

INSIGHT

Trial Exam Paper

2006

MATHEMATICAL METHODS

Written examination 1

Worked solutions

This book presents:

- correct answers
- worked solutions, giving you a series of points to show you how to work through the questions and achieve top marks
- mark allocation details.

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Question 1

1a. Use the factor theorem to show that $(x + 2)$ is a factor of $9x^3 + 12x^2 - 11x + 2$.

Answer

$$\begin{aligned} p(-2) &= 9 \times -8 + 12 \times 4 + 22 + 2 \\ &= -72 + 48 + 24 = 0 \therefore x + 2 \text{ is a factor} \end{aligned}$$

Marks

- 1 mark for using $x = -2$
- 1 mark for showing equals to zero.

1b. The equation $y = 9x^3 + 12x^2 - 11x + 2$ can be written in the form $y = (x + 2)(ax - b)^2$ where $\{a, b\} > 0$. State the values of a and b .

Answer

By inspection, $a^2 = 9 \Rightarrow a = 3$. Equating the coefficients gives $2 \times b^2 = 2 \Rightarrow b = 1$

Marks

- 1 mark for $a = 3$
- 1 mark for $b = 1$

2 + 2 = 4 marks

Question 2

2a. The graph of a function g is obtained from the graph of the function f which has the rule $f(x) = 2(x - 2)^5$ by a performing a translation of -4 units parallel to the x -axis. Write down the rule for g .

Answer

Replace x with $x + 4$ to give $g(x) = 2(x + 2)^5$

2b. The graph of a function h is obtained from the graph of g by a reflection in the y -axis. Write down the rule for h .

Answer

Replace x with $-x$ to give $h(x) = 2(2 - x)^5$

2c. The graph of a function k is obtained from the graph of h by a dilation by a scale factor of $\frac{1}{2}$ along the y -axis. Write down the rule for k .

Answer

Replace x with $2x$ to give $k(x) = 2(2 - 2x)^5$

Marks

- 1 mark for each correct equation

1 + 1 + 1 = 3 marks

Question 3

Solve the equation $\sqrt{3} \sin(2x) + \cos(2x) = 0$ for $x \in [0, 2\pi]$, giving exact values in terms of π .

Answer

$$\sqrt{3} \sin 2x = -\cos 2x$$

$$\tan 2x = \frac{-1}{\sqrt{3}}$$

$$2x = \frac{5\pi}{6}, \frac{11\pi}{6}, \frac{17\pi}{6}, \frac{23\pi}{6}$$

$$x = \frac{5\pi}{12}, \frac{11\pi}{12}, \frac{17\pi}{12}, \frac{23\pi}{12}$$

Marks

- 1 mark for rearranging to get $\tan(2x)$
- 1 mark for finding $\frac{5\pi}{6}, \frac{11\pi}{6}$
- 1 mark for all four correct angles

3 marks

Question 4

Let $f(x) = x^2$ and $g(x) = 3x - 5$.

4a. Write down the rule of $f(g(x))$.

Answer

$$f(g(x)) = (3x - 5)^2$$

4b. Find the derivative of $f(g(x))$.

Answer

Let $y = f(g(x))$,

$$\frac{dy}{dx} = 2(3x - 5) \times 3 = 6(3x - 5) \quad \text{using the chain rule.}$$

4c. Hence, find the coordinates of the point P on the curve with the equation $y = f(g(x))$ at which the tangent is parallel to the line $2y - 12x = 7$.

Answer

‘Parallel’ means a line has the same gradient as another.

The gradient of the line $2y - 12x = 7$ is 6 (after rearranging to $y = mx + c$ form), so:

$$6(3x - 5) = 6$$

$$(3x - 5) = 1$$

$$x = 2, y = 1$$

Marks

- 1 mark for the rule in **a**.
- 1 mark for the derivative in **b**.
- 1 mark for equating the derivative to equal 6
- 1 mark for the correct coordinate

1 + 1 + 2 = 4 marks

Question 5

For the function $f : [-\pi, \pi] \rightarrow \mathbb{R}$, $f(x) = -2\cos(2t + \frac{\pi}{2})$

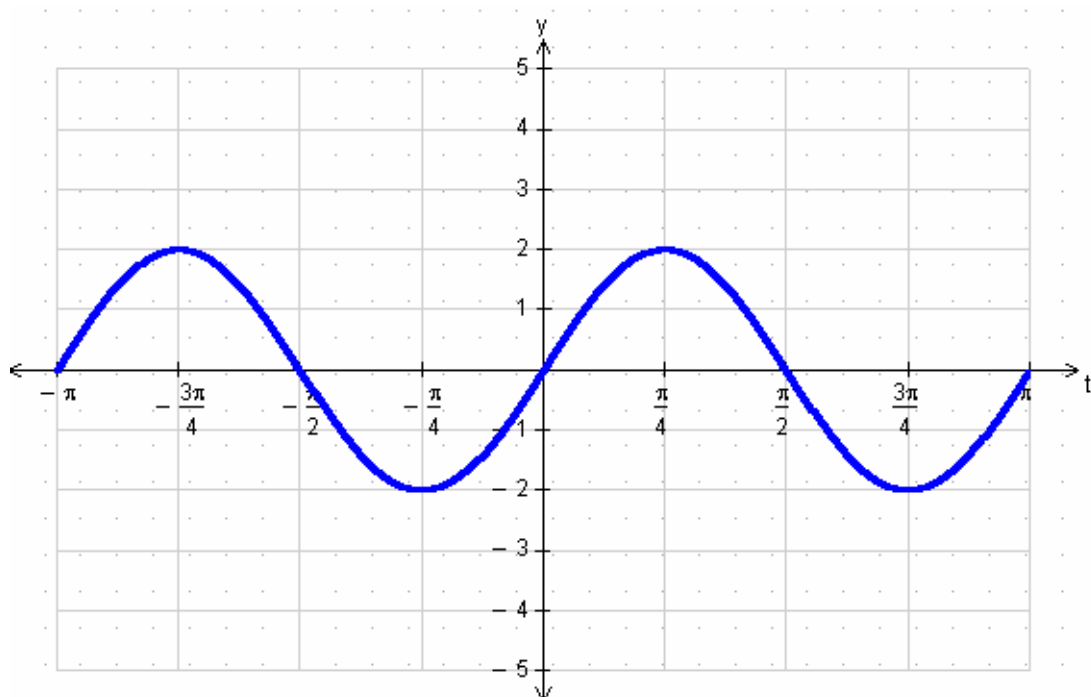
5a. Write down the period of the function.

Answer

$$p = \frac{2\pi}{n} = \frac{2\pi}{2} = \pi$$

5b. On the set of axes below, sketch the graph of the function f .

Answer



5c. State the number of solutions to the equation $\cos(2t + \frac{\pi}{2}) = \frac{1}{2}$, where $-\pi \leq x \leq \pi$.

Answer

Look at the intersection of the graph with the line $y = -1$. There are four points of intersection, and therefore four solutions to the equation.

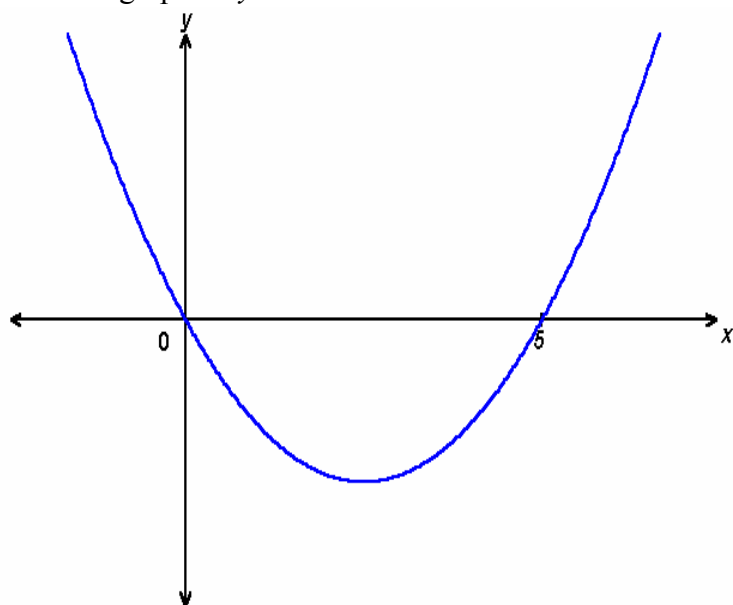
Marks

- 1 mark for correct period
- 1 mark for sketching two cycles of graph
- 1 mark for correct shape and amplitude of graph
- 1 mark for stating '4'

1 + 2 + 1 = 4 marks

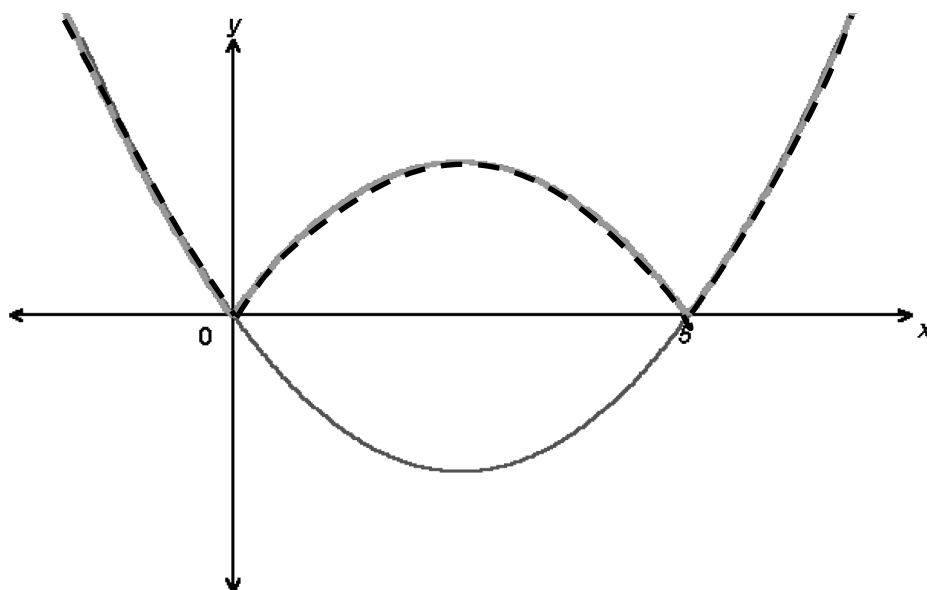
Question 6

Part of the graph of $y = x^2 - 5x$ is shown below.



6a. On the same set of axes sketch the graph of $y = |x^2 - 5x|$.

Answer



6b. Find the set of values of x for which $|x^2 - 5x| \geq 6$.

Answer

Solve

$$x^2 - 5x \geq 6 \text{ and } -x^2 + 5x \geq 6$$

to give $2 \leq x \leq 3$

and $x \geq 6$ or $x \leq -1$

Marks

- 1 mark for the correct sketch
- 1 mark for setting up two equations to solve
- 1 mark for the correct intervals

1 + 2 = 3 marks

Question 7

For the function $f(x) = 2e^{1-x}$,

7a. find the rule of the inverse function f^{-1} .

Answer

Swap x and y to give $x = 2e^{1-y}$, then rearrange to make y the subject, as follows:

$$\frac{x}{2} = e^{1-y}$$

$$\log_e \frac{x}{2} = 1 - y \Rightarrow y = 1 - \log_e \frac{x}{2}$$

7b. find the domain of the inverse function f^{-1} .

Answer

Range of the original function = domain of the inverse function, so the domain of the inverse function = R^+ .

Marks

- 1 mark for knowing to swap x and y and an attempt to find y
- 1 mark for the correct equation
- 1 mark for the correct domain

2 + 1 = 3 marks

Question 8

The random variable X has the following probability distribution.

x	-1	0	1	2
$Pr(X=x)$	a	$2a$	$3a$	0.4

8a. Find the value of a .

Answer

Since this is a probability distribution, the probabilities must total 1. Therefore,

$$6a + 0.4 = 1$$

$$6a = 0.6$$

$$a = 0.1$$

8b. If $Pr(X \leq k) > 0.5$, find the minimum value of k .

Answer

$$Pr(X \leq 0) = 0.3$$

$$Pr(X \leq 1) = 0.6 \text{ so } k = 1$$

Marks

- 1 mark for the correct value of a
- 1 mark for the correct value of k

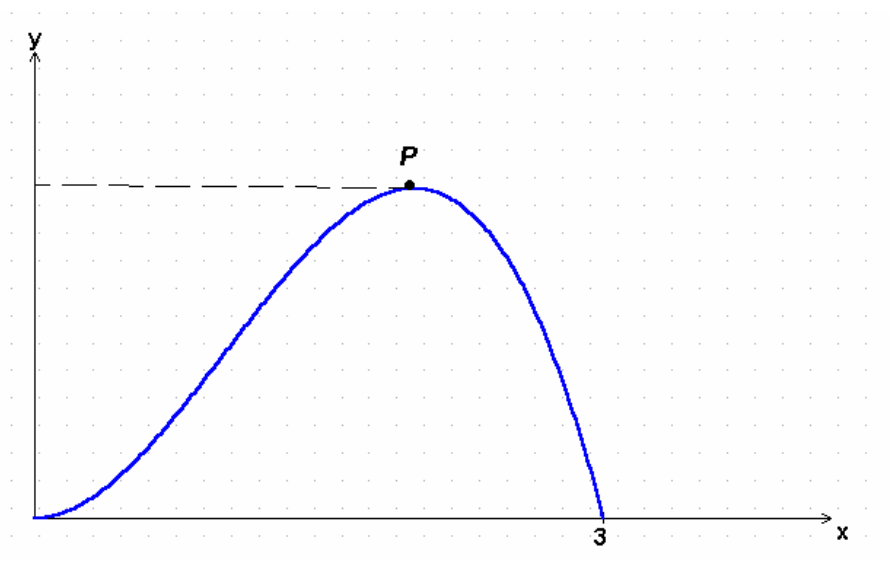
1 + 1 = 2 marks

Question 9

A continuous random variable X has the probability density function given by

$$f(x) = \frac{4}{27}(3x^2 - x^3), \quad 0 \leq x \leq 3$$

The graph of f , as shown below, has a maximum point at P .



9a. Find the value of the x -coordinate of P .

Answer

The maximum occurs when the derivative equals zero.

$$\begin{aligned} f'(x) &= \frac{4}{27}(6x - 3x^2) = 0 \\ \Rightarrow 3x(2 - x) &= 0 \\ \Rightarrow x = 0 \text{ or } x = 2 \text{ so } x &= 2 \end{aligned}$$

Marks

- 1 mark for the correct derivative
- 1 mark for equating derivative to zero
- 1 mark for the correct answer

9b. Find the $\Pr(0 < X < 2)$.

Answer

$$\begin{aligned} \Pr(0 < X < 2) &= \int_0^2 \frac{4}{27}(3x^2 - x^3) dx \\ &= \frac{4}{27} \left[x^3 - \frac{x^4}{4} \right]_0^2 \\ &= \frac{4}{27}(8 - 4) = \frac{16}{27} \end{aligned}$$

Marks

- 1 mark for setting up integral from 0 to 2
- 1 mark for the correct antiderivative
- 1 mark for the correct answer

9c. Find the mean value of X .

Answer

$$\begin{aligned} E(X) &= \int_0^3 xf(x) dx = \frac{4}{27} \int_0^3 (3x^3 - x^4) dx \\ &= \frac{4}{27} \left[\frac{3x^4}{4} - \frac{x^5}{5} \right]_0^3 = \frac{4}{27} \left(\frac{243}{4} - \frac{243}{5} \right) = \frac{9}{5} \end{aligned}$$

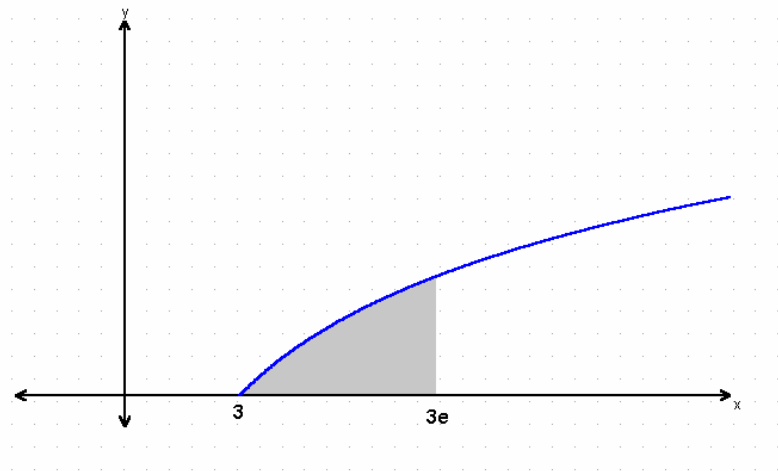
Marks

- 1 mark for setting up integral of $xf(x)$ from 0 to 3
- 1 mark for the correct antiderivative
- 1 mark for the correct answer

$3 + 3 + 3 = 9$ marks

Question 10

The graph of the function $f : [3, \infty) \rightarrow \mathbb{R}$, $f(x) = \log_e \frac{x}{3}$ is shown below.



10a. If $y = x \log_e \frac{x}{3} - x$, find $\frac{dy}{dx}$.

Answer

$$\begin{aligned} \frac{dy}{dx} &= x \frac{3}{x} \frac{1}{3} + \log_e \frac{x}{3} - 1 \\ &= 1 + \log_e \frac{x}{3} - 1 \\ &= \log_e \frac{x}{3} \end{aligned}$$

Marks

- 1 mark for using product rule to differentiate
- 1 mark for the correct answer

10b. Hence, find the exact area of the shaded region.

Answer

$$\begin{aligned} \int_3^{3e} \log_e \frac{x}{3} dx &= \left[x \log_e \frac{x}{3} - x \right]_e^{3e} \quad \text{from part a} \\ &= 3e \log_e e - 3e - 3 \log_e 1 + 3 \\ &= 3e - 3e - 0 + 3 = 3 \text{ sq units.} \end{aligned}$$

Marks

- 1 mark for using result of part a.
- 1 mark for evaluating with $3e$ and e
- 1 mark for the correct answer

2 + 3 = 5 marks

END OF SOLUTIONS