

2004 Mathematical Methods Written Examination 2

Suggested Solutions

Question 1

a. $f(x) = (x-1)^2(x-2) + 1$

Using the Product Rule with the first function being $(x-1)^2$ and the second $(x-2)$, we see that

$$\begin{aligned} f'(x) &= 2(x-1)(x-2) + (x-1)^2 \\ &= (x-1)[2(x-2) + (x-1)] \\ &= (x-1)[2x-4 + x-1] \\ &= (x-1)(3x-5) \end{aligned}$$

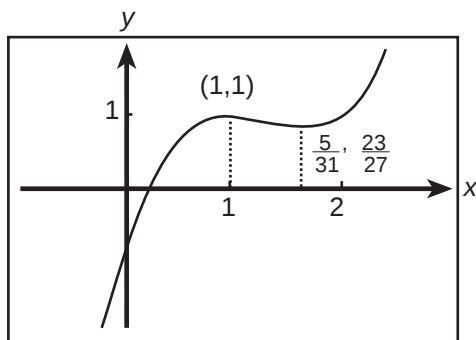
Therefore $u = 3$ and $v = -5$.

b. Now $f'(x) = 0$ for the turning points and so $x-1 = 0$ or $3x-5 = 0$.

If $x = 1$ then $y = 1$ so the point $(1, 1)$ is a turning point. Therefore $a = 1$.

If $x = \frac{5}{3}$ then $y = (\frac{5}{3}-1)^2(\frac{5}{3}-2) + 1 = \frac{23}{27}$. Therefore $b = \frac{5}{3}$.

c.



If $p = \frac{23}{27}$ which is the y -value of the turning point of $f(x)$ then the graph of $f(x) = p$ would be lowered and have the minimum value on the x -axis.

Therefore one set of solutions is $x < \frac{23}{27}$.

However, the other turning point has a y -value of 1.

If $p > 1$, this also gives one solution only.

d. $f(x) - 1 = (x-1)^2(x-2) + 1 - 1$
 $= (x-1)^2(x-2)$

The region bounded by the x -axis and $f(x) - 1$ is given by $\int_1^2 (f(x) - 1) dx$

$$\begin{aligned} &= \int_1^2 x^3 - 4x^2 + 5x - 2 dx \\ &= \left[\frac{1}{4}x^4 - \frac{4}{3}x^3 + \frac{5}{2}x^2 - 2x \right]_1^2 \\ &= \left(\frac{16}{4} - \frac{32}{3} + 10 - 4 \right) - \left(\frac{1}{4} - \frac{4}{3} + \frac{5}{2} - 2 \right) \\ &= \frac{1}{12} \end{aligned}$$

The area is therefore $\frac{1}{12}$ or 0.083 correct to 3 decimal places.

e. i. Dilation by a factor of 2 from the y -axis followed by a translation 1 unit down.

ii. $f(\frac{x}{2}) - 1 = (\frac{x}{2}-1)^2(\frac{x}{2}-2)$ and so the x -intercepts are $x = 2$ and $x = 4$.

iii. 0.17

f. $f(x+h) - 1 = (x+h-1)^2(x+h-2) + 1 - 1$
 $= (x+h-1)^2(x+h-2)$

If one positive solution only is required for this equation then imagine that the first factor is x^2 , i.e. that $h = 1$. There would only be one positive answer, i.e. $x = 2$.

If the graph was translated 2 units to the left then the factors would be $(x+1)^2x$, which has no positive solutions.

Hence the values that h can take are $1 \leq h < 2$.

Question 2

- a. i. The sum of the proportions must be 1 for a probability distribution function.
Therefore $4k^2 + 4k + 5k^2 + 2k + k^2 + k + 2k = 1$
Collecting like terms gives $10k^2 + 9k = 1$ which is the same as $10k^2 + 9k - 1 = 0$.
- ii. Factorising this equation gives $(10k - 1)(k + 1) = 0$.
Hence $k = 0.1$; it is impossible to have a negative proportion and so $k + 1 = 0$ is discarded.
- b. i. This is a Binomial Distribution: $(0.14 + 0.86)^9$
The Expected Value of the Binomial is np which is $0.14 \times 9 = 1.26$
- ii. ${}^9C_2 (0.14)^2 (0.86)^7 = 0.245$ to 3 decimal places.
- iii. $(0.86)^n \leq 0.09$
Take logarithm of both sides gives: $\log_e (0.86)^n \leq \log_e 0.09$
 $n \log_e (0.86) \leq \log_e 0.09$

Dividing both sides of this inequality by $\log_e (0.86)$ which is a negative number

$$n \geq \frac{\log_e 0.09}{\log_e 0.86} \text{ so } n \geq 15.96$$

Answer: $n = 16$

- c. Find the Z-score associated with an area of 0.078 to the left of it.
 $\text{Invnorm}(0.078, 0, 1) = -1.419$

$$\text{Now } Z = \frac{X - \mu}{\sigma}$$

$$-1.419 = \frac{100 - 125}{\sigma} \text{ and so } -1.419\sigma = -25$$

Therefore $\sigma = 17.6$

To the nearest gram $\sigma = 18$.

- d. Hypergeometric with $N = 15$, $n = 3$, $D = 4$
Probability of at least one defective = $1 - \text{Pr}(\text{no defectives})$
 $\text{Pr}(0 \text{ defectives}) = \frac{{}^4C_0 \times {}^{11}C_3}{{}^{15}C_3}$
 $= 0.3626$

Probability of at least one defective = $1 - 0.3626 = 0.637$ to 3 decimal places.

- e. Binomial distribution $(0.08 + 0.92)^{12}$
Probability of fewer than 2 defectives = $\text{Pr}(0 \text{ defectives}) + \text{Pr}(1 \text{ defective})$
 $= {}^{12}C_1 (0.08)^1 (0.92)^{11} + 0.92^{12}$
 $= 0.751$ correct to 3 decimal places.

Question 3

- a. i. $a + bt \geq 0$ and so $bt \geq -a$.
Dividing both sides by b which is a negative number gives $t \leq \frac{-a}{b}$
- ii. At $t = 0$ we know that $P(0) = P_0$ and so $P_0 = P_0 e^{G(0)}$. This means that $e^{G(0)} = 1$.
But $e^0 = 1$ and so, equating indices, $G(0) = 0$.
- iii. If $G'(t) = a + bt$ then anti-differentiating $G(t) = at + \frac{1}{2}bt^2 + c$ where c is a constant.
But when $t = 0$, $G(0) = 0$ and so $c = 0$.
Therefore $G(t) = at + \frac{1}{2}bt^2$
- b. i. Now $P_0 = 3$ and $G(t) = 0.02t + 0.5 \times -0.0002t^2$ and so $P(t) = 3e^{0.02t - 0.0001t^2}$
- ii. Since $t \leq \frac{-a}{b}$ then $t \leq \frac{-0.02}{-0.0002}$ which is 100 years.

$$\text{iii. } \frac{dP}{dt} = P_0 G'(t) eG(t)$$

$$= 3(0.02 - 0.0002t)e^{0.02t - 0.0001t^2}$$

$$\text{iv. } \text{When } t = 20, \frac{dP}{dt} = 3(0.02 - 0.0002 \times 20)e^{0.02 \times 20 - 0.0001 \times 400} = 0.069$$

$$\text{c. i. } \frac{P}{3} = e^{0.01t} \text{ and so } \log_e \frac{P}{3} = 0.01t.$$

$$\text{Therefore } t = 100 \log_e \frac{P}{3}.$$

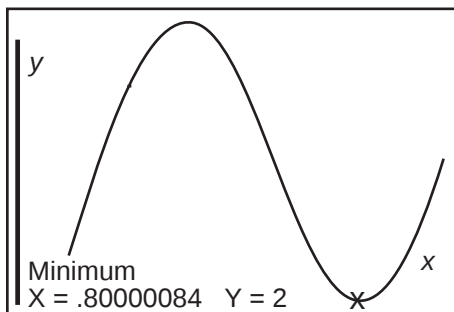
$$\text{ii. } \text{When the population is doubled, } P = 6. \text{ Therefore } t = 100 \log_e 2 = 69.31$$

Answer: During the year 2059.

Question 4

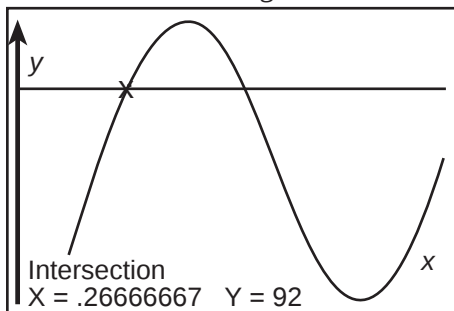
- a. 122 m
b. 2 m

$$\text{c. } \text{The period is } \frac{2\pi}{2.5\pi} \text{ which is 0.8 hours. It returns to the lowest point at 1:48 pm.}$$



$$\begin{aligned} \text{d. i. } 92 &= 62 + 60 \sin \frac{(5t-1)\pi}{2} \\ 30 &= 60 \sin \frac{(5t-1)\pi}{2} \\ 0.5 &= \sin \frac{(5t-1)\pi}{2} \text{ and so } \frac{(5t-1)\pi}{2} = \frac{\pi}{6} \\ 5t - 1 &= \frac{1}{3} \text{ and so } t = \frac{4}{15} \end{aligned}$$

It first reaches a height of 92 m at 1:16 pm.



$$\text{ii. } \text{The next time it reaches a height of 92 m is when } \frac{(5t-1)\pi}{2} = \frac{5\pi}{6}$$

$$\text{Therefore } 5t - 1 = \frac{5}{3} \text{ and so } t = \frac{8}{15}$$

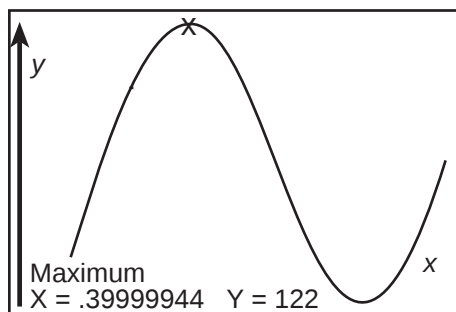
It will be at least 92 m above the ground between $t = \frac{4}{15}$ and $t = \frac{8}{15}$, that is for $\frac{4}{15}$ of an hour which is 16 minutes.

$$\begin{aligned} \text{e. i. } h'(t) &= 60 \times \cos \frac{(5t-1)\pi}{2} \\ &= 150\pi \cos \frac{(5t-1)\pi}{2} \end{aligned}$$

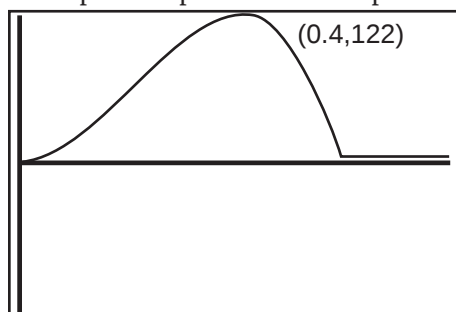
$$\text{ii. } \text{When } t = 1, h'(1) = 150\pi \cos 2\pi = 150\pi$$

To one decimal place this is 471.2 m/h.

- f. i. The co-ordinates of the local maximum are (0.4, 122).



- ii. The spider's path is made up of three parts:



- For the domain $[0, 0.4]$ it follows the path of the ferris wheel.
 - From $(0.4, 0.603]$ the spider follows the addition of ordinates path of the equations for $h(t)$ and $y = -300(t - 0.4)$.
 - It then is at ground level and so follows $y = 0$ for $t > 0.603$.
- iii. The spider reached the ground in $0.60259334 - 0.4 = 0.20259334$ hours, that is 12 minutes (to the nearest minute).