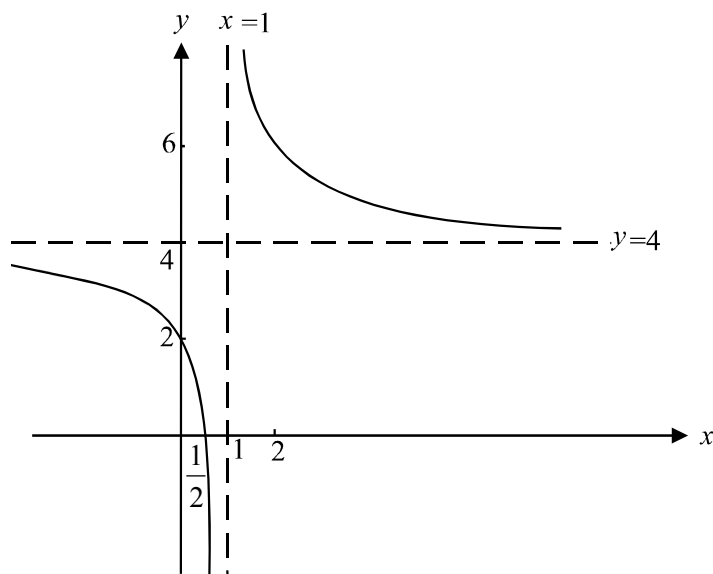


Question 1

a. i.



x-intercepts

$$\begin{aligned} y = 0, \quad 0 &= \frac{-2}{1-x} + 4 \\ -4(1-x) &= -2 \\ -4 + 4x &= -2 \\ 4x &= 2 \\ x &= \frac{1}{2} \end{aligned}$$

y-intercepts

$$\begin{aligned} x = 0, \quad y &= \frac{-2}{1} + 4 \\ &= 2 \end{aligned}$$

(1 mark) correct shape
including asymptotes
(1 mark) intercepts

ii. $d = R \setminus \{1\}$
 $r = R \setminus \{4\}$

(1 mark)

b. i. $y = \frac{-2}{1-x} + 4$
 $= -2(1-x)^{-1} + 4$
 $\frac{dy}{dx} = 2(1-x)^{-2} \times -1$
 $= \frac{-2}{(1-x)^2}$

(1 mark)

- ii. From the graph in part a. i., there are no local minima or maxima and no stationary points of inflection, hence there is no solution to the equation

$$\frac{dy}{dx} = 0.$$

(1 mark)

- c. i. f^{-1} exists because we see from the graph in a. i., that f is a 1:1 function.

(1 mark)

ii. Now, $f(x) = \frac{-2}{1-x} + 4$

Let $y = \frac{-2}{1-x} + 4$

Swap x and y , so $x = \frac{-2}{1-y} + 4$

Rearrange $x - 4 = \frac{-2}{1-y}$

$$(x - 4)(1 - y) = -2$$

$$1 - y = \frac{-2}{x - 4}$$

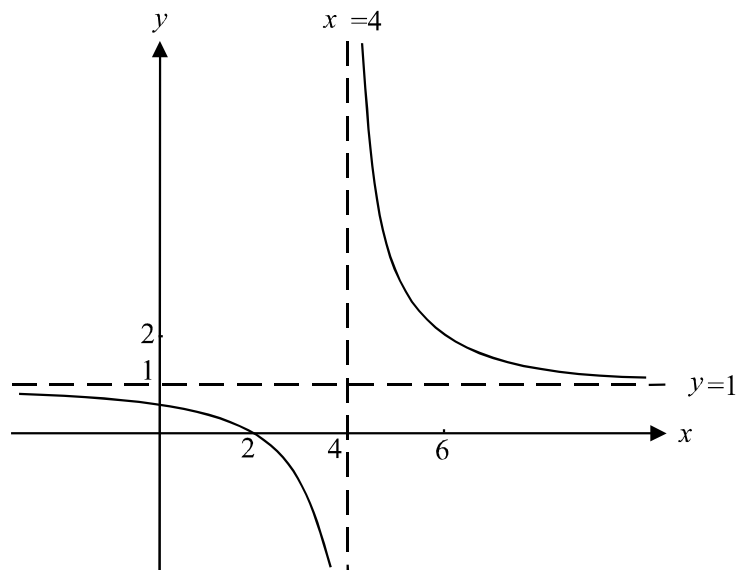
$$-y = \frac{-2}{x - 4} - 1$$

$$y = \frac{2}{x - 4} + 1$$

So $f^{-1}(x) = \frac{2}{x - 4} + 1$ as required.

(1 mark)

- iii. Use the graph you have drawn in part a. i. and reflect it in the line $y = x$.



The asymptotes for the graph of $y = f(x)$ are $x = 1$ and $y = 4$, therefore for the graph of $y = f^{-1}(x)$, the asymptotes are $y = 1$ and $x = 4$.

On the graph of $y = f(x)$, the graph passed through the points $\left(\frac{1}{2}, 0\right)$, $(0, 2)$ and $(2, 6)$.

On the graph of $y = f^{-1}(x)$, the graph passes through the points $\left(0, \frac{1}{2}\right)$, $(2, 0)$ and $(6, 2)$.

(1 mark) shape
including asymptotes
(1 mark) intercepts

d. To Show. $\int_0^{\frac{1}{2}} f(x) dx = \int_0^2 f^{-1}(x) dx$

$$\text{L.S.} = \int_0^{\frac{1}{2}} f(x) dx$$

$$= \int_0^{\frac{1}{2}} \left(\frac{-2}{1-x} + 4 \right) dx$$

$$= \int_0^{\frac{1}{2}} \left\{ 2 \left(\frac{-1}{1-x} \right) + 4 \right\} dx$$

$$= \left[2 \log_e(1-x) + 4x \right]_0^{\frac{1}{2}} \quad (1 \text{ mark})$$

$$= \left\{ \left(2 \log_e \left(\frac{1}{2} \right) + 2 - (2 \log_e(1) + 0) \right) \right\} \quad (1 \text{ mark})$$

$$= 2 \log_e \left(\frac{1}{2} \right) + 2 \quad \text{since } \log_e(1) = 0$$

$$\text{R.S.} = \int_0^2 f^{-1}(x) dx$$

$$= \int_0^2 \left(\frac{2}{x-4} + 1 \right) dx$$

$$= \int_0^2 \left(2 \left(\frac{1}{x-4} \right) + 1 \right) dx \quad (1 \text{ mark})$$

$$= \left[2 \log_e(x-4) + x \right]_0^2$$

$$= \left[\log_e(x-4)^2 + x \right]_0^2$$

$$= \{ (\log_e(4) + 2) - (\log_e(16) + 0) \} \quad (1 \text{ mark})$$

$$= \log_e(4) - \log_e(16) + 2$$

$$= \log_e \left(\frac{4}{16} \right) + 2$$

$$= \log_e \left(\frac{1}{4} \right) + 2$$

$$= \log_e \left(\frac{1}{2} \right)^2 + 2$$

$$= 2 \log_e \left(\frac{1}{2} \right) + 2$$

$$= \text{L.S.}$$

Have shown

(log laws)

$n \log_e(m)$

$= \log_e(m^n)$

(1 mark)

e. Method 1

Use your graphics calculator to graph each of the functions $y = f^{-1}(x)$ and $y = f(x)$.
The values of x for which $f(x) = f^{-1}(x)$ occur at $x = 0.44$ and $x = 4.56$ where each number is correct to 2 decimal places.

(2 marks)**Method 2**

As the points of intersection must lie on the line $y = x$, find the points of intersection

between $y = \frac{2}{x-4} + 1$ and $y = x$.

$$\text{So, } x = \frac{2}{x-4} + 1 \quad (1 \text{ mark})$$

$$(x-1)(x-4) = 2$$

$$x^2 - 5x + 4 = 2$$

$$x^2 - 5x + 2 = 0$$

$$x = \frac{5 \pm \sqrt{25 - 4 \times 1 \times 2}}{2}$$

$$= \frac{5 \pm \sqrt{17}}{2}$$

The values of x for which $f(x) = f^{-1}(x)$ are $x = \frac{5 \pm \sqrt{17}}{2}$. **(1 mark)**

Method 3

$$f(x) = f^{-1}(x)$$

$$\text{So, } \frac{-2}{1-x} + 4 = \frac{2}{x-4} + 1$$

$$\frac{-2}{1-x} - \frac{2}{x-4} = -3$$

$$\frac{-2(x-4) - 2(1-x)}{(1-x)(x-4)} = -3$$

$$-2x + 8 - 2 + 2x = -3(x-4-x^2+4x)$$

$$6 = -3(-x^2 + 5x - 4)$$

$$-2 = -x^2 + 5x - 4$$

$$x^2 - 5x + 2 = 0$$

(1 mark)

$$x = \frac{5 \pm \sqrt{25 - 4 \times 1 \times 2}}{2}$$

$$x = \frac{5 \pm \sqrt{17}}{2}$$

The values of x for which $f(x) = f^{-1}(x)$ are $x = \frac{5 \pm \sqrt{17}}{2}$.

(1 mark)**Total 16 marks**

Question 2

- a. The probability that a randomly selected female donor at the facility will have a haemoglobin level that is considered high is 0.1. (1 mark)

- b. This is a binomial distribution because each time we select one of the 10, the probability that they have a level that is considered high is constant i.e. 0.1.

$$\begin{aligned}\Pr(x \geq 2) &= 1 - \Pr(x < 2) && \text{(1 mark)} \\ &= 1 - \{\Pr(x = 0) + \Pr(x = 1)\} \\ &= 1 - \left({}^{10}C_0 (0.1)^0 (0.9)^{10} + {}^{10}C_1 (0.1)^1 (0.9)^9 \right) \\ &= 0.2639 \quad (\text{to 4 decimal places})\end{aligned}$$

(1 mark)

- c. We require x such that

$$\begin{aligned}\Pr(X > x) &> 0.1 \\ \text{So } \Pr(X < x) &< 0.9 \\ \text{Now } \Pr(z < 1.2815) &< 0.9 \\ (\text{by reading the table inside out}) \\ \text{So } 1.2815 &= \frac{x - 140}{19.5}\end{aligned}$$

$$= 164.98925... \quad \text{(1 mark)}$$

$$= 165 \text{ to the nearest whole number}$$

(1 mark)

- d.

$$\begin{aligned}\Pr(X < 120) &= \Pr(z < -1.025641) && z = \frac{120 - 140}{19.5} \\ &= \Pr(z > 1.025641) && = -1.0256 \\ &= 1 - \Pr(z < 1.025641) && \text{(1 mark)} \\ &= 1 - 0.8474 \\ &= 0.1526\end{aligned}$$

So 15% (to the nearest whole percent) of female donors at the facility are not allowed to donate because their haemoglobin level is too low.

(1 mark)

- e. There are 10% of female donors who have a haemoglobin level that is considered high and from part d. there are 15% of female donors who have haemoglobin levels that are too low to donate. So 75% of female donors don't suffer from a haemoglobin level that is considered high or is too low.

(1 mark)

The expected number of female donors in a day in this category is 225.

(1 mark)

- f. This is a hypergeometric distribution because there is no replacement.

Now, $N = 12$, $D = 10$, $n = 3$ and $x = 3$.

(1 mark)

$$\begin{aligned}\Pr(X = 3) &= \frac{{}^{10}C_3 {}^2C_0}{{}^{12}C_3} && \text{(1 mark)} \\ &= 0.54 && \text{(1 mark)}\end{aligned}$$

Total 12 marks

Question 3

a. $p(x) = \cos(\pi x) \quad x \in [0, 2]$

The amplitude of this function is 1. The normal height of the water is 1 metre.

(1 mark)

b.

i.
$$\begin{aligned} \text{Area} &= - \int_{0.5}^{1.5} \cos(\pi x) dx && \textbf{(1 mark)} \\ &= \frac{-1}{\pi} [\sin(\pi x)]_{0.5}^{1.5} \\ &= \frac{-1}{\pi} (\sin(1.5\pi) - \sin(0.5\pi)) \\ &= 0.6366 \text{ (to 4 decimal places)} && \textbf{(1 mark)} \end{aligned}$$

ii. Hence the volume of water in a 10 m length of fibreglass channel is

$$0.6366 \text{ m}^2 \times 10 \text{ m}$$

$$= 6.366 \text{ m}^3$$

$$= 6.366 \times 1000000 \text{ cm}^3$$

$$= 6.366 \times 10^6 \text{ mL}$$

$$= 6366 \times 10^3 \text{ mL}$$

$$= 6366 \text{ L}$$

(1 mark)

c. i. $p(x) = \cos(\pi x)$

$$\frac{dp}{dx} = -\pi \sin(\pi x)$$

(1 mark)

ii. The maximum value of $\sin(\pi x)$ is 1 and the minimum value is -1 .

Therefore the maximum value of $-\pi \sin(\pi x)$ is π and the minimum value is $-\pi$.

(2 marks)

Note that the question did not ask for the maximum and minimum points on the fibreglass channel i.e. we were not looking to solve $\frac{dp}{dx} = 0$.

iii. When $\frac{dp}{dx} = \pi$, we have

$$\pi = -\pi \sin(\pi x), \quad x \in [0, 2]$$

So, $\sin(\pi x) = -1$

$$\pi x = \frac{3\pi}{2}$$

$$x = \frac{3}{2}$$

(1 mark)

When $\frac{dp}{dx} = -\pi$, we have

$$-\pi = -\pi \sin(\pi x)$$

$$\text{So, } \sin(\pi x) = 1$$

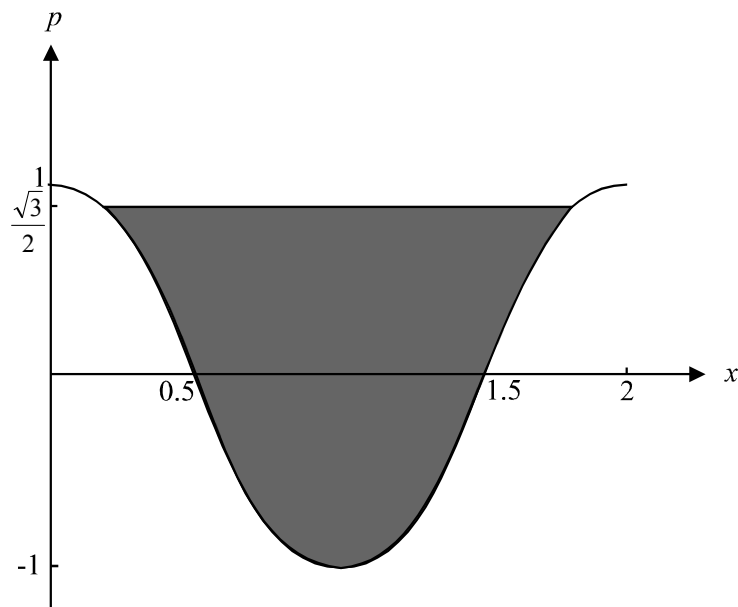
$$\pi x = \frac{\pi}{2}$$

$$x = \frac{1}{2}$$

(1 mark)

So half a metre from either side of the fibreglass channel, the maximum and minimum gradients occur.

d. Sketch a graph.



We need to find the area shaded in the diagram above.

Firstly, find the two values of x where

$$p(x) = \frac{\sqrt{3}}{2}, \quad x \in [0, 2]$$

$$\cos(\pi x) = \frac{\sqrt{3}}{2}$$

$$\pi x = \frac{\pi}{6}, \frac{11\pi}{6}$$

$$\text{So, } x = \frac{1}{6}, \frac{11}{6}$$

$$\begin{aligned}
 \text{So area required} &= \int_{\frac{1}{6}}^{\frac{11}{6}} \left(\frac{\sqrt{3}}{2} - \cos(\pi x) \right) dx && \text{(1 mark) correct integrand} \\
 &= \left[\frac{\sqrt{3}}{2} x - \frac{1}{\pi} \sin(\pi x) \right]_{\frac{1}{6}}^{\frac{11}{6}} && \text{(1 mark) correct terminals} \\
 &= \left\{ \left(\frac{\sqrt{3}}{2} \times \frac{11}{6} - \frac{1}{\pi} \sin\left(\frac{11\pi}{6}\right) \right) - \left(\frac{\sqrt{3}}{12} - \frac{1}{\pi} \sin\left(\frac{\pi}{6}\right) \right) \right\} \\
 &= \frac{11\sqrt{3}}{12} + \frac{1}{2\pi} - \frac{\sqrt{3}}{12} + \frac{1}{2\pi} \\
 &= \frac{5\sqrt{3}}{6} + \frac{1}{\pi} \text{ m}^2
 \end{aligned}$$

(1 mark)

Total 13 marks**Question 4**

- a. Now, $V(0) = 10$, since the piece of machinery normally has 10 litres of coolant in it.

$$\text{So } ke^0 + ke^0(0-1)^2 = 10$$

$$k + k = 10 \quad \text{since } e^0 = 1$$

$$k = 5$$

(1 mark)

- b.

$$\begin{aligned}
 V(t) &= 5e^{-t} + 5e^{-t}(t-1)^2 \\
 &= 5e^{-t}(1 + (t-1)^2)
 \end{aligned}$$

As $t \rightarrow \infty$, $e^t \rightarrow \infty$ and hence $\frac{1}{e^t} \rightarrow 0$. Since $e^{-t} = \frac{1}{e^t}$, $e^{-t} \rightarrow 0$.

So $5e^{-t}(1 + (t-1)^2) \rightarrow 0$ since it doesn't matter how large $(1 + (t-1)^2)$ becomes if it is being multiplied by a value close to zero then the value of the product of the two approaches zero also. So $V(t) \rightarrow 0$

(1 mark)

- c. We need to solve $V(t) = 5$

$$\text{So } 5e^{-t} + 5e^{-t}(t-1)^2 = 5.$$

Graph the function $y = 5e^{-t} + 5e^{-t}(t-1)^2 - 5$ on your graphics calculator and find the value of t when $y = 0$.

Alternatively, find the t -coordinate where the functions $y = 5e^{-t} + 5e^{-t}(t-1)^2$ and $y = 5$ intersect.

(1 mark)

So, $t = 0.35$ minutes (correct to 2 decimal places)

(1 mark)

d. Rate of change in first minute

$$\begin{aligned}
 &= \frac{V(1) - V(0)}{1 - 0} \\
 &= \frac{5e^{-1} + 5e^{-1}(1-1)^2 - \{5e^0 + 5e^0(0-1)^2\}}{1} \\
 &= 5e^{-1} - 10 \\
 &= -8.1606 \quad (\text{to 4 decimal places})
 \end{aligned}$$

Rate of change in second minute

$$\begin{aligned}
 &= \frac{V(2) - V(1)}{2 - 1} \\
 &= \frac{5e^{-2} + 5e^{-2}(2-1)^2 - 5e^{-1}}{1} \\
 &= 10e^{-2} - 5e^{-1} \\
 &= -0.4860 \quad (\text{to 4 decimal places})
 \end{aligned}$$

So the rate of change of V with respect to t is greater in the first minute.

(Note that technically $-0.4860 > -8.1606$ but the negative refers to a decrease in the volume of fluid present and so the fluid is decreasing at a higher rate in the first hour.)

(1 mark) correct rates of change

(1 mark) correct conclusion

e. i. $V(t) = 5e^{-t} + 5e^{-t}(t-1)^2$, $t \geq 0$

$$V'(t) = -5e^{-t} - 5e^{-t}(t-1)^2 + 5e^{-t} \times 2(t-1) \times 1 \quad (\text{product rule}) \quad \textbf{(1 mark)}$$

$$\begin{aligned}
 &= -5e^{-t}(1 + (t-1)^2 - 2(t-1)) \\
 &= -5e^{-t}(1 + t^2 - 2t + 1 - 2t + 2) \\
 &= -5e^{-t}(t^2 - 4t + 4) \\
 &= -5e^{-t}(t-2)^2
 \end{aligned}$$

(1 mark)

A stationary point occurs when

$$V'(t) = 0$$

$$\text{So } -5e^{-t}(t-2)^2 = 0$$

$$\text{Now, } -5e^{-t} \neq 0$$

$$\text{So, } (t-2)^2 = 0$$

$$t = 2$$

A stationary point occurs on the graph of $y = V(t)$ when $t = 2$.

(1 mark)

- ii. From the graph it appears that at $x = 2$ there could be a stationary point of inflection. You must confirm that however. Use the first derivative test.

$$\text{At } t = 1, \quad V'(1) = -5e^{-1}(1-2)^2$$

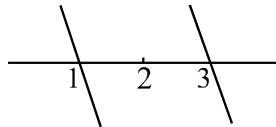
$$= -5e^{-1}$$

$$< 0$$

$$\text{At } t = 3, \quad V'(3) = -5e^{-3}(3-2)^2$$

$$= -5e^{-3}$$

$$< 0$$



At $t = 2$, there is a stationary point of inflection.

(1 mark) answer

(1 mark) first derivative test

- iii. If an employee momentarily stops the liquid from leaking then the gradient of the graph of $y = V(t)$ will be zero for an instant. This corresponds to the stationary point of inflection on the graph.

Hence when $t = 2$,

$$V(2) = 5e^{-2} + 5e^{-2}(2-1)^2$$

$$= 10e^{-2}$$

Hence the exact volume present is $10e^{-2}$ litres.

(1 mark)

- f. From part e. i.,

$$V'(t) = -5e^{-t}(t-2)^2$$

Now, $e^{-t} > 0$ for all t ,

and $(t-2)^2 > 0$ for all t .

(1 mark)

The product $-5 \times e^{-t} \times (t-2)^2$ must therefore be negative for all t .

(1 mark)

Total 14 marks