

VCE
2006 Mathematical Methods
Trial Examination 2

Suggested Solutions

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PURPOSE OF THIS TRIAL EXAMINATION

This Mathematics Methods Trial Examination is designed to assess

- understanding and communication of mathematical ideas
- interpretation, analysis and solution of routine problems
- interpretation, analysis and solution of non-routine problems

Assessment is by multiple-choice questions and extended answer questions involving multi-stage solutions of increasing complexity.

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Question 1 D Sketch the graphs on the calculator. There are three solutions at $x = -0.77$, 2 and 4.	Question 2 C $g\{f(x)\} = \log_e(x^2 - 9)$ $x^2 - 9 > 0$ $\Rightarrow x^2 > 9$ $\Rightarrow \pm x > 3$ $\Rightarrow x < -3$ or $x > 3$ Domain $(-\infty, -3) \cup (3, \infty)$												
Question 3 E The inverse only exists for a one-to-one function. The graph is one-to-one for $-\frac{\pi}{2} \leq 3x \leq \frac{\pi}{2} \Rightarrow -\frac{\pi}{6} \leq x \leq \frac{\pi}{6}$	Question 4 A <table><tr><td>x^2</td><td>1</td><td>4</td><td>9</td><td>16</td><td>25</td></tr><tr><td>$\Pr(X = x)$</td><td>k^2</td><td>$4k^2$</td><td>$9k^2$</td><td>$16k^2$</td><td>$25k^2$</td></tr></table> $\sum \Pr(X = x) = 1$ $\Rightarrow 55k^2 = 1$ $\Rightarrow k^2 = \frac{1}{55}$ $\Rightarrow k = \pm \frac{1}{\sqrt{55}} \times \frac{\sqrt{55}}{\sqrt{55}} = \pm \frac{\sqrt{55}}{55}$	x^2	1	4	9	16	25	$\Pr(X = x)$	k^2	$4k^2$	$9k^2$	$16k^2$	$25k^2$
x^2	1	4	9	16	25								
$\Pr(X = x)$	k^2	$4k^2$	$9k^2$	$16k^2$	$25k^2$								
Question 5 E $2(1 - \cos^2 \theta) - 7 \cos \theta + 2 = 0$ $\Rightarrow 2 - 2 \cos^2 \theta - 7 \cos \theta + 2 = 0$ $\Rightarrow -2 \cos^2 \theta - 7 \cos \theta + 4 = 0$ $\Rightarrow 2 \cos^2 \theta + 7 \cos \theta - 4 = 0$ $\Rightarrow (2 \cos \theta - 1)(\cos \theta + 4) = 0$ $\Rightarrow 2 \cos \theta - 1 = 0$ or $\cos \theta + 4 = 0$ $\cos \theta = \frac{1}{2}$ or $\cos \theta = -4$ But $-1 \leq \cos \theta \leq 1$ $\Rightarrow \cos \theta = \frac{1}{2}$ $\Rightarrow \theta = \frac{\pi}{3}, 2\pi - \frac{\pi}{3} = \frac{\pi}{3}, \frac{5\pi}{3}$ Sum $= \frac{\pi}{3} + \frac{5\pi}{3} = 2\pi$	Question 6 A $f(x) = a(x + 1)^2(x - 3)$ When $x = 0$, $f(x) = 6$ $\Rightarrow 6 = a \times 1 \times -3$ $\Rightarrow a = -2$ $f(x) = -2(x + 1)^2(x - 3)$ $\Rightarrow f(x) = 2(x + 1)^2(3 - x)$												

Question 7 B $y = 4 - \frac{1}{x-3}$ Interchange x and y $\Rightarrow x = 4 - \frac{1}{y-3}$ $\Rightarrow \frac{1}{y-3} = 4 - x$ $\Rightarrow y - 3 = \frac{1}{4 - x}$ $\Rightarrow y = \frac{1}{4 - x} + 3 = 3 + \frac{-1}{x - 4}$ $\Rightarrow y = 3 - \frac{1}{x - 4}$	Question 8 C $V = \frac{4}{3}\pi r^3$ $\frac{dV}{dr} = 4\pi r^2$ $\frac{dr}{dt} = 0.1$ $\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt} = 4\pi r^2 \times 0.1 = 0.4\pi r^2$ When $r = 7$, $\frac{dV}{dt} = 0.4\pi \times 49 = 61.6\text{cm}^3\text{sec}^{-1}$
Question 9 B Use a graphics calculator to sketch the graphs and find that the points of intersection are $x = 1.57$ and $x = 2.2$ Use the calculator to find the area between the curve and the X axis for $y = 6x - 11$ from $x = 1.57$ to $x = 2.2$ Area = 1.188 Use the calculator to find the area between the curve and the X axis for $y = x$ from $x = 1.57$ to $x = 2.2$ Area = 0.576 Area between these graphs $= 1.188 - 0.576 = 0.612$	Question 10 C $\int \frac{1}{(3x+2)^{1/2}} dx$ $= \int (3x+2)^{-1/2} dx$ $= \frac{2}{3}(3x+2)^{1/2} + c$ $= \frac{2\sqrt{3x+2}}{3} + c$ $= \frac{2\sqrt{3x+2}}{3}$

Question 11 E When $t = 0, A = A_0 = 200$ $A = 200e^{-kt}$ When $t = 20, A = 100$ $0.5 = e^{-20k}$ $\log_e 0.5 = -20k$ $k = 0.035$ $A = 200e^{-0.035t}$ $\frac{dA}{dt} = 200 \times (-0.035)e^{-0.035t}$ When $t = 30, \frac{dA}{dt} = -2.45$ The isotope is decaying at 2.45 g/day.	Question 12 E The area in answer B can be checked as incorrect by using a graphics calculator. $\text{gradient} = \frac{dy}{dx} = xe^x + e^x \times 1$ $= e^x(x+1) = 0$ when $x = -1$ When $x < -1, \frac{dy}{dx} < 0$ Therefore, negative gradient when $x < -1$
Question 13 A $k \int_0^9 x^2 dx = 1$ $\Rightarrow \left[\frac{kx^3}{3} \right]_0^9 = 1$ $\Rightarrow \frac{729k}{3} - 0 = 1$ $\Rightarrow 243k = 1$ $\Rightarrow k = \frac{1}{243} = 0.004$	Question 14 B $\sqrt{f(x)}$ only exists when $f(x) \geq 0$. That is, when y on the original graph is above or on the X axis.

Question 15 B

The gradient is positive everywhere except when $x \leq -1$ and when $x = 0$. $x = -1$ is a minimum as the graph goes from a negative to a positive gradient. $x = 0$ is a stationary point of inflexion as the graph goes from a positive to a positive gradient.

Question 16 D

$$\frac{dy}{dx} = \frac{2}{2x-1} + 4 = \frac{2+8x-4}{2x-1} = \frac{8x-2}{2x-1}$$

When $x = 1$, $\frac{dy}{dx} = 6$

So, gradient of normal $= -\frac{1}{6}$

Equation of normal is $y = -\frac{1}{6}x + c$

When $x = 1$ on curve $y = \log_e 1 + 4 = 4$

So point $(1, 4)$ lies on the normal.

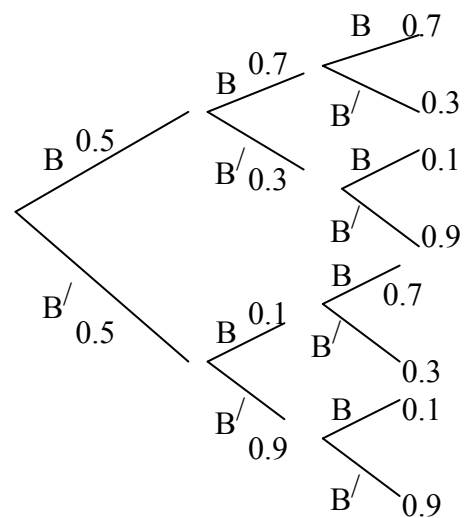
Therefore, $4 = -\frac{1}{6} + c \Rightarrow \frac{25}{6} = c$

$\Rightarrow y = -\frac{1}{6}x + \frac{25}{6} \Rightarrow 6y + x - 25 = 0$

Question 17 E

$$\begin{aligned}
 A &= \int_1^2 \frac{12}{12-5x} dx \\
 &= \frac{2}{-5} \int_1^2 \frac{-5}{12-5x} dx \\
 &= -\frac{2}{5} \log_e(12-5x) \Big|_1^2 \\
 &= -\frac{2}{5} \log_e(2) + \frac{2}{5} \log_e(7) \\
 &= \frac{2}{5} [\log_e(7) - \log_e(2)] \\
 &= 0.4 \log_e\left(\frac{7}{2}\right) \\
 &= 0.4 \log_e 3.5
 \end{aligned}$$

Question 18 B



$$\begin{aligned}
 B B B &= 0.5 \times 0.7 \times 0.7 = 0.245 \\
 B B' B &= 0.5 \times 0.3 \times 0.1 = 0.015 \\
 B' B B &= 0.5 \times 0.1 \times 0.7 = 0.035 \\
 B' B' B &= 0.5 \times 0.9 \times 0.1 = 0.045 \\
 \text{Total} &= 0.34
 \end{aligned}$$

Question 19 C

$$\begin{aligned}
 E(\cos x) &= \int_0^{\pi/2} \frac{2}{\pi} \cos x dx \\
 &= \left[\frac{2}{\pi} \sin x \right]_0^{\pi/2} \\
 &= \frac{2}{\pi} \left[\sin \frac{\pi}{2} - \sin 0 \right] \\
 &= \frac{2}{\pi} [1 - 0] = \frac{2}{\pi} = 0.637
 \end{aligned}$$

Question 20 E

$$\begin{aligned}
 \Pr(A) &= \frac{1}{6} \\
 \Pr(B) &= \frac{1}{2} \\
 \Pr(A \cap B) &= \frac{3}{36} = \frac{1}{12} \\
 \Pr(A \cap B) &= \Pr(A) \times \Pr(B) = \frac{1}{12} \\
 \text{Therefore, independent events.} \\
 \Pr(A \cup B) &= \Pr(A) + \Pr(B) - \Pr(A \cap B) \\
 &= \frac{1}{6} + \frac{1}{2} - \frac{1}{12} = \frac{7}{12} \\
 \text{Therefore, not mutually exclusive.}
 \end{aligned}$$

Question 21 A Use a graphics calculator. Enter data in the list menu. Use binompdf to get probabilities. Then use graph to get A.	Question 22 D $\Pr(B GG)$ $= \frac{\Pr(B \cap GG)}{\Pr(B \cap GG) + \Pr(Y \cap GG)}$$= \frac{\frac{3}{6} \times \frac{2}{5}}{\frac{3}{6} \times \frac{2}{5} + \frac{2}{6} \times \frac{1}{5}}$$= 0.75$
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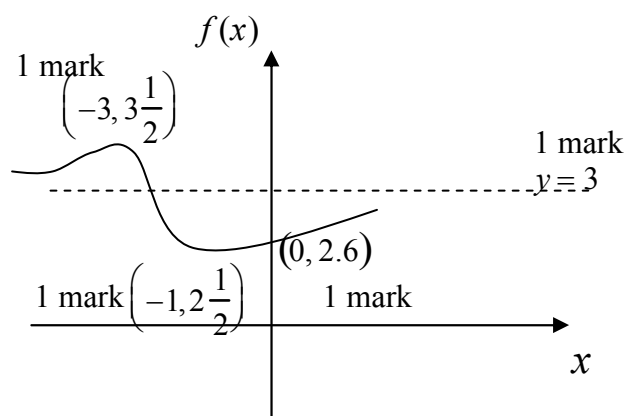
Suggested Solutions Part II

Question 1

a. $f(x) = \frac{x^2}{1+x^2}$ $f'(x) = \frac{(1+x^2)1 - x \times 2x}{(1+x^2)^2}$ $= \frac{1+x^2 - 2x^2}{(1+x^2)^2}$ $= \frac{1-x^2}{(1+x^2)^2} = 0 \text{ for turning point (1 mark)}$ $\Rightarrow 1 - x^2 = 0$ $\Rightarrow x = \pm 1 \text{ (1 mark)}$ When $x = 1$, $f(x) = \frac{1}{2}$ When $x = -1$, $f(x) = \frac{1}{2}$ When $x < -1$, $f'(x) < 0$ When $-1 < x < 1$, $f'(x) > 0$ When $x > 1$, $f'(x) < 0$ Maximum $\left(1, \frac{1}{2}\right)$ Minimum $\left(-1, \frac{1}{2}\right)$ (1 mark)	b. - Graph is reflected in the X axis. - Graph is translated 2 units to the left parallel to the X axis. - Graph is translated 3 units up parallel to the Y axis. (1 mark for each)
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Question 1 (continued)

c.



d.

$(-1, 1)$ (1 mark)

e.

$$y = \frac{x}{1+x^2}$$

Interchange x and y .

$$x = \frac{y}{1+y^2}$$

$$x(1+y^2) = y \quad (1 \text{ mark})$$

$$x + xy^2 - y = 0$$

$$xy^2 - y + x = 0$$

$$y = \frac{+1 \pm \sqrt{1-4x^2}}{2x}$$

$$\text{But } y > 0 \therefore y = \frac{+1 + \sqrt{1-4x^2}}{2x} \quad (1 \text{ mark})$$

$$\text{Domain } \left(0, \frac{1}{2}\right) \quad (1 \text{ mark})$$

Question 2

a. Maximum value of $\sin\left(\frac{2\pi t}{3}\right) = 1$ Max length of spring = $(a + b)$ cm (1 mark)	b. Minimum value of $\sin\left(\frac{2\pi t}{3}\right) = -1$ Min length of spring = $(a - b)$ cm (1 mark)
c. $8 \sin\left(\frac{2\pi t}{3}\right) = 4$ $\Rightarrow \sin\left(\frac{2\pi t}{3}\right) = \frac{1}{2}$ (1 mark) $\Rightarrow \frac{2\pi t}{3} = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}$ (1 mark) $\Rightarrow t = \frac{\pi}{6} \times \frac{3}{2\pi}, \frac{5\pi}{6} \times \frac{3}{2\pi}, \frac{13\pi}{6} \times \frac{3}{2\pi}, \frac{17\pi}{6} \times \frac{3}{2\pi}$ $\Rightarrow t = \frac{1}{4}, 1\frac{1}{4}, 3\frac{1}{4}, 4\frac{1}{4}$ (1 mark)	d. Period = $2\pi \div \frac{2\pi}{3} = 3$ seconds (1 mark)
e. $64 + 8 \sin\left(\frac{2\pi t}{3}\right) > 60$ This is true when $8 \sin\left(\frac{2\pi t}{3}\right) > -4 \Rightarrow \sin\left(\frac{2\pi t}{3}\right) > -\frac{1}{2}$ Let $\sin\left(\frac{2\pi t}{3}\right) = \frac{1}{2}$ $\Rightarrow \frac{2\pi t}{3} = \pi + \frac{\pi}{6}, 2\pi - \frac{\pi}{6} = \frac{7\pi}{6}, \frac{11\pi}{6}$ $\Rightarrow t = \frac{7\pi}{6} \times \frac{3}{2\pi}, \frac{11\pi}{6} \times \frac{3}{2\pi}$ $\Rightarrow t = \frac{7}{4}, \frac{11}{4}$ (1 mark) $64 + 8 \sin\left(\frac{2\pi t}{3}\right) > 60$ from 0 to 1.75 seconds. Between 1.75 seconds and 2.75 seconds, the function is less than 60.	f. When $t = 0$, length $(x) = 64$ When $t = 0.25$, length $(x) = 68$ (1 mark) Average rate of change $= \frac{\Delta x}{\Delta t} = \frac{4}{0.25} = 16$ cm/sec (1 mark)
g. $\frac{dx}{dt} = \frac{16\pi}{3} \cos\left(\frac{2\pi t}{3}\right)$ (1 mark) When $t = 2.75$, $\frac{dx}{dt} = \frac{16\pi}{3} \cos\left(\frac{2\pi \times 2.75}{3}\right) = \frac{8\pi\sqrt{3}}{3}$ cm/sec (1 mark)	

Question 3

a. $y = a(x - b)^2 + c \quad (b = 0)$ $y = ax^2 + c$ When $x = 0, y = 1$ so $y = ax^2 + 1$ (1 mark) When $x = -3, y = 3.25$ so $3.25 = 9a + 1$ $\Rightarrow 2.25 = 9a$ $\Rightarrow \frac{2.25}{9} = a$ $\Rightarrow a = 0.25$ (1 mark)	b. Diameter = distance from -10 to $+10 = 20$ (1 mark)
c. Express x in terms of y. $y = \frac{x^2}{4} + 1$ $\Rightarrow y - 1 = \frac{x^2}{4}$ $\Rightarrow x^2 = 4(y - 1)$ $\Rightarrow x = \pm\sqrt{4(y - 1)}$ (1 mark)	d. $\int_a^b 4\pi(y - 1)dy$ (1 mark) a and b are the y values. $a = 1$ (1 mark) b is the value of y when $x = 10$ (1 mark) When $x = 10, y = \frac{1}{4} \times 100 + 1 = 26$ $\Rightarrow b = 26$ $V = 4\pi \left[\frac{y^2}{2} - y \right]_1^{26} = 3927$ cubic units (1 mark)
e. $r = 3 \Rightarrow x = 3$ When $x = 3, y = \frac{1}{4} \times 9 + 1 = \frac{13}{4}$ $V = 4\pi \int_1^{13/4} (y - 1)dy$ (1 mark) $V = 4\pi \left[\frac{y^2}{2} - y \right]_1^{13/4}$ $V = 4\pi \left[\frac{169}{32} - \frac{13}{4} - \frac{1}{2} + 1 \right]$ $V = 4\pi \times \frac{81}{32} = \frac{81\pi}{8}$ cubic units (1 mark)	f. $\frac{dy}{dx} = \frac{1}{9} [e^{3x} \times 3 + (3x + 2) \times 3e^{3x}]$ (1 mark) $= \frac{1}{3} e^{3x} \times [1 + 3x + 2]$ $= \frac{1}{3} e^{3x} \times [3x + 3]$ $= e^{3x}(x + 1)$ (1 mark)

g. i. $\frac{dh}{dt} = 4te^{3t} \Rightarrow h = \int_0^1 4te^{3t} dt$ Now $\int te^{3t} + \int e^{3t} = \frac{1}{9}e^{3t}(3t+2) + c$ (1 mark) $\int te^{3t} = \frac{1}{9}e^{3t}(3t+2) - \frac{1}{3}e^{3t} + c$ $h = 4 \int_0^1 te^{3t} dt = 4 \left[\frac{1}{9}e^{3t}(3t+2) - \frac{1}{3}e^{3t} \right]_0^1$ (1 mark) $h = 4 \left[\frac{5}{9}e^3 - \frac{1}{3}e^3 - \frac{2}{9} + \frac{1}{3} \right]$ $h = 18.2983 \text{ units}$ (1 mark)	g. ii. $\frac{dV}{dt} = \frac{dV}{dh} \frac{dh}{dt}$ $h = y$ $\frac{dV}{dt} = \frac{dV}{dy} \frac{dy}{dt}$ $\frac{dV}{dy} = 4\pi(y-1)$ (1 mark) $\frac{dh}{dt} = \frac{dy}{dt} = 4te^{3t} = 4e^3 \text{ when } t = 1$ $\frac{dV}{dt} = 4\pi(18.2983 - 1) \times 4e^3$ $= 17464.52 \text{ cubic units/time unit}$ (1 mark)
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Question 4

a. i. $E(X) = \sum x \Pr(X = x)$ $= 0 + 0.01 + 0.1 + 0.6 + 1.6 + 1.5$ $= 3.81 \quad (1 \text{ mark})$	a. ii. $\sigma^2 = E(X^2) - \mu^2$ $E(X^2) = 0.01 + 0.2 + 1.8 + 6.4 + 7.5$ $= 15.91 \quad (1 \text{ mark})$ $\sigma^2 = 15.91 - (3.81)^2$ $\sigma = \sqrt{15.91 - (3.81)^2} = 1.18 \quad (1 \text{ mark})$																				
b. i. $1 - 0.85 = 0.15 \quad (1 \text{ mark})$	b. ii. Probability(will not flower in 2008) <table><tr><td>2006</td><td>2007</td><td>2008</td><td></td></tr><tr><td>F</td><td>F</td><td>F/</td><td>$= 0.8 \times 0.8 \times 0.2$</td></tr><tr><td>F</td><td>F/</td><td>F/</td><td>$= 0.8 \times 0.2 \times 0.85$</td></tr><tr><td>F/</td><td>F/</td><td>F/</td><td>$= 0.2 \times 0.15 \times 0.2$</td></tr><tr><td>F/</td><td>F</td><td>F/</td><td>$= 0.2 \times 0.85 \times 0.85$</td></tr></table> (1 mark) $= 0.128 + 0.136 + 0.006 + 0.1445$ $= 0.4145 \quad (1 \text{ mark})$	2006	2007	2008		F	F	F/	$= 0.8 \times 0.8 \times 0.2$	F	F/	F/	$= 0.8 \times 0.2 \times 0.85$	F/	F/	F/	$= 0.2 \times 0.15 \times 0.2$	F/	F	F/	$= 0.2 \times 0.85 \times 0.85$
2006	2007	2008																			
F	F	F/	$= 0.8 \times 0.8 \times 0.2$																		
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F/	F/	F/	$= 0.2 \times 0.15 \times 0.2$																		
F/	F	F/	$= 0.2 \times 0.85 \times 0.85$																		
c. i. $a \int_0^1 (t^2 - t^3) dt = 1$ $\Rightarrow a \left[\frac{t^3}{3} - \frac{t^4}{4} \right]_0^1 = 1$ $\Rightarrow a \left(\frac{1}{3} - \frac{1}{4} \right) = 1$ $\Rightarrow \frac{a}{12} = 1 \Rightarrow a = 12 \quad (1 \text{ mark})$	c. ii. $12 \int_0^{0.25} (t^2 - t^3) dt$ $= 12 \left[\frac{t^3}{3} - \frac{t^4}{4} \right]_0^{0.25}$ $= 0.05 \quad (1 \text{ mark})$																				

Question 4 (continued)

c. iii.

$$\Pr(X \leq 3) | \Pr(X \leq 6) = \frac{\Pr(X \leq 3) \cap \Pr(X \leq 6)}{\Pr(X \leq 6)}$$

$$= \frac{\Pr(X \leq 3)}{\Pr(X \leq 6)}$$

$$\Pr(X \leq 6) = 12 \left[\frac{t^3}{3} - \frac{t^4}{4} \right]_0^{0.5} = 0.3125 \quad (1 \text{ mark})$$

$$\Pr(X \leq 3) | \Pr(X \leq 6) = \frac{0.05078125}{0.3125}$$

$$= 0.1625$$

$$= 0.16 \text{ to 2 decimal places} \quad (1 \text{ mark})$$

Question 4 (continued)

d. i.

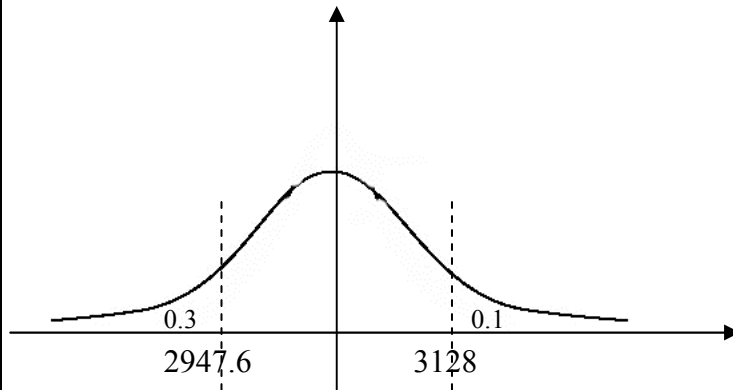
Binomial $n = 10$, $p = 0.3$

$\text{binompdf}(10, 0.3, 1) = 0.1493$

Probability = 0.1493

(1 mark)

d. ii.



$$\Pr(X < 3128) = \Pr(Z < a) = 0.9$$

$$a = 1.2816 \quad (1 \text{ mark})$$

$$\Pr(X < 2947.6) = \Pr(Z < -b) = 0.3$$

$$\Pr(Z > b) = 0.3 \Rightarrow \Pr(Z < b) = 0.7$$

$$b = 0.5244$$

$$-b = -0.5244 \quad (1 \text{ mark})$$

$$Z = \frac{x - \mu}{\sigma}$$

$$1.2816 = \frac{3128 - \mu}{\sigma} \dots (1)$$

$$-0.5244 = \frac{2947.6 - \mu}{\sigma} \dots (2)$$

$$\mu + 1.2816\sigma = 3128 \dots (1a)$$

$$\mu - 0.5244\sigma = 2947.6 \dots (2a) \quad (1 \text{ mark})$$

$$(1a) - (2a) \text{ gives } 1.806\sigma = 180.4$$

$$\sigma = 99.98 = 100 \text{ g to the nearest g} \quad (1 \text{ mark})$$

Substituting in (1a) gives

$$\mu = 3128 - 128.018 = 2999.98 = 3000 \text{ g to the nearest g} \quad (1 \text{ mark})$$

END OF SUGGESTED SOLUTIONS

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