

1a. $f: (-1, \infty) \rightarrow \mathbb{R}, f(x) = \frac{e^{x+1}}{2}$.

Equation of f is $y = \frac{e^{x+1}}{2}$. At $x = -1$, $y = \frac{e^0}{2} = \frac{1}{2}$; as $x \rightarrow \infty$, $y \rightarrow \infty$. $\therefore$ range of f is $\left(\frac{1}{2}, \infty\right)$.

Equation of f^{-1} is $x = \frac{e^{y+1}}{2}$, $2x = e^{y+1}$, $y = \log_e(2x) - 1$,
 $\therefore f^{-1}(x) = \log_e(2x) - 1$.

1b. The domain of f^{-1} is the range of f , i.e. $\left(\frac{1}{2}, \infty\right)$.

2a. Apply the product rule,

$$\begin{aligned} \frac{d}{dx}(2\sqrt{x} \cos(x)) &= (\cos(x))\left(\frac{1}{\sqrt{x}}\right) + (2\sqrt{x})(-\sin(x)) \\ &= \frac{\cos(x) - 2x \sin(x)}{\sqrt{x}}. \end{aligned}$$

2b. $f'(x) = \frac{1}{(2-x)^2}$,

$$\therefore f(x) = \int \frac{1}{(2-x)^2} dx = \int (2-x)^{-2} dx = \frac{1}{2-x} + C.$$

$$f(3) = -1 + C = 3, \therefore C = 4, \text{ and } f(x) = \frac{1}{2-x} + 4.$$

3. To find the x -intercept, let $y = 0$, $\therefore \sqrt{3} \sin(x) + \cos(x) = 0$,

$$\therefore \sqrt{3} \sin(x) = -\cos(x), \frac{\sin(x)}{\cos(x)} = -\frac{1}{\sqrt{3}}, \tan(x) = -\frac{1}{\sqrt{3}},$$

$\therefore x = \frac{5\pi}{6}$ is the smallest positive value.

4a. $f(x) = 2 \sin(3x)$, where $x \in [0, 2\pi]$,

$$g(x) = \frac{1}{3} f\left(\frac{x}{2}\right) = \frac{1}{3} \left(2 \sin 3\left(\frac{x}{2}\right)\right) = \frac{2}{3} \sin\left(\frac{3}{2}x\right). \text{ Amplitude} = \frac{2}{3}$$

and period $= \frac{4\pi}{3}$.

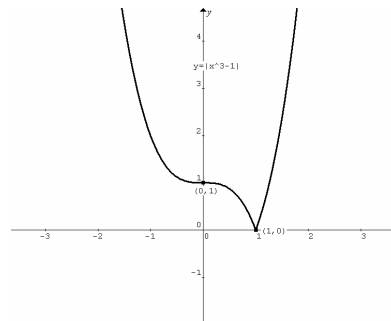
4b. $h(x) = g(x) - \frac{2}{3} = \frac{2}{3} \sin\left(\frac{3}{2}x\right) - \frac{2}{3}$.

Domain: $[0, 2\pi]$; range: $\left[-\frac{4}{3}, 0\right]$.

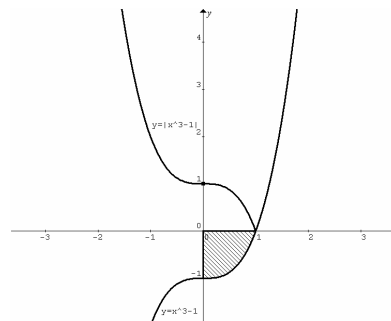
5. Solve simultaneously $y = ax - 1$ and $y = x^2$.

$x^2 = ax - 1$, $x^2 - ax + 1 = 0$. For $y = ax - 1$ to be a tangent to $y = x^2$, $x^2 - ax + 1 = 0$ has only one solution,
i.e. $\Delta = 0$, $(-a)^2 - 4(1)(1) = 0$, $\therefore a = 2$ given that $a > 0$.

6a.



6b.



Area of the required region bounded by the two curves
 $= 2 \times$ area of the shaded region (graph above)

$$= \left| 2 \int_0^1 (x^3 - 1) dx \right| = \left| 2 \left[\frac{x^4}{4} - x \right]_0^1 \right| = \left| -2 \left(\frac{1}{4} - 1 \right) \right| = \frac{3}{2}.$$

6c. $y = |x^3 - 1|$ is differentiable for $x \neq 1$, $\therefore$ maximal domain is $\mathbb{R} \setminus \{1\}$.

7a. Given $u(x) = \log_e(x)$ and $f(x) = x^2 + 1$,

$$\therefore f(u(x)) = f(\log_e(x)) = (\log_e(x))^2 + 1.$$

7b. Apply the chain rule: $f'(u(x)) = f'(u) \times u'(x)$

$$= 2u(x) \times \frac{1}{x} = \frac{2 \log_e(x)}{x}.$$

7c. Since $\frac{2 \log_e(x)}{x} = f'(u(x))$, $\therefore \int \frac{2 \log_e(x)}{x} dx = \int f'(u(x)) dx$,

$$\therefore \int \frac{\log_e(x)}{x} dx = \frac{1}{2} \int f'(u(x)) dx = \frac{1}{2} f(u(x)) + c = \frac{1}{2} (\log_e(x))^2 + C$$

An anti-derivative is $\frac{1}{2} (\log_e(x))^2$.

8. The water is pumped from the vessel at a constant rate of 2 m^3 per minute. $\therefore$ air is filling the vessel at a constant rate of 2 m^3 per minute.

Relationship between the height (h) of air above the water surface and the side-length of water surface (x):

$$\frac{x}{h} = \frac{3}{4}, \therefore x = \frac{3h}{4}.$$

Volume of air above the water inside the vessel:

$$V = \frac{1}{3}x^2h = \frac{1}{3}\left(\frac{3h}{4}\right)^2h = \frac{3h^3}{16}.$$

Related rate: $\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt} = \frac{9h^2}{16} \times \frac{dh}{dt}.$

When the depth of water is 1 m, the height of air above is 3 m,

$$\therefore 2 = \frac{9(3)^2}{16} \times \frac{dh}{dt}, \therefore \frac{dh}{dt} = \frac{32}{81}, \text{ i.e. the height of air above is}$$

rising at $\frac{32}{81}$ m per minute. Hence the depth of water is falling at

$$\frac{32}{81} \text{ m per minute.}$$

9a. Since $\mu = 5$, $\therefore \Pr(X < 5) = 0.5$.

9b. Since $\mu = 5$ and $\sigma = 1.5$,

$$\therefore \Pr(X \geq 8) = \Pr(X \geq \mu + 2\sigma) = \frac{1}{2}(1 - 0.95) = 0.025.$$

$$10a. f(x) = \begin{cases} 0.3 & \text{if } 0 \leq x < 2 \\ p & \text{if } 2 \leq x \leq 6 \\ 0 & \text{if } x < 0 \text{ or } x > 6 \end{cases}$$

$$0.3 \times 2 + p \times 4 = 1, 4p = 0.4, \therefore p = 0.1.$$

10b. $\Pr(-2 < X \leq 3)$

$$= \Pr(-2 < X < 0) + \Pr(0 \leq X < 2) + \Pr(2 \leq X \leq 3)$$

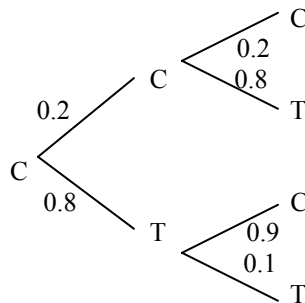
$$= 0 + 0.3 \times 2 + 0.1 \times 1 = 0.7.$$

11. X has a binomial distribution with $n = 3$, $p = \frac{1}{3}$ and

$$\Pr(X = x) = {}^3C_x \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{3-x}.$$

x	0	1	2	3
$\Pr(X = x)$	$\frac{8}{27}$	$\frac{4}{9}$	$\frac{2}{9}$	$\frac{1}{27}$

12.



$$\Pr(3rdTEA) = \Pr(CCT) + \Pr(CTT) = 0.2 \times 0.8 + 0.8 \times 0.1 = 0.24.$$

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