

2002 Mathematical Methods

Written examination 1 (facts, skills and applications)

Suggested answers and solutions

Part I (Multiple-choice) Answers

- | | | | | |
|-------|-------|-------|-------|-------|
| 1. E | 2. E | 3. D | 4. E | 5. C |
| 6. A | 7. A | 8. C | 9. B | 10. A |
| 11. D | 12. D | 13. C | 14. B | 15. D |
| 16. C | 17. A | 18. B | 19. A | 20. B |
| 21. B | 22. D | 23. D | 24. A | 25. A |
| 26. B | 27. E | | | |

1. Amplitude = 1 [E]

Period = 4π

$$\Rightarrow \frac{2\pi}{n} = 4\pi$$

$$n = \frac{2\pi}{4\pi}$$

$$n = \frac{1}{2}$$

This eliminates A and C.

Shape is a cosine curve translated 1 unit

up, so the answer is $y = 1 + \cos\left(\frac{x}{2}\right)$

OR

Can use graphics calculator in RAD mode or substitute $x = 0$ in remaining answers to find when $y = 2$.

2. $\sin(2x) = 1 \quad x \in [0, 4\pi]$ [E]

$$2x = \sin^{-1}(1) \quad 2x \in [0, 8\pi]$$

$$2x = \frac{\pi}{2}, \frac{\pi}{2} + 2\pi, \frac{\pi}{2} + 4\pi, \frac{\pi}{2} + 6\pi, \frac{\pi}{2} + 8\pi$$

$$2x = \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, \frac{13\pi}{2}$$

$$\therefore x = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \frac{13\pi}{4}$$

$$\text{sum} = \frac{\pi}{4} + \frac{5\pi}{4} + \frac{9\pi}{4} + \frac{13\pi}{4}$$

$$= \frac{28\pi}{4}$$

$$= 7\pi$$

3. $y = 18 - 5\sin\left(\frac{\pi t}{12}\right)$ [D]

max occurs when:

$$\sin\left(\frac{\pi t}{12}\right) = -1 \quad \text{i.e. } y = 18 - 5(-1)$$

$$= 18 + 5$$

$$\frac{\pi t}{12} = \sin(-1) \quad = 23^\circ \text{C}$$

$$\frac{\pi t}{12} = \frac{3\pi}{2}$$

$t = 18$ hours after midnight is 6 pm

OR

Can use graphics calculator in RAD mode.

$$\text{Type } y_1 = 18 - 15\sin\left(\frac{\pi x}{12}\right), X_{\min} = 0,$$

$$X_{\max} = 24, \text{ZOOMFIT}$$

Find max turning point at (18, 23).

18 hours after midnight is 6 pm.

4. x -intercepts are $x = -2, -1, 1, 3$ [E]

So factors are: $(x+2)(x+1)(x-1)(x-3)$
 $(x+3)$ is not a factor so this eliminates A, B and C.

The graph 

is the shape of a negative quartic function which eliminates D and can be presented as

$$y = -(x+2)(x+1)(x-1)(x-3)$$

$$\text{or } y = (x+2)(x+1)(x-1)(3-x)$$

Can substitute $x = 0$ to check for a negative intercept.

5. Vertical asymptote at $x = 3$, so $b = -3$ [C]
 Horizontal asymptote at $y = 2$, so $c = 2$
 Eliminate A, D and E.

$$y = \frac{a}{x-3} + 2$$

Graph passes through $(4, 0)$ substituting gives,

$$0 = \frac{a}{4-3} + 2$$

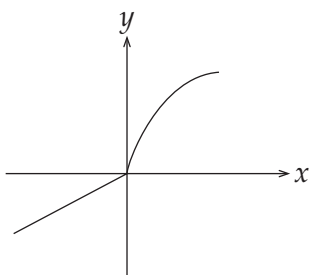
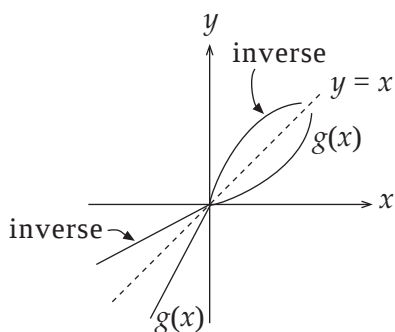
$$0 = a + 2$$

$$a = -2$$

6. $y = f(-x)$ gives a reflection about the y -axis. [A]

So the vertical asymptote will be reflected to become $x = -1$.

7. The graph of the inverse function is a reflection of $g(x)$ about the line $y = x$. [A]



8. Using the graphics calculator store the x values in L1 [C]

$\{1, 2, 3, 4, 5, 6, 7, 8\} \rightarrow L1$

and the y values in L2

$\{1.6, 2.6, 4.3, \dots\} \rightarrow L2$

Shape does not show linear, circular or logarithmic function, so eliminate A, D and E.

Either a power or exponential function.

To check which is the better fit press:

STAT CALC A: PwrReg ENTER

$$r = 0.9596 \quad r^2 = 0.9208$$

STAT CALC 0: ExpReg ENTER

$$r^2 = 0.9999$$

Exponential has r^2 closer to 1.

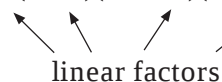
9. $x^4 + x^3 - 3x^2 - 3x$ [B]

$$= x(x^3 + x^2 - 3x - 3) \quad \text{common factor}$$

$$= x[x^2(x+1) - 3(x+1)] \quad \text{grouping}$$

$$= x(x+1)(x^2 - 3)$$

$$= x(x+1)(x + \sqrt{3})(x - \sqrt{3}) \quad \text{DOPS}$$

 linear factors

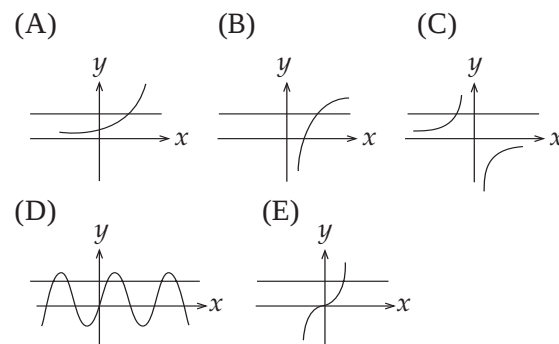
10. Let $m = 2^{\log_e(x+5)}$ [A]

Take the $\log_2$ of both sides

$$\log_2 m = \log_2(2^{\log_e(x+5)})$$

$$\text{so } m = x + 5$$

11. A sketch of each function in the given domain: [D]

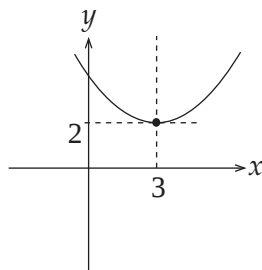


Shows that all functions except $f(x) = \sin x$ have one x value for every y value. i.e. are one-to-one.

For $f(x) = \sin x$, $x \in \mathbb{R}$, the horizontal line test shows that there are many x values for the same y value, therefore not one-to-one.

12. For $f(x)$ to have an inverse function, it must be one-to-one. [D]

$f(x) = (x-3)^2 + 2$ is a many-to-one function if $x \in \mathbb{R}$.



$f(x)$ is a quadratic function with turning point at $(3, 2)$.

Use x values on one side of the turning point as the domain to create a one-to-one function.

Therefore the domain can be $x \leq 3$

13. The graph of $f(x)$ yields the following observations for its gradient. [C]

when $x = -1$ steep positive gradient

when $x = 0$ positive gradient, steepness decreasing

when $x = 1$ positive gradient closer to 0

when $x \geq 2$ small, positive gradient approaching 0

14. $y = \log_e(\cos(2x))$ [B]

let $u = \cos 2x$, $\frac{du}{dx} = -2 \sin 2x$

then $y = \log_e u$, $\frac{dy}{du} = \frac{1}{u}$

Using the chain rule

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= \frac{1}{u} \times -2 \sin 2x \\ &= \frac{1}{\cos 2x} \times -2 \sin 2x \\ &= \frac{-2 \sin 2x}{\cos 2x} \\ &= -2 \tan 2x \end{aligned}$$

15. Equation of normal $y = mx + c$ on curve $y = x \sin x$. [D]

Let $u = x$ and $v = \sin x$

$$\therefore \frac{du}{dx} = 1 \quad \therefore \frac{dv}{dx} = \cos x$$

$$y = uv$$

$$\frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx} \quad (\text{product rule})$$

$$= x \cos x + \sin x \times 1$$

$$= x \cos x + \sin x$$

when $x = \pi$

$$y = \pi \times \sin \pi$$

$$= \pi \times 0$$

$$= 0$$

$(\pi, 0)$ point

$$\frac{dy}{dx} = \pi \cos \pi + \sin \pi$$

$$= \pi(-1) + 0$$

$$= -\pi \text{ gradient of tangent}$$

Gradient of normal

$$m = \frac{-1}{-\pi} = \frac{1}{\pi}$$

substitute $(\pi, 0)$ and $m = \frac{1}{\pi}$

in $y = mx + c$

$$0 = \frac{1}{\pi} \times \pi + c$$

$$0 = 1 + c$$

$$c = -1$$

Equation $y = \frac{1}{\pi}x - 1$

$$y = \frac{1}{\pi}(x - \pi)$$

16. $\frac{dy}{dx} = -e^{-x}$ [C]

when $x = 0$

$$\frac{dy}{dx} = -e^{-0}$$

$$= -1$$

17. $f(x+h) \approx f(x) + hf'(x)$ [A]
 $f(3.02) = f(3 + 0.02)$ where $x = 3$ $h = 0.02$
 $\approx f(3) + 0.02f'(3)$

18. $f(0) = 0$ and $f(-3) = 0$ [B]

The function passes through the points $(0, 0)$ and $(-3, 0)$

$$f'(0) = 0 \text{ and } f'(-1) = 0.$$

There are stationary points at $x = 0$ and at $x = -1$

Therefore eliminate C.

$$f'(x) > 0 \text{ for } x < -1$$

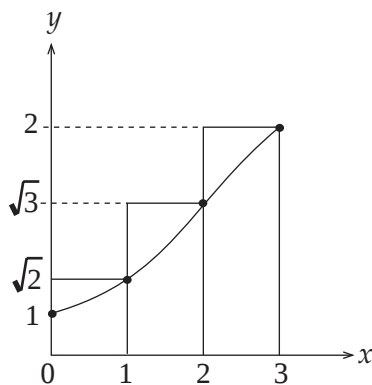
The function is increasing when $x < -1$, eliminate A and E.

$$f'(x) < 0 \text{ for } x > -1 \setminus \{0\}$$

The function is decreasing for $x > -1 \setminus \{0\}$ eliminate D.

19. $y = \sqrt{1+x}$ [A]

x	0	1	2	3
y	1	$\sqrt{2}$	$\sqrt{3}$	2



Area of rectangles

$$= 1 \times \sqrt{2} + 1 \times \sqrt{3} + 1 \times 2$$

$$= \sqrt{2} + \sqrt{3} + 2$$

20. $f'(x) = 2 \cos(5x)$ [B]

antidifferentiating gives

$$f(x) = \frac{2}{5} \sin(5x) + c$$

To check differentiate answer.

21. $\frac{dy}{dx} = \frac{3}{(2x+1)^{\frac{1}{2}}}$ [B]

$$\frac{dy}{dx} = 3(2x+1)^{-\frac{1}{2}}$$

antidifferentiating gives

$$y = \frac{3(2x+1)^{\frac{1}{2}}}{\frac{1}{2} \times 2} + c$$

$$y = 3(2x+1)^{\frac{1}{2}} + c$$

22. Area from a to b: $\int_a^b f(x)dx$ [D]

Area from b to c:

$$\left| \int_b^c f(x)dx \right| = -\int_b^c f(x)dx = \int_c^b f(x)dx$$

$$\text{Total Area} = \int_a^b f(x)dx + \int_c^b f(x)dx$$

23. A discrete random variable is a 'counting' number (not a measurement). [D]

Goals are countable, you cannot have half a goal.

24. Let X be the number of \$10 chips drawn. [A]

Therefore $X = 0, 1, 2, 3$ or 4 .

Hypergeometric distribution (without replacement).

$$N = 20, n = 4, D = 5$$

Pr(at least one \$10 chip)

$$= \Pr(X \geq 1)$$

$$= 1 - \Pr(X = 0)$$

$$= 1 - \frac{{}^5C_0 \times {}^{15}C_4}{{}^{20}C_4}$$

$$= 1 - \frac{{}^{15}C_4}{{}^{20}C_4}$$

25. $\mu = 10$ and $\sigma = 3$

[A]

For a binomial distribution $\mu = np$ and

$$\sigma = \sqrt{npq}, q = 1 - p$$

$$10 = np \text{ and } 3 = \sqrt{npq}$$

$$9 = npq$$

$$9 = 10q$$

$$\therefore q = \frac{9}{10} = 0.9$$

$$p = 1 - q$$

$$p = 1 - 0.9$$

$$p = 0.1$$

26. $X \sim N(4.7, 1.2^2)$

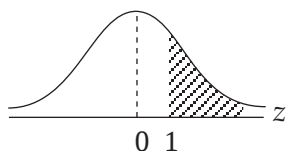
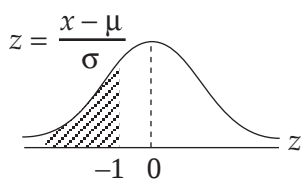
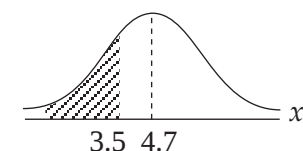
[B]

$$\Pr(X < 3.5)$$

$$= \Pr\left(Z < \frac{3.5 - 4.7}{1.2}\right)$$

$$= \Pr(Z < -1)$$

$$= \Pr(Z > 1)$$



27. $\mu = ?$ and $\sigma = 3$

[E]

$$\Pr(X < 250) = 0.01$$

$$\Pr\left(Z < \frac{250 - \mu}{3}\right) = 0.01$$

Using the graphics calculator,

press **2nd** **VARS** and choose

3: invNorm (0.01) ENTER

$$\frac{250 - \mu}{3} = -2.326$$

$$250 - \mu = -6.979$$

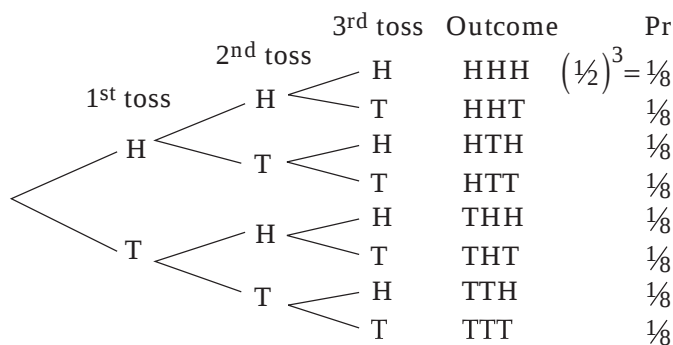
$$\mu = 250 + 6.979$$

$$= 256.979$$

$$\approx 257$$

Part 2: Short-answers

1. a. Using a tree diagram:



$$\Pr(X = 0) = \Pr(TTT) = \frac{1}{8}$$

$$\Pr(X = 1) = \Pr(TTH) + \Pr(THT) + \Pr(HTT) = \frac{3}{8}$$

$$\Pr(X = 2) = \Pr(THH) + \Pr(HTH) + \Pr(HHT) = \frac{3}{8}$$

$$\Pr(X = 3) = \Pr(HHH) = \frac{1}{8}$$

OR Using the Binomial Distribution: $X \sim Bi\left(3, \frac{1}{2}\right)$

$$\Pr(X = 0) = {}^3C_0 \times \left(\frac{1}{2}\right)^0 \times \left(\frac{1}{2}\right)^3 = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$$

$$\Pr(X = 1) = {}^3C_1 \times \left(\frac{1}{2}\right)^1 \times \left(\frac{1}{2}\right)^2 = 3 \times \left(\frac{1}{2}\right)^3 = \frac{3}{8}$$

$$\Pr(X = 2) = {}^3C_2 \times \left(\frac{1}{2}\right)^2 \times \left(\frac{1}{2}\right)^1 = 3 \times \left(\frac{1}{2}\right)^3 = \frac{3}{8}$$

$$\Pr(X = 3) = {}^3C_3 \times \left(\frac{1}{2}\right)^3 \times \left(\frac{1}{2}\right)^0 = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$$

OR Using the graphics calculator,

2nd **VARS** 0: binompdf $\left(3, \frac{1}{2}\right)$

will list the probability distribution.

Answer can be converted to fractions using MATH 1: Frac

X Pr(X = x)

0 $\frac{1}{8}$

1 $\frac{3}{8}$

2 $\frac{3}{8}$

3 $\frac{1}{8}$

b. $E(X) = \sum xp(x)$

$$= 0 \times \frac{1}{8} + 1 \times \frac{3}{8} + 2 \times \frac{3}{8} + 3 \times \frac{1}{8}$$

$$= 0 + \frac{3}{8} + \frac{6}{8} + \frac{3}{8}$$

$$= \frac{12}{8}$$

$$= 1.5$$

OR

$E(X) = np$ for binomial distribution

$$= 3 \times \frac{1}{2}$$

$$= 1.5$$

2. Without replacement therefore Hypergeometric.

$$N = 100, n = 5, D = 5$$

Let X represent the number of defective alarms in the sample,

$$X = 0, 1, 2, 3, 4, 5$$

Accepted when $X = 0$

$$\Pr(X = 0) = \frac{{}^5C_0 {}^{95}C_5}{{}^{100}C_5}$$

$$= 0.770 \text{ (3 decimal places)}$$

Note: there are calculator programs that are available which can calculate this.

3. a. Period = 8

b. Amplitude = $\frac{1.5 + 0.5}{2} = 1.0$

4. a. $2 \times 2^{-2x} = 2002$

$$2^{-2x} = 1001 \quad (\div 2 \text{ both sides})$$

$$\log_e 2^{-2x} = \log_e 1001 \quad (\log_e \text{ both sides})$$

$$-2x \log_e 2 = \log_e 1001 \quad (\div -2 \log_e 2 \text{ both sides})$$

$$x = \frac{\log_e 1001}{-2 \log_e 2}$$

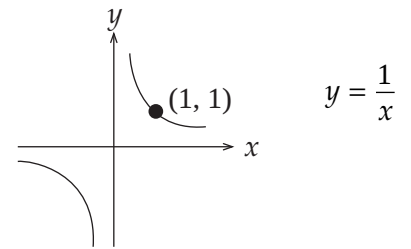
$$x = -4.984 \quad (3 \text{ decimal places})$$

b. $2 \log_e(3x + 1) - \log_e(x)$

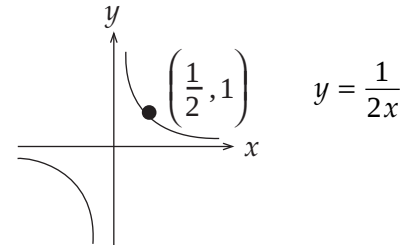
$$= \log_e(3x + 1)^2 - \log_e(x)$$

$$= \log_e \frac{(3x + 1)^2}{x}$$

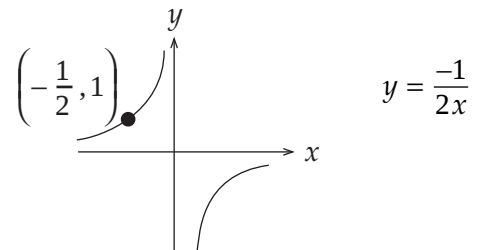
5. a.



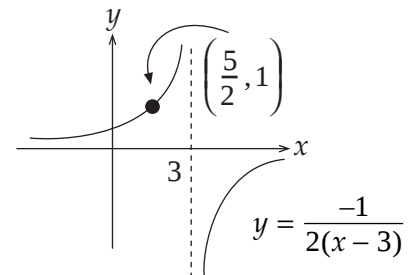
dilation by a factor of $\frac{1}{2}$ from y -axis



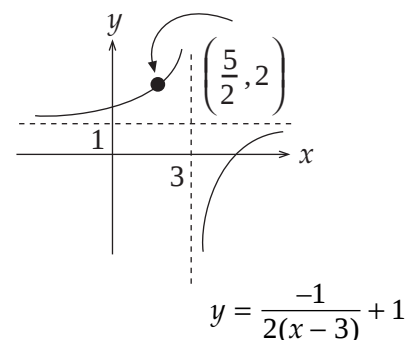
reflection in the y -axis



translation +3 units parallel to the x -axis



translation of +1 unit parallel to the y -axis



b. domain = $\mathbb{R} \setminus \{3\}$

range = $\mathbb{R} \setminus \{1\}$

6. a. $2 \sin(3x) - 1 = -0.5 \quad x \in \left[0, \frac{\pi}{2}\right]$

$$2 \sin(3x) = 0.5$$

$$\sin(3x) = 0.25$$

$$3x = \sin^{-1}(0.25)$$

2nd quad. 1st quad.

$$3x = 0.253, \pi - 0.253 \quad 3x \in \left[0, \frac{3\pi}{2}\right]$$

$$x = \frac{0.253}{3}, \frac{\pi - 0.253}{3}$$

$$x = 0.084, 0.963 \quad (3 \text{ decimal places})$$

OR Using graphics calculator in RADIAN mode.

Enter $y_1 = 2 \sin(3x) - 1$

$$y_2 = -0.5$$

WINDOW Xmin = 0

$$X_{\max} = \frac{\pi}{2}$$

ZOOMFIT

Graph and find points of intersection

Press **2nd** **TRACE** 5: Intersect

Answer $x = 0.084, 0.963$ (3 decimal places)

b. $f(x) = 2 \sin(3x) - 1$

$$f'(x) = 6 \cos(3x)$$

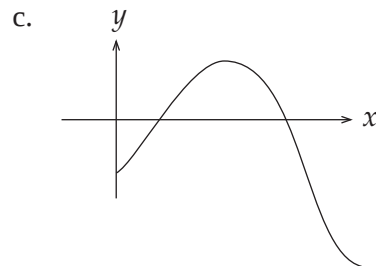
When $x = 1 \quad f'(x) = 6 \cos(3)$

$$= -5.940 \quad (3 \text{ d.p.})$$

OR Using graphics calculator

Press **2nd** **TRACE** 6: $\frac{dy}{dx} x = 1$ **ENTER**

$$\frac{dy}{dx} = -5.940 \quad (3 \text{ decimal places})$$



The rate of change is positive to the left of the turning point as shown on the graph.

Find the turning point:

Press **2nd** **TRACE** 4: Maximum etc.

This gives an x -coordinate of 0.523

The interval over which the rate of change is positive is $(0, 0.523)$ (3 decimal places)

OR To find the turning point:

$$f'(x) = 0$$

$$6 \cos(3x) = 0 \quad x \in \left[0, \frac{\pi}{2}\right]$$

$$\cos(3x) = 0 \quad 3x \in \left[0, \frac{3\pi}{2}\right]$$

$$3x = \cos^{-1}(0)$$

$$3x = \frac{\pi}{2}, \frac{3\pi}{2}$$

$$x = \frac{\pi}{6}, \frac{3\pi}{6}$$

x	$< \frac{\pi}{6}$	$\frac{\pi}{6}$	$> \frac{\pi}{6}$
$f'(x)$	$+ve$	0	$-ve$
	$\nearrow$	$\overline{}$	$\searrow$

Rate is positive when $x \in \left(0, \frac{\pi}{6}\right)$ i.e.

$$x \in (0, 0.524) \quad (3 \text{ decimal places})$$

7. a. $f'(x)$ does not exist at the points of discontinuity i.e. at $x = -2, 0, 4$.

$f'(x)$ does not exist where curves are not smooth i.e. at $x = 2$.

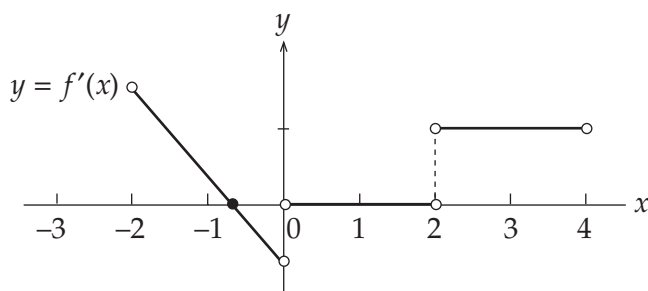
$f'(x) = 0$ when $x \approx -0.6$ and when $0 < x < 2$.

$f'(x) > 0$ when $-2 < x < -0.6$ and when $2 < x < 4$.

$f'(x) < 0$ when $-0.6 < x < 0$ and becomes more negative from left to right.

For $x \in (-2, -0.6)$, the positive gradient decreases from left to right approaching zero.

For $2 < x < 4$ $f'(x) = \frac{\text{rise}}{\text{run}} = \frac{2}{2} = 1$



- b. $\text{Dom } f' = (-2, 0) \cup (0, 2) \cup (2, 4)$
8. a. Algebraic method, solve simultaneously

$$2x^2 + 4x - 5 = 3x + 1$$

$$2x^2 + x - 6 = 0$$

$$(2x - 3)(x + 2) = 0$$

$$2x - 3 = 0 \text{ or } x + 2 = 0$$

$$x = \frac{3}{2}, x = -2$$

OR Using graphics calculator

$$y_1 = 2x^2 + 4x - 5$$

$$y_2 = 3x + 1$$

Zoomstandard

Find intersection by pressing

2nd TRACE 5: Intersect etc.

$$x = \frac{3}{2} \text{ and } x = -2$$

- b. When $x \in \left[-2, \frac{3}{2}\right]$

$$y_2 > y_1 \text{ where } y_1 = 2x^2 + 4x - 5$$

$$\text{and } y_2 = 3x + 1$$

$$\text{Area} = \int_{-2}^{\frac{3}{2}} (3x + 1) - (2x^2 + 4x - 5) dx$$

$$= \int_{-2}^{\frac{3}{2}} (3x + 1 - 2x^2 - 4x + 5) dx$$

$$= \int_{-2}^{\frac{3}{2}} (-2x^2 - x + 6) dx$$

$$= \left[\frac{-2x^3}{3} - \frac{x^2}{2} + 6x \right]_{-2}^{\frac{3}{2}}$$

$$= \left(\frac{-2\left(\frac{3}{2}\right)^3}{3} - \frac{\left(\frac{3}{2}\right)^2}{2} + 6\left(\frac{3}{2}\right) \right)$$

$$- \left(\frac{-2(-2)^3}{3} - \frac{(-2)^2}{2} + 6(-2) \right)$$

$$= \left(\frac{-27}{9} - \frac{9}{8} + 9 \right) - \left(\frac{16}{3} - 2 - 12 \right)$$

$$= \frac{45}{8} + \frac{26}{3}$$

$$\approx 14.292 \text{ square units}$$