

UNIT 4 MATHEMATICAL METHODS 2005 WRITTEN EXAMINATION 2 - SOLUTIONS

QUESTION 1

- a. (i) Using the Pythagorean Theorem:

$$d^2 + w^2 = 15^2$$

$$d^2 = 225 - w^2$$

$$S = kd^2w = k(225 - w^2)w = k(225w - w^3)$$

(ii) $d^2 = 225 - w^2$

$$d = \pm\sqrt{225 - w^2}$$

As dimensions must be positive in value, $d \neq -\sqrt{225 - w^2}$

$$\therefore d = \sqrt{225 - w^2}$$

$$\therefore w^2 < 225$$

$$w < \pm 15$$

As dimensions must be positive in value, $0 < w(\text{cm}) < 15$.

- b. (i) $S = k(225w - w^3) = 225wk - kw^3$

As $k > 0$, and $0 < w(\text{cm}) < 15$, the graph of S lies entirely above the w axis for all accepted values of w .

$$\int_0^5 k(225w - w^3) dw = 425$$

$$k \int_0^5 (225w - w^3) dw = 425$$

$$k \left[\frac{225w^2}{2} - \frac{w^4}{4} \right]_0^5 = 425$$

$$k \left(\frac{225(25)}{2} - \frac{(5)^4}{4} \right) - (0) = 425$$

$$k \left(\frac{10625}{4} \right) = 425$$

$$\therefore k = \frac{4}{25}$$

$$(ii) \quad S = \frac{4}{25}(225w - w^3)$$

A maximum can occur at an end point of a domain or a stationary point.
Using the calculator – the following points can be obtained:

When $w = 0$, $S = 0$.

When $w = 7$, $S = 197.12$.

Stationary point occurs at (8.660, 207.846) - which lies outside the possible domain.

Therefore, maximum strength obtained when $w = 7$ cm.

$$\text{Therefore, } d = \sqrt{225 - w^2} = \sqrt{225 - 49} = \sqrt{176} = \sqrt{16 \times 11} = 4\sqrt{11} \text{ cm}$$

$$c. \quad (i) \quad C(\$) = 0.2 \times \text{Area Face}$$

$$C(\$) = 0.2(dw) = 0.2w\sqrt{225 - w^2}$$

$$(ii) \quad C = 0.2w(225 - w^2)^{1/2}$$

$$\begin{aligned} \text{Product Rule: } \frac{dC}{dw} &= 0.2w \times \left(\frac{1}{2} \times -2w \times (225 - w^2)^{-1/2} \right) + (225 - w^2)^{1/2} \times 0.2 \\ &= \frac{-0.2w^2}{\sqrt{225 - w^2}} + 0.2\sqrt{225 - w^2} \end{aligned}$$

$$\text{Let } \frac{dC}{dw} = 0$$

$$\frac{-0.2w^2}{\sqrt{225 - w^2}} + 0.2\sqrt{225 - w^2} = 0$$

$$\frac{0.2w^2}{\sqrt{225 - w^2}} = 0.2\sqrt{225 - w^2}$$

$$0.2w^2 = 0.2(\sqrt{225 - w^2})^2$$

$$w^2 = 225 - w^2$$

$$2w^2 = 225$$

$$w = \sqrt{\frac{225}{2}} = \frac{15}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{15\sqrt{2}}{2} \text{ cm}$$

Maximum cost occurs when $w = \frac{15\sqrt{2}}{2}$ cm and is equal to \$22.50

QUESTION 2

a. (i) $\Pr(\text{tuna from Boat 1}) = \frac{1800}{3000} = \frac{3}{5}$

(ii) Let X = Number of tuna

$$E(X) = \frac{3}{5} \times 500 = 300$$

b. (i) $\Pr(\text{Boat 1} / \text{tuna}) = \frac{\Pr(\text{Boat 1} \cap \text{tuna})}{\Pr(\text{tuna})} = \frac{\frac{1800}{5000}}{\frac{2600}{5000}} = \frac{1800}{2600} = \frac{9}{13}$

Alternatively – the probability may be obtained directly from the given table.

$$\frac{\text{Number tuna caught by Boat 1}}{\text{Total number tuna caught}} = \frac{1800}{2600} = \frac{9}{13}$$

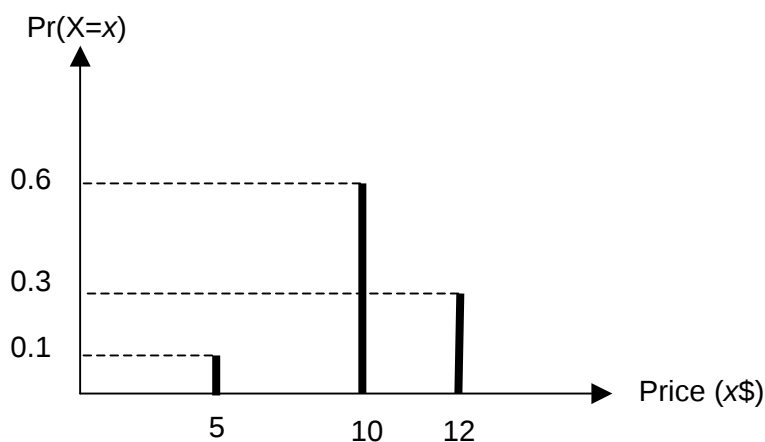
(ii) $\Pr(\text{one is a tuna}) = \Pr(\text{tuna from Boat 1 AND not from Boat 2}) \text{ OR } \Pr(\text{tuna from Boat 2 AND not from Boat 1})$

$$= \left(\frac{1800}{3000} \times \frac{1200}{2000} \right) + \left(\frac{800}{2000} \times \frac{1200}{3000} \right)$$

$$= 0.36 + 0.16 = 0.520$$

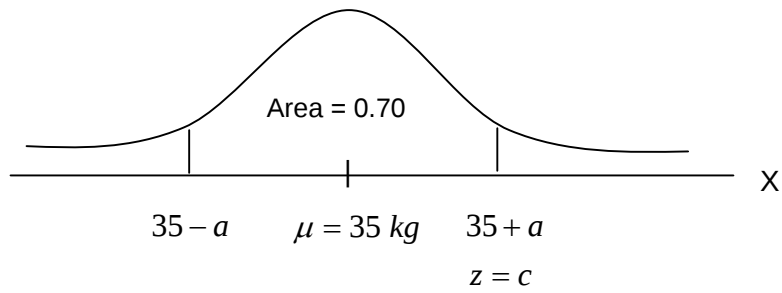
c. (i)

	Tuna	Salmon	Other
Number of fish	1800	900	300
Price per kilo (x\$)	10	12	5
Probability ($\Pr(X=x)$)	$\frac{1800}{3000} = 0.6$	$\frac{900}{3000} = 0.3$	$\frac{300}{3000} = 0.1$



(ii) $E(\$) = (10 \times 0.6) + (12 \times 0.3) + (5 \times 0.1) = \$10.10 / kg$

d. (i)



$$\Pr(X < 35 + a) = 0.85$$

$$\Pr(z < c) = 0.85$$

$$\text{invnorm}(0.85, 0, 1) = 1.0364$$

$$\therefore c = 1.0364$$

$$\frac{X - \mu}{\sigma} = 1.0364$$

$$\frac{35 + a - 35}{\sigma} = 1.0364$$

$$a = 1.0364\sigma$$

(ii) Binomial.

$X = \text{Number acceptable fish}$

$p = \text{probability acceptable} = 0.70$

$n = 5$

$$\text{Find } \Pr(X > 2) = 1 - \Pr(X = 0) - \Pr(X = 1) - \Pr(X = 2)$$

$$= 1 - \Pr(X \leq 2) = 1 - \text{binomcdf}(5, 0.7, 2) = 0.837$$

- e. (i) Let X = number of acceptable fish

$$p = p$$

$$n = 5$$

$$\Pr(X = 2) = \binom{5}{2}(p)^2(1-p)^3 = 10p^2(1-p)^3$$

$$(ii) \quad P'(p) = [10p^2 \times -3(1-p)^2] + [(1-p)^3 \times 20p]$$

$$= -30p^2(1-p)^2 + 20p(1-p)^3$$

$$= 10p(1-p)^2[-3p + 2(1-p)]$$

$$= 10p(1-p)^2[-3p + 2 - 2p]$$

$$= 10p(1-p)^2(2-5p)$$

$$\text{Let } P'(p) = 0$$

$$10p(1-p)^2(2-5p) = 0$$

$$p = 0, 1, \frac{2}{5}$$

$$\text{As } 0 < p < 1, \quad p = \frac{2}{5}.$$

QUESTION 3

- a. (i) As there is no reflection in the X axis, a represents the amplitude.

$$\text{Amplitude} = \frac{1}{2} \times \text{Distance between maximum and minimum values} = \frac{1}{2} \times 8 = 4 \text{ m}$$

- (ii) Period = 12 metres

$$\frac{2\pi}{b} = 12$$

$$b = \frac{\pi}{6}$$

(iii) $h(x) = 4 \cos\left(\frac{\pi x}{6}\right) + c$

As curve passes through the point $(0, 6)$: $6 = 4 \cos(0) + c \quad \therefore c = 2$.

- b. Find x when $h = 2(\sqrt{2} + 1)$

$$4 \cos\left(\frac{\pi x}{6}\right) + 2 = 2(\sqrt{2} + 1)$$

$$\cos\left(\frac{\pi x}{6}\right) = \frac{\sqrt{2}}{2}$$

1st Quadrant angle: $\cos^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$

Solutions to lie in Quadrants 1 and 4.

$$\left(\frac{\pi x}{6}\right) = \frac{\pi}{4} \quad \text{or} \quad \left(\frac{\pi x}{6}\right) = 2\pi - \frac{\pi}{4}$$

$$x = \frac{3}{2}, \frac{21}{2} \text{ metres}$$

c. $f(x+h) \approx f(x) + hf'(x)$

Let $x = 3$
 $x + h = 2.9$
 $\therefore h = -0.1$

$$f(2.9) \approx f(3) - 0.1f'(3)$$

$$f(x) = 4\cos\left(\frac{\pi x}{6}\right) + 2 \quad \therefore f(3) = 2$$

$$f'(x) = -\frac{2\pi}{3}\sin\left(\frac{\pi x}{6}\right) \quad f'(3) = -2.0944$$

As $f(2.9) \approx f(3) - 0.1f'(3)$
 $f(2.9) \approx 2 - 0.1(-2.0944) \approx 2.20922$

Change in height = *final height – initial height*
 $2 - 2.20922 = -0.20922 = -0.209$ metres.

i.e. the height has decreased by 0.209 metres.

- d. (i) To join functions smoothly, the gradients at the point of contact must be equal.

Ramps meet at $(6, -2)$

$$h'(x) = -\frac{2\pi}{3}\sin\left(\frac{\pi x}{6}\right)$$

$$h'(0) = -\frac{2\pi}{3}\sin(0) = 0$$

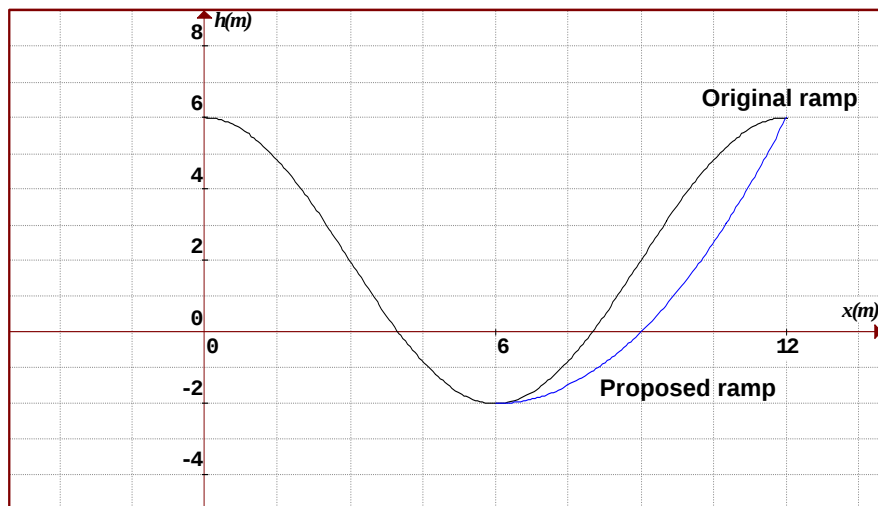
$$g(x) = \frac{2}{9}(x-6)^2 - 2$$

$$g'(x) = \frac{4}{9}(x-6)$$

$$g'(6) = \frac{4}{9}(6-6) = 0$$

As gradients are the same at the point of contact, join is smooth.

(ii)



Find points of Intersection: Coordinates can only be found using the INTERSECT function of the graphics calculator (or via iteration).

Point of Intersection is (12, 6).

$$\int_6^{12} 4 \cos\left(\frac{\pi x}{6}\right) + 2 - \left(\frac{2}{9}(x-6)^2 - 2\right) dx$$

$$\int_6^{12} 4 \cos\left(\frac{\pi x}{6}\right) + 2 - \frac{2}{9}(x-6)^2 + 2 dx$$

$$\int_6^{12} 4 \cos\left(\frac{\pi x}{6}\right) - \frac{2}{9}(x-6)^2 + 4 dx$$

$$\left[\frac{24}{\pi} \sin\left(\frac{\pi x}{6}\right) - \frac{2(x-6)^3}{27} + 4x \right]_6^{12}$$

$$= 32 - 24 = 8 \text{ units}^2$$

QUESTION 4

a. (i) $y = A \log_e(t + b)$

When $t = 0$, $y = 0$

$$A \log_e(b) = 0$$

$$\log_e(b) = 0$$

$$\therefore b = 1$$

(ii) $y = A \log_e(t + 1)$

When $t = 8$, $y = 50$

$$A \log_e(8 + 1) = 50$$

$$A \log_e(9) = 50$$

$$A = \frac{50}{\log_e 9}$$

b. (i) Find y when $t = 15$.

$$y = \left(\frac{50}{\log_e 9} \right) \log_e(t + 1)$$

$$y = \left(\frac{50}{\log_e 9} \right) \log_e(15 + 1) = \left(\frac{50}{\log_e 9} \right) \log_e(16) \text{ units}$$

(ii) $\left(\frac{50}{\log_e 9} \right) \log_e(t + 1) = \frac{1}{\log_e 9}$

$$\log_e(t + 1) = \frac{1}{50}$$

$$e^{1/50} = t + 1$$

$$t = e^{1/50} - 1 \mu \text{sec}$$

c. (i) $y = \left(\frac{50}{\log_e 9} \right) \log_e (t+1)$

$$t = \left(\frac{50}{\log_e 9} \right) \log_e (y+1)$$

$$\frac{t \log_e 9}{50} = \log_e (y+1)$$

$$e^{\frac{t}{50} \log_e 9} = y+1$$

$$y = f^{-1}(t) = e^{\frac{t}{50} \log_e 9} - 1 = e^{\log_e 9^{t/50}} - 1$$

$$y = f^{-1}(t) = 9^{t/50} - 1$$

Alternatively: $\frac{t}{50} \log_e 9 = \log_e (y+1)$

$$\log_e 9^{t/50} = \log_e (y+1)$$

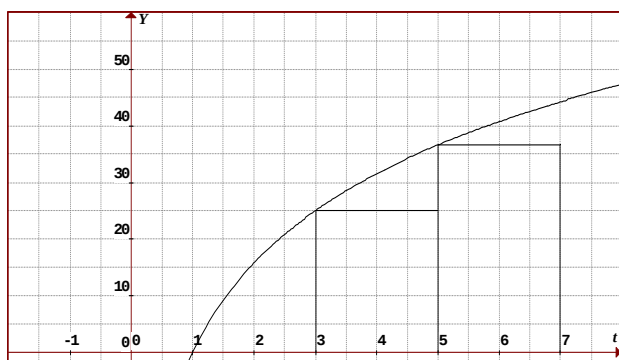
$$9^{t/50} = y+1$$

$$y = f^{-1}(t) = 9^{t/50} - 1$$

- (ii) Range $f^{-1} = \text{Domain } f(t) : (0, \infty)$. Note restriction on domain due to the fact that time cannot be negative.

- d. (i) Translation of 1 unit in the positive direction, parallel to the t axis.

(ii)



$$\text{Area} = (0 \times 2) + \left(\frac{50 \log_e 3}{\log_e 9} \times 2 \right) + \left(\frac{50 \log_e 5}{\log_e 9} \times 2 \right) = 123.259 \text{ units}^2$$

e. (i) $y = t \log_e t$

Product Rule: $\frac{dy}{dt} = \left(t \times \frac{1}{t} \right) + \log_e t = 1 + \log_e t$

(ii) $\int (1 + \log_e t) dt = t \log_e t + c$

$$\int (1) dt + \int (\log_e t) dt = t \log_e t + c$$

$$\int (\log_e t) dt = t \log_e t - \int 1 dt + c$$

$$= t \log_e t - t + d$$

(iii) Area = $\int_1^7 (A \log_e t) dt = A \int_1^7 (\log_e t) dt$

$$= A [t \log_e t - t]_1^7$$

$$= A(6.6214 - -1) = 7.6214A$$

As $A = \frac{50}{\log_e 9}$, Area = $\frac{50}{\log_e 9} \times 7.6214 = 173.432 \text{ units}^2$