



Australian Education Academy 2005

MATHEMATICAL METHODS

WRITTEN EXAMINATION 2 - SOLUTIONS

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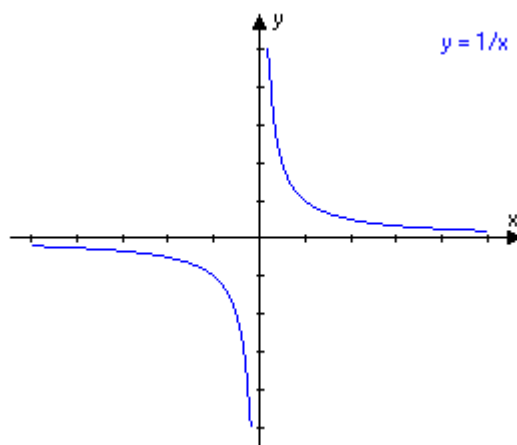
Written Examination 2: Analysis Task Suggested Solutions

Question 1

a. The function $f(x) = \frac{24}{x+3} - 6$ does not allow $x = -3$. Thus the largest domain is $\mathbb{R} \setminus \{-3\}$.

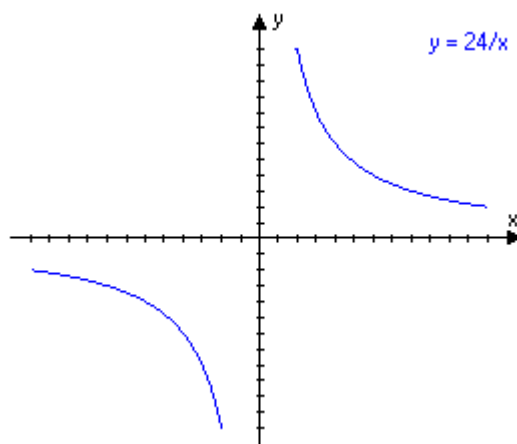
1 mark

b. Applying following transformations for the graph $y = \frac{1}{x}$ we can obtain the graph of $f(x) = \frac{24}{x+3} - 6$



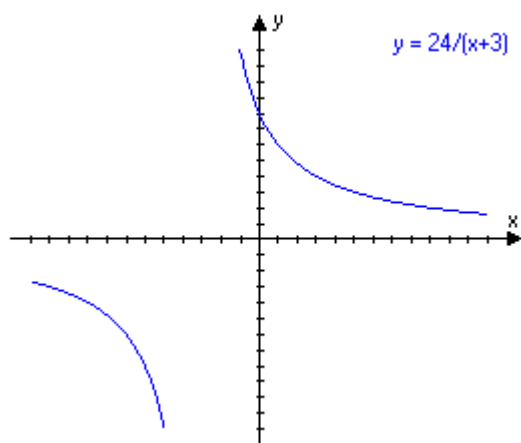
i. First, dilate by a factor of 24 in a vertical direction away from the x -axis.

1 mark



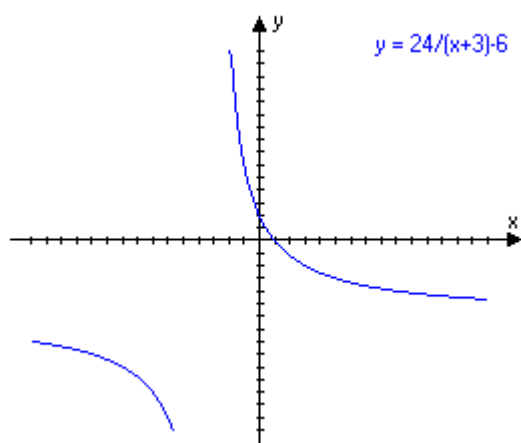
- ii. Second translate the graph to the left (in a negative direction) along the x -axis by 3 units.

1 mark



- iii. Third, translate the graph down (i.e. in a negative y -direction) by 6 units.

1 mark



- c. When $x = 0 \Rightarrow f(0) = \frac{24}{3} - 6 = 2$ and when $f(x) = 0 \Rightarrow \frac{24}{x+3} - 6 = 0 \Rightarrow x = 1$
 $\therefore$ Graph passes through the point $(0, 2)$ and $(1, 0)$.

1 mark

d. i.

$$f(x) = \frac{24}{x+3} - 6$$

$$g: (-3, \infty) \rightarrow \mathbb{R} \text{ such that } g(x) = f(x)$$

$\therefore$ domain of $g(x)$ is $(-3, \infty)$ and

range of $g(x)$ is $(-6, \infty)$

1 mark

ii.

If $y = g(x) = \frac{24}{x+3} - 6$ then inverse function of $g(x)$ is given by

$$x = \frac{24}{y+3} - 6 \Rightarrow y+3 = \frac{24}{x+6}$$

$$\therefore g^{-1}(x) = y = \frac{24}{x+6} - 3$$

2 marks

iii When $x = 0 \Rightarrow g^{-1}(0) = \frac{24}{6} - 3 = 1$ and when

$$f(x) = 0 \Rightarrow \frac{24}{x+6} - 3 = 0 \Rightarrow x = 2$$

$\therefore$ Graph passes through the point $(0, 1)$ and $(2, 0)$.

1 mark

iv. Domain of g^{-1} is the range of g .

$\therefore$ the domain of g^{-1} is $(-6, \infty)$.

1 mark

v. Solutions of $g(x) = x$ is given by

$$\frac{24}{x+3} - 6 = x$$

$$\therefore (x+3)(x+6) = 24$$

$$\Rightarrow x^2 + 9x - 6 = 0$$

$$\therefore x = \frac{-9 \pm \sqrt{105}}{2} \Rightarrow x = -9.62 \text{ or } 0.62$$

1 mark

$\therefore$ Solution in the domain of g is 0.62

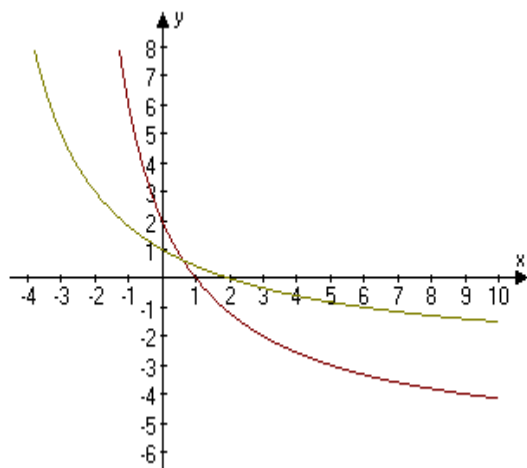
1

mark

$g(x)$, g^{-1} take the same value on the line $y = x$. Hence $g(x) = g^{-1}(x) = x$ is satisfied when $x = 0.62$.

1 mark

vi.



2 marks

Question 2

$$P(t) = 1500e^{-0.05t}$$

a. When $t = 0$, $P(0) = 1500$

1 mark

b. When $P(t) = 750$.

$$750 = 1500e^{-0.05t}$$

$$\therefore e^{0.05t} = 2$$

$$0.05t = \log_e 2$$

$$t = \frac{\log_e 2}{0.05} = 13.86 \approx 14 \text{ weeks}$$

1 mark

c. $\frac{dP}{dt} = 1500 \times (-0.05)e^{-0.05t} = -75e^{-0.05t}$

1 mark

d. i. When $t = 2$

$$\left(\frac{dP}{dt}\right)_{t=2} = -75e^{-0.1} = -67.86$$

$$\approx -68 \text{ rabbits/week}$$

1 mark

ii. When $t = 10$

$$\begin{aligned}\left(\frac{dP}{dt}\right)_{t=10} &= -75e^{-0.5} = -45.49 \\ &\approx -46 \text{ rabbits/week}\end{aligned}$$

1 mark

e. Domain is $[16, \infty)$.

1 mark

f. $P(t) = P_0 + 12(t - 16)\log_e(2t - 31)$

$$\begin{aligned}\text{when } t = 16 \quad P(16) &= P_0 = 1500e^{-0.05 \times 16} (\because \text{Part (a)}) \\ &= 673.99 \approx 674 \\ \therefore P(t) &= 674 + 12(t - 16)\log_e(2t - 31)\end{aligned}$$

1 mark

g. When $t = 32$

$$\begin{aligned}P(32) &= 674 + 12(16)\log_e(33) \\ &= 1345.33 \approx 1346\end{aligned}$$

1 mark

h. $\frac{dP}{dt} = \frac{24(t - 16)}{(2t - 31)} + 12\log_e(2t - 31)$

2 marks

i. i. When $t = 20$

$$\begin{aligned}\left(\frac{dP}{dt}\right)_{t=20} &= \frac{24 \times 4}{9} + 12\log_e 9 \\ &= \frac{32}{3} + 12 \times 2.1972 \\ &= 10.6667 + 26.3667 = 37.0334 \\ &\approx 37 \text{ rabbits/week}\end{aligned}$$

2 marks

$$\begin{aligned}\text{ii. When } t = 32 \quad \left(\frac{dP}{dt}\right)_{t=32} &= \frac{24 \times 16}{33} + 12\log_e 33 &= \frac{128}{11} + 12 \times 3.4965 \\ &= 11.6364 + 41.9580 = 53.5944 \\ &\approx 54 \text{ rabbits/week}\end{aligned}$$

2 marks

- j. Size of the population at time t is given by
 $P(t) = 674 + 12(t-16)\log_e(2t-31)$

Suppose that population takes t weeks to get back to its original number.

Initial size of the population is 1500.

$\therefore$ when $P(t) = 1500$ we have

$$1500 = 674 + 12(t-16)\log_e(2t-31)$$

$$\Rightarrow 413 = 6(t-16)\log_e(2t-31)$$

$\therefore t$ is a solution of the above equation.

1 mark

Question 3

- a. i.

This shape can be considered to be a sphere plus a cylinder. The volume is

$$V = V_{\text{sphere}} + V_{\text{cylinder}}$$

$$= \frac{4\pi r^3}{3} + \pi r^2 h = \frac{\pi r^2}{3}(4r + 3h)$$

2 marks

ii Surface area of this capsule can be considered to be surface area of a sphere plus surface area of the curved surface of a cylinder.

$$S = 4\pi r^2 + 2\pi r h$$

$$= 2\pi r(2r + h)$$

2 marks

- b. i. Given that $V = \pi a^3 \text{ cm}$

$$\therefore \pi a^3 = \frac{\pi r^2}{3}(4r + 3h)$$

$$\Rightarrow h = \frac{a^3}{r^2} - \frac{4r}{3}$$

1 mark

ii. To find the domain for h consider the above expression. we know that

$$h \geq 0 \Rightarrow \frac{a^3}{r^2} - \frac{4r}{3} \geq 0$$

$$\Rightarrow \frac{3a^3}{4} \geq r^3$$

$$\therefore \left(\frac{3}{4}\right)^{\frac{1}{3}} a \geq r > 0$$

2 marks

iii

$$\begin{aligned} S &= 2\pi r \left(2r + \frac{a^3}{r^2} - \frac{4r}{3} \right) \\ &= 2\pi r \left(\frac{a^3}{r^2} + \frac{2r}{3} \right) = 2\pi r^2 \left(\frac{a^3}{r^3} + \frac{2}{3} \right) \end{aligned}$$

1 mark

c. i.

$$\frac{dS}{dr} = -\frac{2\pi a^3}{r^2} + \frac{8\pi r}{3}$$

Turning points are given by $\frac{dS}{dr} = 0$

$$-\frac{2\pi a^3}{r^2} + \frac{8\pi r}{3} = 0$$

$$\therefore \Rightarrow r^3 = \frac{3a^3}{4}$$

$$\Rightarrow r = \left(\frac{3}{4} \right)^{\frac{1}{3}} a$$

2 marks

ii.

$$\text{when } r = \frac{a}{2}, \quad \frac{dS}{dr} = -4\pi a + \frac{4\pi a}{3} < 0$$

$$\therefore \left(\frac{dS}{dr} \right)_{(r < \left(\frac{3}{4} \right)^{\frac{1}{3}} a)} < 0 \text{ and}$$

$$\text{when } r = a, \quad \frac{dS}{dr} = -2\pi a + \frac{8\pi a}{3} > 0$$

$$\therefore \left(\frac{dS}{dr} \right)_{(r > \left(\frac{3}{4} \right)^{\frac{1}{3}} a)} > 0$$

2 marks

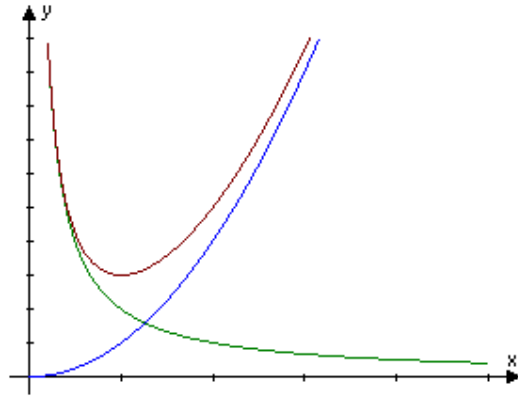
Hence $r = \left(\frac{3}{4} \right)^{\frac{1}{3}} a$ is a minimum point

Corresponding value of S is given by

$$S_{\min} = 4\pi a^2 \left(\frac{3}{4} \right)^{\frac{2}{3}}$$

$$\text{iii. } S = \frac{2\pi a^3}{r} + \frac{4\pi r^2}{3}$$

$$y = \frac{2\pi a^3}{r} + \frac{4\pi r^2}{3}$$



3 marks

Question 4

- a. Let p the probability that he gets a particular answer correct. Then $p = \frac{1}{5}$.

1 mark

- b. Binomial distribution with $n = 25$ and $p = \frac{1}{5}$.

1 mark

- c. Random variable X denote the number of correct answers. Then the distribution of X

is $P(X = x) = \binom{25}{x} \left(\frac{1}{5}\right)^x \left(\frac{4}{5}\right)^{25-x}$ where

$$x = 0, 1, 2, \dots, 25$$

1 mark

- d. The mean is given by

$$E(X) = np$$

$$= 25 \times \frac{1}{5} = 5$$

1 mark

Variance of X is given by

$$\begin{aligned} \text{Var}(X) &= npq \\ &= 25 \times \frac{1}{5} \times \frac{4}{5} \\ &= 4 \end{aligned}$$

1 mark

The standard deviation is given by

$$\begin{aligned} \sigma &= \sqrt{\text{Var}(X)} \\ \therefore &= \sqrt{4} = 2 \end{aligned}$$

1 mark

e.

$$\begin{aligned} P(X=0) &= \binom{25}{0} \left(\frac{1}{5}\right)^0 \left(\frac{4}{5}\right)^{25} \\ &= 0.0038 \end{aligned}$$

1 mark

$$P(X > M) = 0.01 \quad 1 \text{ mark}$$

1 mark

$$P\left(\frac{X-\mu}{\sigma} > \frac{M-\mu}{\sigma}\right) = 0.01$$

$$P\left(Z > \frac{M-5}{2}\right) = 0.01 \quad 1 \text{ mark}$$

1 mark

$$\Rightarrow \frac{M-5}{2} = 2.33$$

$$\therefore M = 9.66 \approx 10 \quad 1 \text{ mark}$$

1 mark