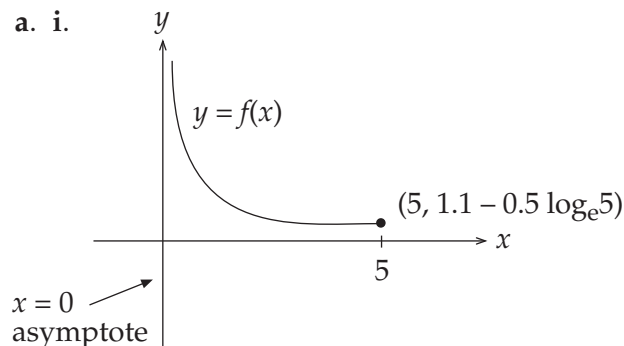


2002 Mathematical Methods

Written examination 2 (analysis task)

Suggested answers and solutions

1. a. i.



ii. The inverse function, f^{-1} , exists because f is a one-to-one function.

iii. To find f^{-1}

$$f(x) = 1.1 - 0.5 \log_e x$$

$$\text{let } y = 1.1 - 0.5 \log_e x$$

For inverse swap x with y

$$x = 1.1 - 0.5 \log_e y$$

$$x - 1.1 = -0.5 \log_e y$$

$$0.5 \log_e y = 1.1 - x$$

$$\log_e y = 2(1.1 - x)$$

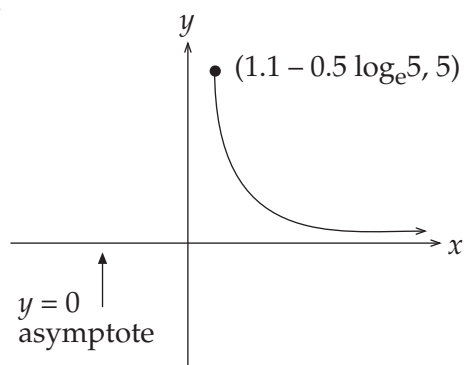
$$y = e^{2(1.1-x)}$$

$$\therefore f^{-1}(x) = e^{2(1.1-x)} = e^{2.2-2x}$$

iv. $\text{dom } f^{-1} = \text{ran } f$

$$= [1.1 - 0.5 \log_e 5, \infty)$$

v.



b. $f(x) = a - b \log_e(x), \quad 0 < x \leq 5$

substituting $(1, 0.5)$ gives

$$0.5 = a - b \log_e 1$$

$$0.5 = a - b \times 0$$

$$0.5 = a$$

substituting $(1.5, 0.3)$ gives

$$0.3 = 0.5 - b \log_e 1.5$$

$$b \log_e 1.5 = 0.2$$

$$b = \frac{0.2}{\log_e 1.5}$$

c. $f(x) = 1.1 - 0.5 \log_e(x)$

Half the button width is given by

$$f\left(\frac{x}{2}\right) = 1.1 - 0.5 \log_e\left(\frac{x}{2}\right)$$

$$= 1.1 - 0.5(\log_e x - \log_e 2)$$

$$= 1.1 - 0.5 \log_e x + 0.5 \log_e 2$$

$$= f(x) + 0.5 \log_e 2$$

$$= f(x) + \log_e 2^{\frac{1}{2}}$$

$$f\left(\frac{x}{2}\right) = f(x) + \log_e \sqrt{2}$$

Hence the time difference is:

$$f\left(\frac{x}{2}\right) - f(x) = \log_e \sqrt{2} \text{ as required to show.}$$

2. a. Let X represent the antenna length of Fhaisi butterflies.

$$X \sim N(20, 2^2)$$

On the graphics calculator:

press 2nd VARS

2: normalcdf $(-1E99, 16, 20, 2)$

ENTER

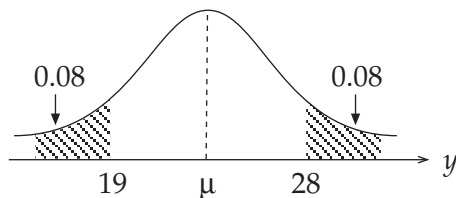
$$\Pr(X < 16) = 0.023 \text{ (3 decimal places)}$$

b. $Y \sim N(\mu, \sigma^2)$

Let Y represent the antenna length of Jojo butterflies.

$$\Pr(Y < 19) = 0.08$$

$$\Pr(Y > 28) = 0.08$$



By symmetry:

$$\mu = \frac{19 + 28}{2}$$

$$\mu = 23.5 \text{ mm}$$

$$\Pr(Y < 19) = 0.08$$

$$\Pr\left(Z < \frac{19 - 23.5}{\sigma}\right) = 0.08$$

using the graphics calculator

3: invNorm (0.08) ENTER

$$\frac{19 - 23.5}{\sigma} = -1.4051$$

$$\sigma = \frac{19 - 23.5}{-1.4051}$$

$$\sigma = 3.2 \text{ mm}$$

OR $\Pr(Y < 19) = 0.08$

$$\Pr(Y > 28) = 0.08$$

$$\Pr\left(Z < \frac{19 - \mu}{\sigma}\right) = 0.08$$

On graphics calculator:

invNorm(0.08) is -1.4051

$$\frac{19 - \mu}{\sigma} = -1.4051$$

$$19 = -1.4051\sigma + \mu \quad \textcircled{1}$$

$$\Pr\left(Z < \frac{19 - \mu}{\sigma}\right) = 0.08$$

$$\Pr\left(Z < \frac{28 - \mu}{\sigma}\right) = 0.92$$

invNorm (0.92) is 1.4051

$$\therefore \frac{28 - \mu}{\sigma} = 1.4051$$

$$28 = 1.4051\sigma + \mu \quad \textcircled{2}$$

$$\textcircled{2} + \textcircled{1} \text{ gives } 47 = 2\mu$$

$$\therefore \mu = 23.5 \text{ mm}$$

substituting into $\textcircled{1}$ gives

$$19 = -1.4051\sigma + 23.5$$

$$1.4051\sigma = 23.5 - 19$$

$$\sigma = \frac{23.5 - 19}{1.4051}$$

$$\therefore \sigma \approx 3.2 \text{ mm}$$

c. $\Pr(\text{Jojos}) = 0.2$

$$\Pr(\text{Fhaisis}) = 0.8$$

Let J represent the number of Jojo butterflies.

$$X \sim \text{Bi}(10, 0.2, 4)$$

$$\begin{aligned} \Pr(J = 4) &= {}^{10}C_4 (0.2)^4 (0.8)^6 \\ &\approx 0.088 \text{ (3 decimal places)} \end{aligned}$$

$$\begin{array}{rcl} \text{d. i.} & \text{Fhasis} & 0.5 \times 0.8 \\ & \text{Jojo} & 0.1370 \times 0.2 \\ & \hline & \Sigma \text{Sum} & 0.427 \end{array}$$

$$\begin{aligned} \text{ii.} & \frac{0.8 \times 0.5}{0.4274} \\ & = 0.936 \text{ (3 decimal places)} \end{aligned}$$

$$3. \text{ a. } y = \frac{1}{2}(2x^4 - x^3 - 5x^2 + 3x)$$

$$\text{For stationary points } \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = 4x^3 - \frac{3}{2}x^2 - 5x + \frac{3}{2} = 0$$

$$\text{b. i. When } x = 1$$

$$\begin{aligned} \frac{dy}{dx} &= 4 - \frac{3}{2} - 5 + \frac{3}{2} \\ &= -1 \text{ (gradient of tangent)} \end{aligned}$$

gradient of normal is

$$m = \frac{-1}{-1} = 1$$

when $x = 1$

$$\begin{aligned} y &= \frac{1}{2}(2 - 1 - 5 + 3) \\ &= \frac{1}{2}(-1) \\ &= -\frac{1}{2} \end{aligned}$$

substitute $\left(1, -\frac{1}{2}\right)$, $m = 1$ into

$$y + \frac{1}{2} = 1(x - 1)$$

$$y = x - \frac{3}{2}$$

$$\therefore y = x - \frac{3}{2} \text{ is equation of normal}$$

$$\text{ii. } y = \frac{1}{2}(2x^4 - x^3 - 5x^2 + 3x)$$

$$y = x - \frac{3}{2}$$

Solve simultaneously to find point(s) of intersection.

$$\frac{1}{2}(2x^4 - x^3 - 5x^2 + 3x) = x - \frac{3}{2}$$

$$2x^4 - x^3 - 5x^2 + 3x = 2x - 3$$

$$2x^4 - x^3 - 5x^2 + x + 3 = 0$$

$$(x - 1)(x + 1)^2(2x - 3) = 0$$

$$\therefore x = 1, -1 \text{ or } \frac{3}{2}$$

Since $(x + 1)$ is a multiple factor then the line will touch when $x = -1$.

$$\begin{aligned} y &= x - \frac{3}{2} \\ &= -1 - \frac{3}{2} \end{aligned}$$

$$y = -\frac{5}{2} \quad \left(-1, -\frac{5}{2}\right)$$

Find the points of intersection of the curve and the normal can also be found using the graphics calculator.

The gradient function of the curve is:

$$\frac{dy}{dx} = \frac{1}{2}(8x^3 - 3x^2 - 10x + 3)$$

$$\text{When } x = -1, \frac{dy}{dx} = 1$$

The gradient of the curve at B $(-1, -2.5)$ is the same as the gradient of the line

$$y = x - \frac{3}{2}, \text{ the point of intersection, so}$$

$$y = x - \frac{3}{2} \text{ is tangent to the point at B.}$$

$$\text{c. i. Shaded area}$$

$$= \int_{-1}^1 \frac{1}{2}(2x^4 - x^3 - 5x^2 + 3x) - \left(x - \frac{3}{2}\right) dx$$

$$= \int_{-1}^1 \left(x^4 - \frac{x^3}{2} - \frac{5x^2}{2} + 0.5x + \frac{3}{2}\right) dx$$

$$\begin{aligned}\text{ii. Area} &= \left[\frac{x^5}{5} - \frac{x^4}{8} - \frac{5x^3}{6} + \frac{x^2}{4} + \frac{3x}{2} \right]_{-1}^1 \\ &= \left(\frac{1}{5} - \frac{1}{8} - \frac{5}{6} + \frac{1}{4} + \frac{3}{2} \right) - \left(-\frac{1}{5} - \frac{1}{8} + \frac{5}{6} + \frac{1}{4} - \frac{3}{2} \right) \\ &= \frac{2}{5} - \frac{10}{6} + \frac{6}{2} \\ &= 1.73 \text{ (2 decimal places)}\end{aligned}$$

$$4. x(t) = 15 + 6 \sin\left(\frac{\pi t}{3}\right)$$

a. i. Maximum occurs when

$$\begin{aligned}\sin\left(\frac{\pi t}{3}\right) &= 1 \\ \therefore x(t) &= 15 + 6\end{aligned}$$

$$= 21 \text{ metres}$$

ii. Minimum height occurs when

$$\begin{aligned}\sin\left(\frac{\pi t}{3}\right) &= -1 \\ \therefore x(t) &= 15 + 6(-1) \\ &= 15 - 6 \\ &= 9 \text{ metres}\end{aligned}$$

$$15 + 6 \sin\left(\frac{\pi t}{3}\right) = 9$$

$$6 \sin\left(\frac{\pi t}{3}\right) = -6$$

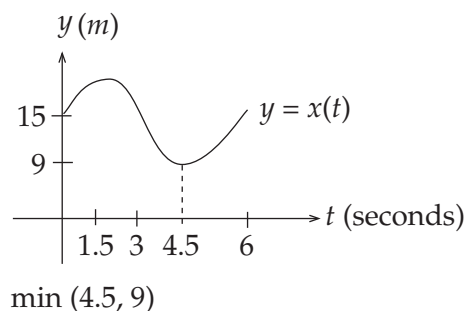
$$\sin\left(\frac{\pi t}{3}\right) = -1$$

$$\frac{\pi t}{3} = \sin^{-1}(-1)$$

$$\frac{\pi t}{3} = \frac{3\pi}{2}$$

$$t = \frac{9}{2} = 4.5 \text{ seconds}$$

Or Using graphics calculator:



$$\text{b. } y(t) = 15 + e^{0.04t} \sin\left(\frac{\pi t}{3}\right), \quad 0 < t \leq 60$$

i. Using graphics calculator

$$y_1 = 15 + e^{0.04t} \sin\left(\frac{\pi t}{3}\right), \quad 0 < t \leq 60$$

$$y_2 = 6$$

2nd CALC 5: Intersect

guess 58 seconds

Platform is first 6m above the ground after 58.03 seconds.

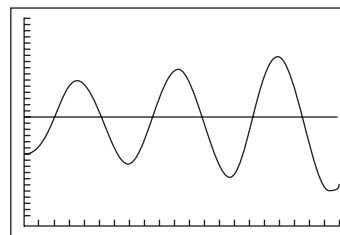
$$\text{ii. } y_1 = 15 + e^{0.04t} \sin\left(\frac{\pi t}{3}\right)$$

$$y_2 = 15$$

In **WINDOW**,

let $X_{\min} = 40$ and $X_{\max} = 59$

ZOOM 0: zoomfit



from the graph, there are 6 points of intersection from $t = 40$ to 59.

$$\text{iii. Let } y_1 = 15 + e^{0.04t} \sin\left(\frac{\pi t}{3}\right)$$

and $y_2 = 24$

Set **WINDOW** $X_{\min} = 0$, $X_{\max} = 60$

ZOOM 0: zoomfit

2nd **TRACE** 5: Intersect

$t = 55$ seconds (to the nearest second)

$$\text{c. i. } y = 15 + e^{0.04t} \sin\left(\frac{\pi t}{3}\right)$$

using the product rule

$$\begin{aligned}\frac{dy}{dt} &= e^{0.04t} \times \frac{\pi}{3} \cos\left(\frac{\pi t}{3}\right) \\ &\quad + 0.04e^{0.04t} \sin\left(\frac{\pi t}{3}\right) \\ &= e^{0.04t} \left(\frac{\pi}{3} \cos\left(\frac{\pi t}{3}\right) + 0.04 \sin\left(\frac{\pi t}{3}\right) \right)\end{aligned}$$

- ii. Platform is closest to the ground when

$$\frac{dy}{dt} = 0$$

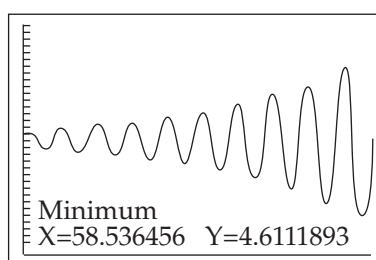
$$\text{i.e. } e^{0.04t} \left(\frac{\pi}{3} \cos\left(\frac{\pi t}{3}\right) + 0.04 \sin\left(\frac{\pi t}{3}\right) \right) = 0$$

Using the graphics calculator, enter

$$y_1 = 15 + e^{0.04t} \sin\left(\frac{\pi t}{3}\right)$$

and find the minimum in the domain $[0, 60]$

Graphics calculator display



The platform is closest to the ground when $t = 58.54$ seconds (2 decimal places) and when $y = 4.61$ metres (2 decimal places)

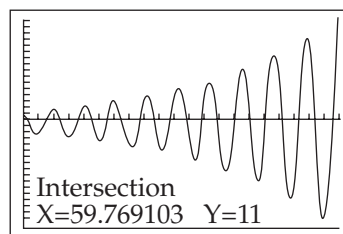
d. $-11 \leq \frac{dy}{dt} \leq 11$

$$-11 \leq e^{0.04t} \left(\frac{\pi}{3} \cos\left(\frac{\pi t}{3}\right) + 0.04 \sin\left(\frac{\pi t}{3}\right) \right) \leq 11$$

y_1

y_2

y_3



$$X_{\min} = 0$$

$$X_{\max} = 60$$

Zoomfit

for $t \in (0, 60]$, y_2 is always greater than y_1 , and y_2 is less than y_3 for $t \in (0, 59.769]$

$$\text{so } -11 \leq \frac{dy}{dt} \leq 11 \text{ for } t \in (0, 59.769]$$

(to 3 decimal places)

e. When $t = 60$, $\frac{dy}{dt} = 11.543443$

$$a \frac{dy}{dt} = \frac{dh}{dt}$$

$$\text{when } t = 60, \frac{dh}{dt} = 11$$

At $t = 60$

$$a \frac{dy}{dt} = \frac{dh}{dt}$$

$$a \times 11.543443 = 11$$

$$a = \frac{11}{11.543443}$$

$$\therefore a = 0.953$$

$$\frac{dh}{dt} \leq 11 \text{ for } t \in (0, 60] \text{ when } a = 0.953$$