

2003 Mathematical Methods

Written Examination 1 (facts, skills and applications)

Suggested answers and solutions

Part 1 (Multiple-choice) Answers

- | | | | | |
|-------|-------|-------|-------|-------|
| 1. D | 2. B | 3. B | 4. D | 5. E |
| 6. A | 7. C | 8. D | 9. D | 10. C |
| 11. C | 12. A | 13. A | 14. E | 15. E |
| 16. C | 17. E | 18. A | 19. E | 20. C |
| 21. A | 22. E | 23. D | 24. D | 25. B |
| 26. B | 27. B | | | |

Part 1 (Multiple-choice) Solutions

Question 1 [D]

Because the graph touches the x -axis at $x = 2$ with a minimum value, there needs to be a factor of $(x - 2)^2$.

If $x > 2$, the graph is positive. The only alternative satisfying both of these conditions is D.

Question 2 [B]

If $x = 2$ then $\log_2 2$ is 1. All other alternatives are true.

Question 3 [B]

$\frac{x^3 + 1}{x} = \frac{x^3}{x} + \frac{1}{x} = x^2 + \frac{1}{x}$ and so the two rules that could have been added are $g(x) = x^2$ and

$$h(x) = \frac{1}{x}.$$

Question 4 [D]

The value of P in $y = e^{kx} + P$ represents the horizontal asymptote that the graph approaches as x tends to negative infinity (k is positive). Since it is positive, the horizontal asymptote must be above the x -axis.

The y -intercept is 1 unit (e^0) above

$$y = P$$

The graph also tends to ∞ as x increases.

Question 5 [E]

For reflection in the y -axis, change any x to $-x$.

Therefore $\sqrt{x}$ becomes $\sqrt{-x}$.

A dilation factor of 2 from the x -axis doubles each y -value. The new equation becomes $y = 2\sqrt{-x}$.

Question 6 [A]

Dividing both sides of $0.5 \cos(2x) = 1$ by 0.5 gives $\cos(2x) = 2$. There are no real solutions to this equation, regardless of the domain.

Question 7 [C]

The graph repeats itself after π . This can be seen from the difference of the x -intercepts.

Question 8 [D]

The period of the function is 4.

$$\therefore \frac{2\pi}{k\pi} = 4 \text{ and so } k = 0.5$$

The graph oscillates around $y = 1$ and so Q is 1. Substituting $(-1, 3)$ into the equation gives

$$P \sin\left(\frac{-\pi}{2}\right) + 1 = 3$$

so $-P + 1 = 3$ giving $P = -2$.

Question 9 [D]

If $y = 2 \tan(2x)$ then $\frac{dy}{dx} = 4 \sec^2(2x)$.

$$\begin{aligned} 4 \sec^2(2x) &= 4 \times \frac{1}{\cos^2(2x)} \\ &= \frac{4}{\cos^2(2x)} \end{aligned}$$

Question 10**[C]**

Using the Product Rule

$$\frac{dy}{dx} = u'(x) \log_e(x) + u(x) \cdot \frac{1}{x}$$

At $x = 2$ this becomes

$$\frac{dy}{dx} = u'(2) \log_e(2) + \frac{u(2)}{2}$$

Question 11**[C]**

At $x = -1$ the graph has a stationary value because the gradient is zero. The gradient is positive on both sides of $x = -1$, that is, it does not change signs. It must be a stationary point of inflexion.

Question 12**[A]**

Since $g'(x) = f(x)$ the curve of $f(x)$ represents the gradient of $g(x)$. On the interval (a, b) the curve of $f(x)$ is negative. Hence the gradient of $g(x)$ is negative in this interval.

Question 13**[A]**

$$\begin{aligned} f(x) &= \int 4e^{2x} dx \\ &= \frac{4}{2} e^{2x} + c \text{ where } c \text{ is a constant.} \\ &= 2e^{2x} + c \end{aligned}$$

The first alternative satisfies this equation with $c = 3$.

Question 14**[E]**

The first integral can be divided in the following way:

$$\int_1^4 (2f(x) + 3) dx = \int_1^4 2f(x) dx + \int_1^4 3 dx$$

$$\text{Now } \int_1^4 2f(x) dx = 2 \times \int_1^4 f(x) dx = 4$$

$$\begin{aligned} \text{and } \int_1^4 3 dx &= [3x]_1^4 \\ &= (12 - 3) \end{aligned}$$

The answer is therefore $4 + 9 = 13$.

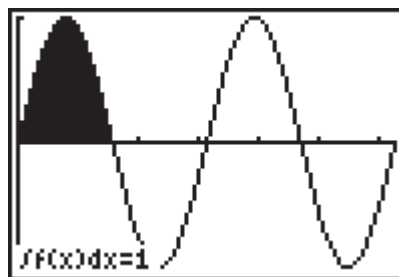
Question 15**[E]**

$$\int_0^5 g(x) dx = \int_0^5 f'(x) dx$$

Now the integral of a derivative takes us back to the function itself and so the integral on the RHS of this equation is $[f(x)]_0^5$, that is $f(5) - f(0)$.

Question 16**[C]**

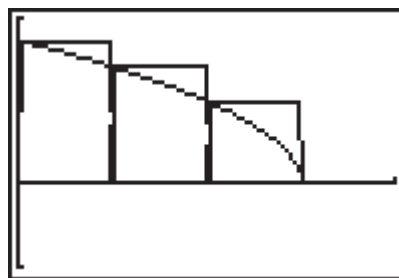
The graph of $y = \sin(2x)$ between 0 and 2π is shown below. The area between 0 and $\pi/2$ has been highlighted and the value is 1.



The total area between 0 and 2π is therefore 4.

Question 17**[E]**

The left-rectangles are shown on the curve. From left to right, their heights are: $\sqrt{3}$, $\sqrt{2}$ and $\sqrt{1}$ respectively.



The widths are all 1 unit and so the area is $\sqrt{3} + \sqrt{2} + 1$

Question 18**[A]**

$f(x) = 2(x + 3)^2 - 8$ is already written in turning-point format. The co-ordinates of the minimum value is $(-3, -8)$.

Question 19**[E]**

The expansion of $(2x - 3)^5$ is:

$$(2x)^5 - {}^5C_1(2x)^4(3)^1 + {}^5C_2(2x)^3(3)^2 - {}^5C_3(2x)^2(3)^3 + \dots$$

The last term here is the x^2 term and the coefficient is ${}^5C_3(2)^2(3)^3 = -1080$.

Question 20**[C]**

If a is a positive number then $x^2 + a$ will not reach the x -axis. Therefore, for $y = 0$ the values of x will be c and $-b$.

Question 21**[A]**

$f(x)$ has a vertical and horizontal asymptote of $x = 3$ and $y = 1$ respectively.

Interchange the x and y variables to find the inverse function.

Therefore the inverse function has a vertical and horizontal asymptote of $x = 1$ and $y = 3$ respectively.

Question 22**[E]**

$$2 \log_e(x) - \log_e(x + 2) = 1 + \log_e(y)$$

$$2 \log_e(x) - \log_e(x + 2) - 1 = \log_e(y)$$

Now $1 = \log_e e$ and $2 \log_e(x) = \log_e x^2$ and so

$$\log_e(y) = \log_e x^2 - \log_e(x + 2) - \log_e e$$

$$\log_e(y) = \log_e \frac{x^2}{e(x + 2)}$$

$$\therefore y = \frac{x^2}{e(x + 2)}$$

Question 23**[D]**

Normal distributions are symmetrical about the mean. Both of these are symmetrical about the same line and so $\mu_1 = \mu_2$.

X_2 is wider than X_1 and so $\sigma_1 < \sigma_2$.

Question 24**[D]**

Probability distribution functions must have both of the following properties:

$$\sum \Pr(X) = 1 \text{ and } 0 \leq \Pr(X) \leq 1.$$

Alternative I has probabilities adding to 1.2 and Alternative IV has negative probabilities.

The rest fulfil the requirements.

Question 25**[B]**

$\text{Var}(X) = E(X^2) - [E(X)]^2$ and the standard deviation is $\sqrt{\text{Variance}}$.

$$E(X) = 1 - p \text{ and } E(X^2) = 1^2(1 - p)$$

$$\begin{aligned} \therefore \text{Var}(X) &= 1 - p - (1 - p)^2 \\ &= 1 - p - (1 - 2p + p^2) \\ &= p - p^2 \\ &= p(1 - p) \end{aligned}$$

The standard deviation is $\sqrt{p(1 - p)}$.

Question 26**[B]**

Binomial distribution.

$$n = 20 \text{ and } p = 0.6$$

$$\Pr(x = 12) = {}^{20}C_{12} (0.6)^{12} (0.4)^8$$

Question 27**[B]**

Hypergeometric distribution.

$$N = 12, D = 4, n = 2$$

$\Pr(\text{At least 1 multigrain})$ is the same as $1 - \Pr(0 \text{ multigrain})$

$\Pr(0 \text{ multigrain})$ means that both rolls chosen are white.

$$\Pr(\text{Both white}) = \frac{{}^8C_2}{{}^{12}C_2}.$$

$$\text{The answer is } 1 - \frac{{}^8C_2}{{}^{12}C_2}.$$

PART II

Question 1

If the polynomial is divisible by the factor $(x + 1)$ then $f(-1) = 0$.

$$\begin{aligned} f(-1) &= 2(-1)^4 - 3(-1)^3 + 7(-1) + 11 \\ &= 2 + 3 - 7 + 11 \\ &= 9 \end{aligned}$$

It is not divisible by this factor $(x + 1)$.

Question 2

- a The point on the curve is where the gradient is 3 since it is parallel to the given straight line.

$$\frac{dy}{dx} = 2x - 2 = 3$$

Therefore $x = 2.5$

$$y = (2.5)^2 - 2(2.5) - 1 = 0.25.$$

Point P is $(2.5, 0.25)$.

- b The gradient of the normal $= -\frac{1}{3}$ as the product of the gradients of the normal and tangent is -1 .

$$\text{Equation: } y - 0.25 = -\frac{1}{3}(x - 2.5)$$

$$\text{Answer: } 12y + 4x = 13$$

Question 3

$$\sin(2\pi x) = -\sqrt{3} \cos(2\pi x)$$

Dividing both sides by $\cos(2\pi x)$ gives:

$$\tan(2\pi x) = -\sqrt{3} \quad \text{for } 0 \leq x \leq 1.$$

Tan is negative in the 2nd and 4th quadrants.

$$\text{Hence } 2\pi x = \pi - \frac{\pi}{3} \text{ and } 2\pi - \frac{\pi}{3}$$

$$2\pi x = \frac{2\pi}{3} \text{ and } \frac{5\pi}{3}$$

$$x = \frac{1}{3} \text{ and } \frac{5}{6}$$

Question 4

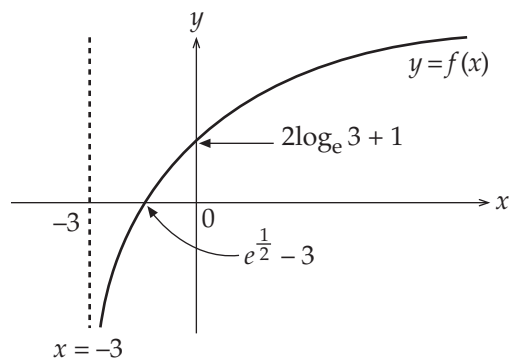
- a When $y = 0$: $2 \log_e(a + 3) = -1$

$$a + 3 = e^{-0.5}$$

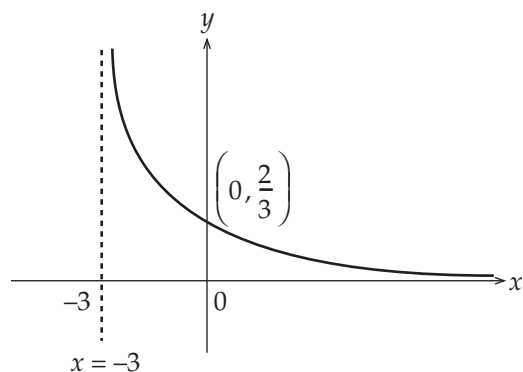
$$a = e^{-0.5} - 3$$

$$\text{When } x = 0: \quad b = 2 \log_e 3 + 1$$

- b The vertical asymptote below must be labelled with its equation: $x = -3$ and the two intercepts should be labelled with their co-ordinates $(0, 2 \log_e 3 + 1)$ and $(e^{-0.5} - 3, 0)$



- c The derivative function is shown below. Two asymptotes need to have their equations placed on the graph: vertical asymptote $x = -3$ and horizontal asymptote $y = 0$. The y -intercept has co-ordinates $(0, \frac{2}{3})$



Question 5

- a**
- Solving the intersection of

$$y_1 = -x + 1 \text{ and}$$

$$y_2 = 1 - e^{-x}$$

on the graphics calculator gives the x value of 0.567 to three decimal places.

$$\mathbf{b} \quad \int_0^{0.567} (-x + 1) - (1 - e^{-x}) dx$$

$$= \int_0^{0.567} (e^{-x} - x) dx$$

$$= \left[-e^{-x} - \frac{x^2}{2} \right]_0^{0.567}$$

$$= 0.272$$

The answer is 0.27 to two decimal places.

Question 5

$$\mathbf{a} \quad \text{Normalcdf}(-E99, 10, 20, 5) = 0.0228$$

$$\text{Normalcdf}(10, 30, 20, 5) = 0.9545$$

$$\text{Normalcdf}(30, E99, 20, 5) = 0.0228$$

give the proportions for small, medium and large respectively.

$$\begin{aligned} \mathbf{b} \quad & \text{Expected cost per plant is found from} \\ & \text{multiplying the proportions by the item costs:} \\ & = \$1.50 \times 0.0228 + \$2.50 \times 0.9545 + \$4.00 \times 0.0228 \\ & = \$2.5115 \end{aligned}$$

$$\begin{aligned} & \text{The expected cost of 100 plants is: } \$2.5115 \times 100 \\ & = \$251 \text{ to the nearest dollar.} \end{aligned}$$