)) = 2e^{-x+1} = 2e^{-(x-1)}$$

reflection in the y -axis

translation 1 unit parallel to x -axis

Question 6**Answer E**

$f(x) = 3e^{-2x} + 1$ with domain R and range $(1, \infty)$

Let $y = 3e^{-2x} + 1$

Inverse: swap x and y and solve for y

$$x = 3e^{-2y} + 1$$

$$\frac{x-1}{3} = e^{-2y}$$

$$y = -\frac{1}{2} \log_e \left(\frac{x-1}{3} \right) \quad \text{with domain } (1, \infty)$$

$$= -\frac{1}{2} \log_e \left(\frac{x-1}{3} \right)^{-\frac{1}{2}}$$

$$= \log_e \left(\sqrt{\frac{3}{x-1}} \right)$$

$$\text{Therefore, } f^{-1}(x) = \log_e \left(\sqrt{\frac{3}{x-1}} \right)$$

Question 7**Answer B**

$$2^{2x} - 9 \times 2^x + 8 = 0$$

$$(2^x - 8)(2^x - 1) = 0$$

$$2^x = 8 \text{ or } 2^x = 1$$

$$\therefore x = 3 \text{ or } x = 0$$

Question 8**Answer C**

$$2 \log_e |x-1| + \log_e (9) = \log_e (a^2)$$

$$\log_e |x-1|^2 + \log_e (9) = \log_e (a^2)$$

$$\log_e (9|x-1|^2) = \log_e (a^2)$$

$$9|x-1|^2 = a^2$$

$$|x-1|^2 = \frac{a^2}{9}$$

$$|x-1| = \frac{a}{3}, a > 0$$

$$x-1 = \frac{a}{3} \text{ and } -x+1 = \frac{a}{3}$$

$$x = 1 + \frac{a}{3} \text{ and } x = 1 - \frac{a}{3}$$

Question 9**Answer B**

$$\sqrt{3} \tan(2\theta) + 1 = 0$$

$$\tan(2\theta) = -\frac{1}{\sqrt{3}}$$

$$2\theta = \frac{5\pi}{6}, \frac{11\pi}{6}, \frac{5\pi}{6} + 2\pi, \frac{11\pi}{6} + 2\pi$$

$$= \frac{5\pi}{6}, \frac{11\pi}{6}, \frac{17\pi}{6}, \frac{23\pi}{6}$$

$$\theta = \frac{5\pi}{12}, \frac{11\pi}{12}, \frac{17\pi}{12}, \frac{23\pi}{12}$$

Sum of the solutions

$$\frac{5\pi}{12} + \frac{11\pi}{12} + \frac{17\pi}{12} + \frac{23\pi}{12} = \frac{56\pi}{12} = \frac{14\pi}{3}$$

Question 10**Answer E**

$$\text{Period} = \frac{2\pi}{\pi/12} = 24$$

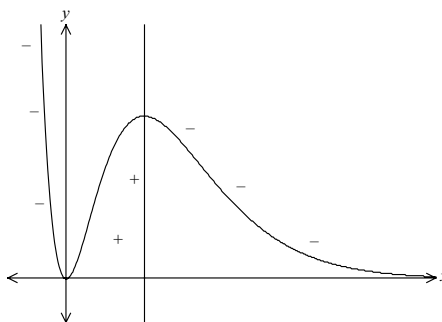
Maximum value of f is $1 + 5 = 6$

Minimum value of f is $1 - 5 = -4$

$$\text{Range} = [-4, 6]$$

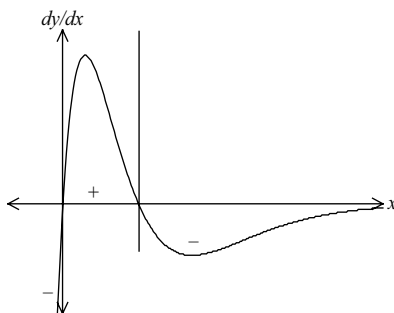
Question 11**Answer B**

Consider the sign of the gradient



x intercepts of $\frac{dy}{dx}$ are the stationary points of y

Gradient $\rightarrow 0$ (from the negative, i.e. from below) as $x \rightarrow \infty$



Question 12

$$\begin{aligned}
 f: [0, \pi] \rightarrow \mathbb{R}, f(x) &= 4\cos\left(\frac{x}{2}\right) \\
 f'(x) &= -1 \\
 -2\sin\left(\frac{x}{2}\right) &= -1 \\
 \sin\left(\frac{x}{2}\right) &= \frac{1}{2} \\
 \frac{x}{2} &= \frac{\pi}{6} \\
 x &= \frac{\pi}{3}
 \end{aligned}$$

Question 13

$$\begin{aligned}
 f(x) &= \frac{1}{\sqrt[3]{x}}, h = 0.1, x = 1 \\
 f(1) &= 1 \\
 f'(x) &= -\frac{1}{3x^{\frac{4}{3}}}, f'(1) = -\frac{1}{3} \\
 f(x+h) &\approx f(x) + hf'(x) \\
 \frac{1}{\sqrt[3]{1.1}} &\approx 1 + 0.1 \times -\frac{1}{3} \\
 &\approx \frac{29}{30}
 \end{aligned}$$

Question 14

The domain of f is $(-\infty, 2]$ and the domain of g is $[-3, \infty)$.

Hence the domain of h is $[-3, 2]$.

The domain of the derivative is $(-3, 2)$.

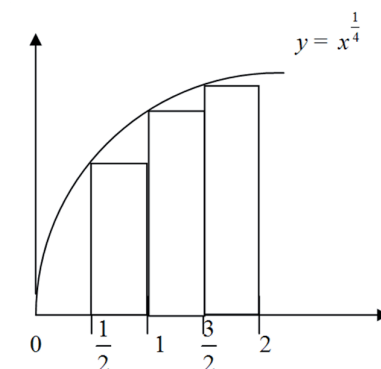
Question 15

$$\begin{aligned}
 \frac{dy}{dt} &= \frac{dy}{dx} \times \frac{dx}{dt} \\
 y &= \left(\frac{x}{2} - 3\right)^8 \\
 \frac{dy}{dx} &= 4\left(\frac{x}{2} - 3\right)^7 \text{ and } \frac{dx}{dt} = 3 \\
 \frac{dy}{dt} &= 4\left(\frac{x}{2} - 3\right)^7 \times 3 \\
 &= 12\left(\frac{x}{2} - 3\right)^7
 \end{aligned}$$

Answer C**Answer D****Answer C****Answer A****Question 16**

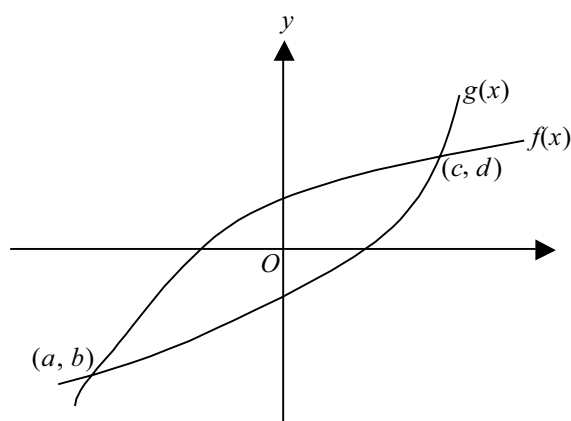
$$\begin{aligned}
 \int \frac{1}{1-2x} dx &= -\int \frac{1}{2x-1} dx \\
 &= -\frac{1}{2} \int \frac{2}{2x-1} dx \\
 &= -\frac{1}{2} \log_e(|2x-1|) + c, \\
 &\text{for } \mathbb{R} \setminus \left\{\frac{1}{2}\right\}
 \end{aligned}$$

c omitted, anti-derivative only required

Question 17

The area of the rectangles

$$\begin{aligned}
 &= \frac{1}{2} \left(f\left(\frac{1}{2}\right) + f(1) + f\left(\frac{3}{2}\right) \right) \\
 &= \frac{1}{2} \left(\left(\frac{1}{2}\right)^{\frac{1}{4}} + 1 + \left(\frac{3}{2}\right)^{\frac{1}{4}} \right) \text{ square units}
 \end{aligned}$$

Question 18

The area between the curves is

$$\begin{aligned}
 &\int_a^c (\text{top curve} - \text{bottom curve}) dx \\
 &= \int_a^c (f(x) - g(x)) dx
 \end{aligned}$$

Answer A**Answer A****Answer D**

Question 19**Answer A**

$$2t^2 + t = 1$$

$$2t^2 + t - 1 = 0$$

$$(2t - 1)(t + 1) = 0$$

$$2t = 1 \text{ (reject } t = -1 \text{ because } 0 \leq p(x) \leq 1)$$

$$t = \frac{1}{2}$$

Question 20**Answer D**

$$k \int_{-\infty}^{\infty} e^{-(x^2/2)} dx = 1$$

$$k \times \sqrt{2\pi} = 1$$

$$k = \frac{1}{\sqrt{2\pi}}$$

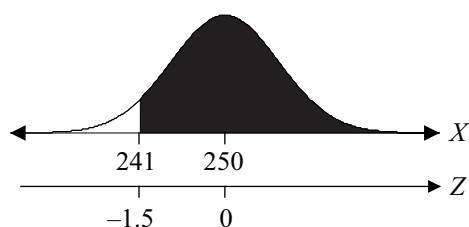
$$E(X) = \mu$$

$$= k \int_{-\infty}^{\infty} x e^{-(x^2/2)} dx$$

$$= k \times 0$$

$$= 0$$

$$\text{Hence, } k = \frac{1}{\sqrt{2\pi}} \text{ and } E(X) = 0$$

Question 21**Answer C**

$$250 - 241 = 9$$

$$1.5\sigma = 9$$

$$\sigma = \frac{9}{1.5}$$

$$\sigma = 6$$

Question 22**Answer B**

$$X \sim Bi\left(3, \frac{1}{3}\right)$$

$$\Pr(X = 2 | X \geq 1) = \frac{\Pr(X = 2)}{1 - \Pr(X = 0)}$$

$$= \frac{3 \times \left(\frac{1}{3}\right)^2 \times \frac{2}{3}}{1 - \left(\frac{2}{3}\right)^3}$$

$$= \frac{\frac{2}{9}}{\frac{19}{27}}$$

$$= \frac{6}{19}$$

Mathematical Methods Exam 2: SOLUTIONS

Section 2: Extended answers

Question 1

a. i. $a = \frac{6}{3} = 2$
 $b = -6$

ii. $x^2 + y^2 = 9$
 $y^2 = 9 - x^2$
 $y = \sqrt{9 - x^2}$
 $c = 9$

b. i. $-2 \int_0^3 (2x - 6) dx$
 $= -2[x^2 - 6x]_0^3$
 $= -2(9 - 18 - 0 + 0)$
 $= 18 \text{ cm}^2$

ii. $2 \int_0^3 \sqrt{9 - x^2} dx$
 $= 2 \left[\frac{9 \sin^{-1}\left(\frac{x}{\sqrt{9}}\right)}{2} + \frac{x\sqrt{9 - x^2}}{2} \right]_0^3$
 $= 2 \left(\frac{9 \sin^{-1}(1)}{2} + \frac{3\sqrt{9 - 9}}{2} - \left(\frac{9 \sin^{-1}(0)}{2} + \frac{0\sqrt{9 - 0}}{2} \right) \right)$
 $= \frac{9\pi}{2} \text{ cm}^2$

c. i. $\frac{dr}{dt} = \frac{dv}{dt} \times \frac{dr}{dv}$
 $v = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi r^2 2r = \frac{2}{3}\pi r^3$
 $\frac{dv}{dr} = 2\pi r^2, \frac{dr}{dv} = \frac{1}{2\pi r^2}$
 $\frac{dr}{dt} = -\pi \times \frac{1}{2\pi r^2} = -\frac{1}{2r^2}$

At $r = 2 \text{ cm}$

$$\frac{dr}{dt} = -\frac{1}{8} \text{ cm/s}$$

Decreasing at $\frac{1}{8} \text{ cm/s}$

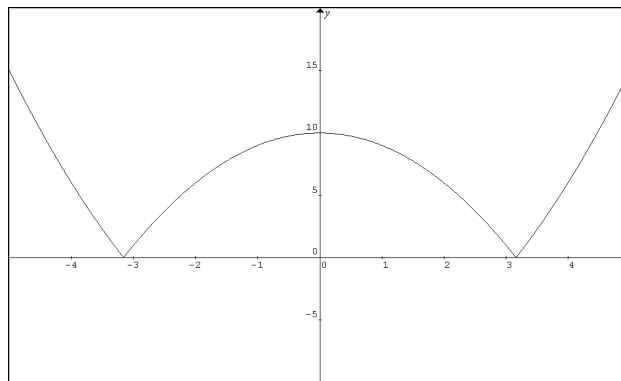
ii. $v = 18\pi \text{ cm}^3, \frac{dv}{dt} = -\pi \text{ cm}^3/\text{s}$
 $t = \frac{18\pi}{\pi} = 18 \text{ s}$
 $t \geq 18 \text{ s}$

Question 2

a. i. $g(f(x)) = |x^2 - 10|$ 1A

ii. Domain of $g(f(x)) = \text{Domain of } f(x)$
 $= [-5, 5]$ 1A

b.



Correct endpoints 1A

Correct cusps and local maximum 1A

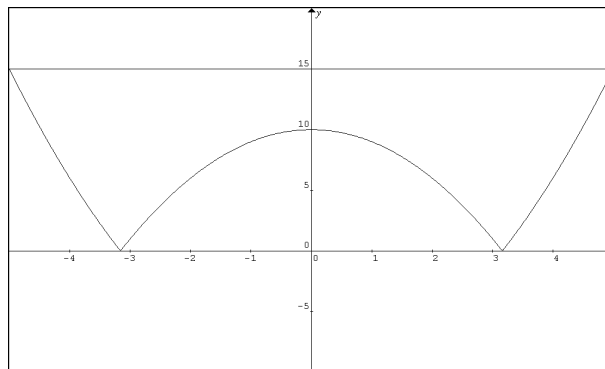
Correct x intercepts 1A

c. i. $0 < k < 10$ 1A

ii. $k = 0$ or 1A

$10 < k \leq 15$ 1A

d.



Cost $= 50 \times \int_{-5}^5 (15 - |x^2 - 10|) dx$ 1M

$\approx \$4117$ 1A

`50*fnInt(15-abs(x^2-10),x,-5,5)`
`4116.962985`

e. i. Substitute (0,0) into the equation.

$$0 = 5e^0 + C$$

$$C = -5$$

ii. Substitute (14, 100) into the equation.

$$100 = 5e^{14B} - 5$$

$$e^{14B} = \frac{105}{5}$$

$$= 21$$

$$B = \frac{\log_e(21)}{14}$$

f. Solve $200 = 5e^{\frac{\log_e(21)}{14}t} - 5$

$$t \approx 17.1$$

During the 18th day.

g. Let $y = 5e^{\frac{\log_e(21)}{14}t} - 5$

Inverse: swap t and y .

$$t = 5e^{\frac{\log_e(21)}{14}y} - 5$$

$$\frac{t+5}{5} = e^{\frac{\log_e(21)}{14}y}$$

$$y = \frac{14}{\log_e(21)} \log_e\left(\frac{t+5}{5}\right)$$

$$r^{-1}(t) = \frac{14}{\log_e(21)} \log_e\left(\frac{t+5}{5}\right), t \geq 0$$

$$\text{h. } r^{-1}(t) = \frac{14}{\log_e(21)} \log_e\left(\frac{20+5}{5}\right) \approx 7.4$$

7 foxes

Question 3

a. $x_{\max} = 30 + 20 = 50 \text{ cm}$

$$x_{\min} = 30 - 20 = 10 \text{ cm}$$

b. Period = $\frac{2\pi}{5\pi/4} = \frac{2\pi}{1} \times \frac{4}{5\pi} = 1.6$

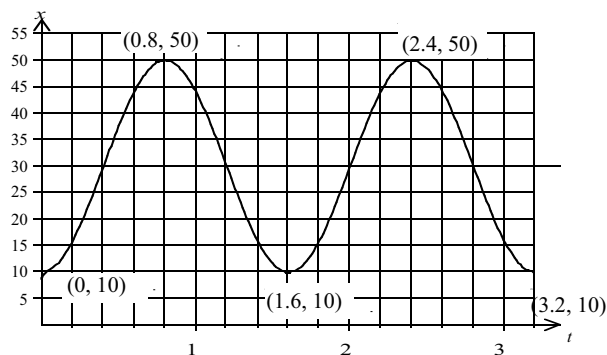
The period is 1.6 seconds.

c.

1M

1M

1M



Shape

1 mark

Endpoints

1 mark

Correct coordinates for maximum and minimum values

1 mark

1A

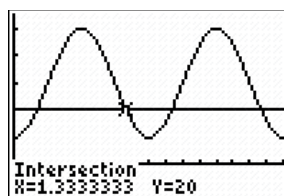
d. Time = $\frac{4}{3} - \frac{4}{15}$

1M

$$= \frac{16}{15} \text{ seconds}$$

1A

1M



1A

$$\begin{array}{l} \times \\ \text{Ans} \rightarrow \text{Frac} \\ 1.333333333 \\ 4/3 \end{array}$$

e. $m = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$

1A

From the graph in part c,

$$m = \frac{50 - 10}{0.8 - 0} = 50$$

1M

The average rate of change is 50 cm/s.

1A

1A

f. Product rule

1A

$$\frac{d}{dt} \left(30 - 20e^{-t/10} \cos\left(\frac{5\pi}{4}t\right) \right)$$

1M

$$= \frac{-20}{-10} e^{-t/10} \cos\left(\frac{5\pi}{4}t\right) -$$

$$\frac{20 \times 5\pi}{4} e^{-t/10} \sin\left(\frac{5\pi}{4}t\right)$$

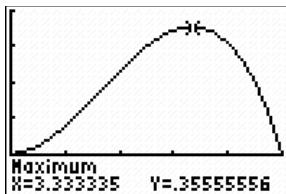
$$= 2e^{-t/10} \cos\left(\frac{5\pi}{4}t\right) + 25\pi e^{-t/10} \sin\left(\frac{5\pi}{4}t\right)$$

1A

Question 4

a. i. $-k \int_0^5 (t^2(t-5))dt = 1$ 1M
 $-k \times -\frac{625}{12} = 1$ 1M
 $k = \frac{12}{625} = 0.0192$

- ii. The mode occurs at the maximum value of the function, which in this case, is a turning point 1M



$$\text{fMax}(Y_1, X, 0.5)$$

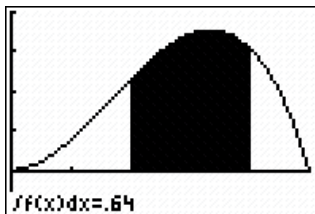
$$3.333330988$$

Mode = 3.33 1A

iii. $\Pr(2 < T < 4) =$

$$\int_2^4 (0.0192x^2(x-5))dx$$
 1M

$\Pr(2 < T < 4) = 0.64$ 1A

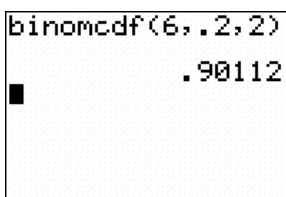


$$\text{fnInt}(Y_1, X, 2, 4)$$

$$.64$$

- iv. Let X be the number of people from area A.

$X \sim \text{Bi}(6, 0.2)$ 1M



$$\Pr(X \leq 2) = \sum_{x=0}^2 {}^6C_x (0.2)^x (0.8)^{6-x}$$

$$= 0.9011$$
 1A

b. $X \sim N(6, 1.5^2)$

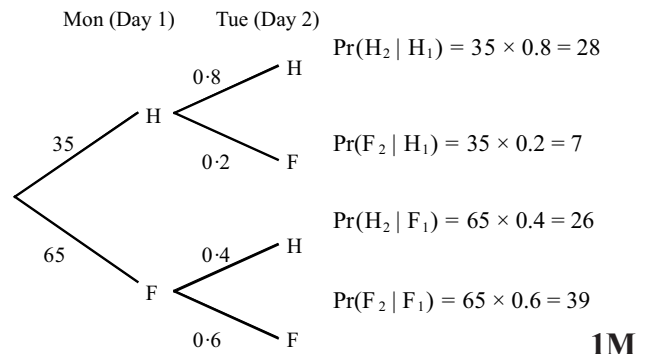
Conditional probability $\Pr(X < 8 | X > 5)$ 1M

$$\Pr(X < 8 | X > 5) = \frac{\Pr(5 < X < 8)}{\Pr(X > 5)}$$
 1M

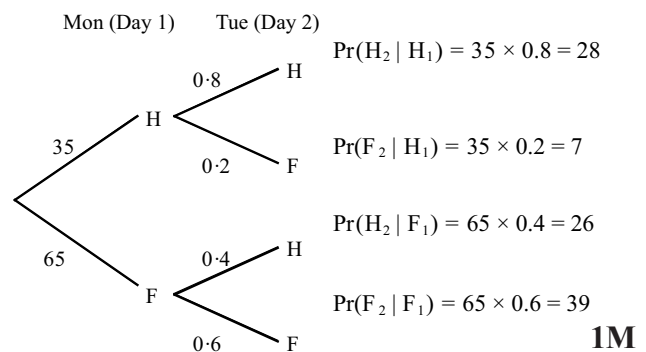
```
normalcdf(5,8,6,
1.5)÷A
.6562962511
normalcdf(5,10^9
9,6,1.5)÷B
.747507533
A÷B
.8779794479
```

$\Pr(X < 8 | X > 5) = 0.8780$ 1A

- c. Let H denote choosing from *Healthy* menu and F denote choosing from *Fast* menu.



$\Pr(H_2) = 28 + 26 = 54$ and
 $\Pr(F_2) = 7 + 39 = 46$ 1A



$\Pr(H_3) = 43.2 + 18.4 = 61.6$
On Wednesday, 62 people will choose from the *Healthy* menu. 1A